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Integral means inequality of certain analytic functions

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Abstract

We obtain the integral means inequality of $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ $(|z|<1)$ and $k_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}}$ $(0\leqq\alpha<1,|z|<1)$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

that are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.

Let $g(z)$, $p_\alpha(z)$, $t(z)$, $q_\alpha(z)$ and $k_\alpha(z)$ denote the analytic functions in $\mathbb{U}$ by

$$g(z)=\frac{zf'(z)}{f(z)}, \tag{1.1}$$

$$p_\alpha(z)=\frac{1+(1-2\alpha)z}{(1-z)} \quad (0\leqq\alpha<1), \tag{1.2}$$

$$t(z)=\frac{f(z)}{z}, \tag{1.3}$$

$$q_\alpha(z)=\frac{1}{(1-z)^{2(1-\alpha)}} \quad (0\leqq\alpha<1) \tag{1.4}$$

and

$$k_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}} \quad (0\leqq\alpha<1), \tag{1.5}$$

respectively.

In this paper, we obtain the integral means inequality of the functions $f(z)$ in $\mathcal{A}$ and $k_\alpha(z)$.

Here we recall the concept of subordination between analytic functions. Let functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $f(z)$ is subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(|z| < 1)$. We denote this subordination by $f(z) \prec g(z)$. Let $g(z)$ be univalent in $\mathbb{U}$. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$(see CH. Pommerenke [3]).

We need the following subordination theorem of J. E. Littlewood.

Lemma 1.1(Littlewood [1]) *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |g(z)|^\mu \, d\theta.$$

Applying the lemma of Littlewood above, H. Silverman ([6]) showed the integral means inequalities for univalent functions with negative coefficients. S. Owa and T. Sekine ([4]) proved integral means inequalities with coefficients inequalities for normalized analytic functions and polynomials(see also Sekine et al. [5]).

In addition we need the following lemma of S. S. Miller and P. T. Mocanu.

Lemma 1.2(Miller and Mocanu [2]) *Let $g(z) = g_n z^n + g_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{U}$ with $g(z) \neq 0$ and $n \geqq 1$. If $z_0 = r_0 e^{i\theta_0}$ $(r_0 < 1)$ and*

$$|g(z_0)| = \max_{|z| \leqq |z_0|} |g(z)|$$

then

(i) $\dfrac{z_0 g'(z_0)}{g(z_0)} = k$

and

(ii) $\operatorname{Re}\left(\dfrac{z_0 g''(z_0)}{g'(z_0)}\right) + 1 \geqq k,$

where $k \geqq n \geqq 1$.

2. Integral means inequality for $f(z)$ and $k_\alpha(z)$.

Lemma 2.1. *Let $f(z)$ be in $\mathcal{A}$, $g(z)$ be the function given by* (1.1) and $p_\alpha(z)$ *be the function given by* (1.2). *If the function $f(z)$ satisfies*

$$\operatorname{Re}\left\{\beta \frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \begin{cases} \alpha\left(\beta + \frac{(1-2\alpha)\gamma}{2(1-\alpha)}\right), & (0 \leqq \alpha < 1/2) \\ \alpha\beta + \frac{(2\alpha^2+\alpha-1)\gamma}{2\alpha}, & (1/2 \leqq \alpha < 1) \end{cases} \tag{2.1}$$

for some real numbers $\beta > 0$ and $\gamma > 0$, then we have

$$g(z) \prec p_\alpha(z).$$

Proof. First, we shall prove Lemma 2.1 for $\alpha(0 \leqq \alpha < 1/2)$. Let us define the function $w(z)$ by

$$g(z) = \frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} \quad (w(z) \neq 1). \tag{2.2}$$

Thus we have an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1 (z \in \mathbb{U})$ for

$$\begin{aligned}
&\operatorname{Re}\left\{\beta\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \\
&= \operatorname{Re}\left\{\beta\left(\frac{1+(1-2\alpha)w(z)}{1-w(z)}\right)\right. \\
&\qquad \left. +\gamma\left(\frac{1+(1-2\alpha)w(z)}{1-w(z)} + \frac{z(1-2\alpha)w'(z)}{1+(1-2\alpha)w(z)} + \frac{zw'(z)}{1-w(z)}\right)\right\} \\
&> \alpha\left(\beta + \frac{(1-2\alpha)\gamma}{2(1-\alpha)}\right) \quad \left(\beta > 0,\ \gamma > 0,\ 0 \leqq \alpha < \frac{1}{2}\right).
\end{aligned}$$

If there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1, \tag{2.3}$$

then we have by Lemma 1.2,

$$w(z_0) = e^{i\theta}, \quad \frac{z_0 w'(z_0)}{w(z_0)} = k, \quad \operatorname{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) + 1 \geqq k \ (k \geqq 1).$$

For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\begin{aligned}
&\operatorname{Re}\left\{\beta\frac{z_0 f'(z_0)}{f(z_0)} + \gamma\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\} \\
&= \operatorname{Re}\left\{\left(\beta\frac{1+(1-2\alpha)w(z_0)}{1-w(z_0)}\right)\right. \\
&\qquad \left. +\gamma\left(\frac{1+(1-2\alpha)w(z_0)}{1-w(z_0)} + \frac{z_0(1-2\alpha)w'(z_0)}{1+(1-2\alpha)w(z_0)} + \frac{z_0 w'(z_0)}{1-w(z_0)}\right)\right\} \\
&= \operatorname{Re}\left\{-(1-2\alpha)(\beta+\gamma) + \frac{2(1-\alpha)(\beta+\gamma)}{1-w(z_0)} + \gamma\frac{(1-2\alpha)z_0 w'(z_0)}{1+(1-2\alpha)w(z_0)} + \gamma\frac{z_0 w'(z_0)}{1-w(z_0)}\right\} \\
&= -(1-2\alpha)(\beta+\gamma) + 2(1-\alpha)(\beta+\gamma)\operatorname{Re}\left\{\frac{1}{1-w(z_0)}\right\} \\
&\qquad +\gamma(1-2\alpha)\operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1+(1-2\alpha)w(z_0)}\right\} + \gamma\operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1-w(z_0)}\right\}
\end{aligned}$$

$$= -(1-2\alpha)(\beta+\gamma)+2(1-\alpha)(\beta+\gamma)\,\mathrm{Re}\left\{\frac{1}{1-w(z_0)}\right\}$$
$$+\gamma k(1-2\alpha)\,\mathrm{Re}\left\{\frac{w(z_0)}{1+(1-2\alpha)w(z_0)}\right\}+\gamma k\,\mathrm{Re}\left\{\frac{w(z_0)}{1-w(z_0)}\right\}$$
$$\leqq -(1-2\alpha)(\beta+\gamma)+(1-\alpha)(\beta+\gamma)+\frac{\gamma k(1-2\alpha)}{2(1-\alpha)}-\frac{\gamma k}{2}$$
$$= \alpha\beta+\alpha\gamma+\frac{\alpha\gamma}{2(1-\alpha)}(-k)$$
$$\leqq \alpha\beta+\alpha\gamma+\frac{\alpha\gamma}{2(1-\alpha)}(-1)$$
$$= \alpha\left(\beta+\frac{(1-2\alpha)\gamma}{2(1-\alpha)}\right),$$

which contradicts the hypothesis (2.1) for $\alpha(0 \leqq \alpha < 1/2)$. Therefore there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus we have

$$g(z) \prec p_\alpha(z).$$

Second, we shall prove Lemma 2.1 for $\alpha(1/2 \leqq \alpha < 1)$ in the same way. We show that the analytic function $w(z)$ defined by (2.2) satisfies $|w(z)| < 1 (z \in \mathbb{U})$ for

$$\mathrm{Re}\left\{\beta\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{f''(z)}{f'(z)}\right)\right\} > \alpha\beta+\frac{(2\alpha^2+\alpha-1)\gamma}{2\alpha}\quad\left(\beta>0,\ \gamma>0,\ \frac{1}{2}\leqq\alpha<1\right).$$

By Lemma 1.2, for the point z_0 satisfying (2.3), we have the following.

$$\mathrm{Re}\left\{\beta\frac{z_0f'(z_0)}{f(z_0)}+\gamma\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\right\}$$
$$= -(1-2\alpha)(\beta+\gamma)+2(1-\alpha)(\beta+\gamma)\,\mathrm{Re}\left\{\frac{1}{1-w(z_0)}\right\}$$
$$+\gamma k(1-2\alpha)\,\mathrm{Re}\left\{\frac{w(z_0)}{1+(1-2\alpha)w(z_0)}\right\}+\gamma k\,\mathrm{Re}\left\{\frac{w(z_0)}{1-w(z_0)}\right\}$$
$$\leqq -(1-2\alpha)(\beta+\gamma)+(1-\alpha)(\beta+\gamma)+\gamma k(2\alpha-1)\frac{1}{2\alpha}-\frac{\gamma k}{2}$$
$$= \alpha\beta+\alpha\gamma+\frac{(1-\alpha)\gamma}{2\alpha}(-k)$$
$$\leqq \alpha\beta+\alpha\gamma+\frac{(\alpha-1)\gamma}{2\alpha}$$
$$= \alpha\beta+\frac{(2\alpha^2+\alpha-1)\gamma}{2\alpha},$$

which contradicts the hypothesis (2.1) for $\alpha(1/2 \leqq \alpha < 1)$. Therefore there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus we have

$$g(z) \prec p_\alpha(z).$$

Lemma 2.2. *Let $f(z)$ be in $\mathcal{A}$, $g(z)$ be the function defined by* (1.1) *and $p_\alpha(z)$ be the function defined by* (1.2). *If the function $f(z)$ satisfies*

$$g(z) = \frac{zf'(z)}{f(z)} \prec p_\alpha(z) \quad (z \in \mathbb{U}),$$

then we have

$$\mathrm{Re}\,\{g(z)\} > \alpha.$$

Proof. Since $p_\alpha(z)$ is univalent in $\mathbb{U}$, we have $g(\mathbb{U}) \subset p_\alpha(\mathbb{U})$ by the assumption of this lemma. Thus we have $\mathrm{Re}\,\{g(z)\} > \alpha$, because $\mathrm{Re}\,\{p_\alpha(z)\} > \alpha$ in $\mathbb{U}$.

Lemma 2.3. *Let $f(z)$ be in $\mathcal{A}$. Let $g(z)$, $t(z)$ and $q_\alpha(z)$ be the functions defined by* (1.1), (1.3) and (1.4), *respectively. If the function $f(z)$ satisfies*

$$\mathrm{Re}\,\{g(z)\} = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (\alpha \in \mathbb{U}),$$

then we have

$$t(z) \prec q_\alpha(z).$$

Proof. Let us define the function $v(z)$ by

$$t(z) = \frac{f(z)}{z} = \frac{1}{(1 - v(z))^{2(1-\alpha)}}. \tag{2.4}$$

Thus we have an analytic function $v(z)$ in $\mathbb{U}$ such that $v(0) = 0$. Further, we prove that the analytic function $v(z)$ satisfies $|v(z)| < 1 (z \in \mathbb{U})$ for

$$\mathrm{Re}\{g(z)\} = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha. \tag{2.5}$$

If there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |v(z)| = |v(z_0)| = 1,$$

then we have by Lemma 1.2,

$$v(z_0) = e^{i\theta}, \ \frac{z_0 v'(z_0)}{v(z_0)} = k \geqq 1, \ \mathrm{Re}\left(\frac{z_0 v''(z_0)}{v'(z_0)}\right) + 1 \geqq k.$$

For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\begin{aligned}\frac{z_0 f'(z_0)}{f(z_0)} - 1 &= \frac{2(1-\alpha)z_0(1-v(z_0))^{1-2\alpha}(v'(z_0))}{(1-v(z_0))^{2(1-\alpha)}} \\ &= \frac{2(1-\alpha)z_0 v'(z_0)}{(1-v(z_0))} \\ &= \frac{2(1-\alpha)kv(z_0)}{(1-v(z_0))}.\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} &= 1 + 2(1-\alpha)k \operatorname{Re}\left\{\frac{v(z_0)}{1-v(z_0)}\right\} \\ &= 1 + 2(1-\alpha)k\left(-\frac{1}{2}\right) \\ &= 1-(1-\alpha)k \\ &\leqq \alpha,\end{aligned}$$

which contradicts the hypothesis (2.5). Therefore there is no $z_0 \in \mathbb{U}$ such that $|v(z_0)| = 1$. This implies that $|v(z)| < 1$ for all $z \in \mathbb{U}$. Thus we have

$$t(z) \prec q_\alpha(z).$$

We obtain the following result by using Lemmas 2.1, 2.2, 2.3 and 1.1.

Theorem 2.1. *Let $f(z)$ be in $\mathcal{A}$. Let $k_\alpha(z)$ be the function given by* (1.5). *If the function $f(z)$ satisfies*

$$\operatorname{Re}\left\{\beta\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \begin{cases}\alpha\left(\beta + \frac{(1-2\alpha)\gamma}{2(1-\alpha)}\right), & (0 \leqq \alpha < 1/2) \\ \alpha\beta + \frac{(2\alpha^2+\alpha-1)\gamma}{2\alpha}, & (1/2 \leqq \alpha < 1)\end{cases}$$

for some real numbers $\beta > 0$ and $\gamma > 0$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$, we have

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|k_\alpha\left(re^{i\theta}\right)\right|^\mu d\theta.$$

Proof. By Lemmas (2.1), (2.2) and (2.3), we have

$$t(z) \prec q_\alpha(z),$$

under the hypothesis of the theorem. Then by Lemma 1.1, we have

$$\int_0^{2\pi} \left|t\left(re^{i\theta}\right)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|q_\alpha\left(re^{i\theta}\right)\right|^\mu d\theta \tag{2.6}$$

for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$, under the hypothesis of the theorem.
Thus we obtain the following integral means inequality by (2.6)

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|k_{\alpha}\left(re^{i\theta}\right)\right|^{\mu} d\theta.$$

for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$, under the hypothesis of the theorem.

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