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Small-time Solvability of Primitive Equations of the Coupled Atmosphere and Ocean *

慶應義塾大学大学院理工学研究科 本多 泰理 (Hirotada Honda)
Graduate School of Science and Technology, Keio University

Abstract

In this paper, we discuss the two phase free boundary problem of the primitive equations for coupled atmosphere and ocean in three-dimensional strip with surface tension. Using the so-called p-coordinates, we prove temporally local existence of the unique strong solution to the transformed problem in Sobolev-Slobodetskiĭ spaces.

1 Introduction

The idea of numerical weather forecast was executed by Richardson in 1920's [17]. He derived a system of equations describing the motion of atmosphere, which is similar to Navier-Stokes equations. His attempt failed because of mainly the lack of stability of the calculations, however many attempts have carried on him.

In 1969 Bryan [1] formulated the model of the ocean circulation similar to the Richardson's model of the atmosphere by applying the hydrostatic approximation. In this model Boussinesq approximation and rigid lid hypothesis, which means that the ocean surface is fixed and flat, were used. On the other hand, Crowly [2] studied the free surface case, not the rigid lid, of the ocean numerically.

Mathematical arguments of primitive equations were begun in 1990's. One of the main features of primitive equations is the fact that the vertical velocity is determined by the horizontal velocities via the continuity equation, since the vertical velocity disappears in the vertical component of equations of motion due to the hydrostatic approximation. In [9], [10], [12]-[14], they also studied the evolutionary 3D coupled atmosphere and ocean model with rigid lid. In [14] they showed the well-posedness of the model formulated in [12] in the same function space as above.

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All the results cited above were obtained under the rigid lid hypothesis and the turbulent viscosity terms were added in primitive equations as an empirical claim. In addition to the turbulent viscosity we take into account the effect of the surface movement following Crowly [2], and adopt f-plane approximation, *i.e.* Coriolis parameter is a constant, where f is the deformation angle from the sphere over the plane. Furthermore, it is to be noted that in the literature cited above, ocean and atmosphere models are described in Cartesian and p-coordinates, respectively. Lions *et al.* [9], [10], [12]-[14] formulated the coupled atmosphere and ocean model by assuming that the height of pressure isobar coincides with the hight of the rigid lid. Besides, the assumption that the wind sheer and the stress force are equal is applied. To avoid such physically unrealistic assupmtions, we use p-coordinates for the ocean model in this paper.

Here, we enumerate the major features of this paper:

i. We consider two phase free boundary problem, in which the surface of the ocean is free, not the rigid lid.

ii. The boundary conditions on the free surface are described by the stress tensor and the effect of vapour.

iii. The original equations are transformed by the p-coordinates system.

iv. The existence of the strong solution is proved in the Sobolev-Slobodetskiĭ spaces.

The remainder of this paper is organized as follows. In section 2, we describe the mathematical formulation of our problem. Then the original problem is rewritten in p-coordinates, and a mapping is defined to transform it into the problem on the fixed time independent region. In section 3, we introduce function spaces. We use anisotropic Sobolev-Slobodetskiĭ spaces $W_2^{l,l/2}(Q_T \equiv \Omega \times (0,T))$, which are adequate to parabolic problems. In section 4, we solve the non-homogeneous linear problem in the original region. In section 5, we solve the nonlinear problem, making use of the iteration method for a small time interval.

2 Formulation of the problem

In this section, we formulate the problem considered in this paper as stated in §1. We are concerned with the free boundary problem of primitive equations for coupled atmosphere and the ocean. By adopting f-plane approximation, our problem can be considered in the strip-like region. By $x = (x_1, x_2, x_3)$, we denote orthogonal coordinate system with x_3 being the vertical direction and with $x_3 = 0$ being the equilibrium free surface. Let the surface and the bottom of the ocean be described by $x_3 = d(t, x')$ and $x_3 = -b(x')$ $(x' = (x_1, x_2))$, respectively, where $b(x')$ is a positive function satisfying $d(t, x') > -b(x')$. Then the domain $\Omega^s(t)$ of the ocean at time t is considered as $\{(x', x_3) | -b(x') < x_3 < d(t, x'), x' \in \mathbf{R}^2\}$, $\Omega^a(t)$ of the atmosphere at time t is considered as $\{(x', x_3) | d(x') < x_3 < u, x' \in \mathbf{R}^2\}$, with a positive constant u, representing the scale hight . The equations that we consider in this paper are as follows:

$$\begin{cases} \dfrac{\partial \mathbf{v}^a}{\partial t} + (\mathbf{v}^a \cdot \nabla)\mathbf{v}^a + w^a \dfrac{\partial \mathbf{v}^a}{\partial x_3} - \dfrac{1}{\varrho^a}\left[\mu_1^a \Delta \mathbf{v}^a + \mu_2^a \dfrac{\partial^2 \mathbf{v}^a}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v}^a \\ = -\dfrac{1}{\varrho^a}\nabla p^a + \mathbf{F}_1^a, \\ \dfrac{\partial p^a}{\partial x_3} = -\dfrac{p^a}{R\theta^a} g, \\ \dfrac{\partial \theta^a}{\partial t} + (\mathbf{v}^a \cdot \nabla)\theta^a + w^a \dfrac{\partial \theta^a}{\partial x_3} - \dfrac{1}{\varrho^a C_p}\left[\mu_3^a \Delta \theta^a + \mu_4^a \dfrac{\partial^2 \theta^a}{\partial x_3^2}\right] = F_2^a, \\ \dfrac{\partial q}{\partial t} + (\mathbf{v}^a \cdot \nabla) q + w^a \dfrac{\partial q}{\partial x_3} - \dfrac{1}{\varrho^a C_q}\left[\mu_5^a \Delta q + \mu_6^a \dfrac{\partial^2 q}{\partial x_3^2}\right] = F_3^a, \\ \dfrac{\partial \varrho^a}{\partial t} + \nabla \cdot (\varrho^a \mathbf{v}^a) + \dfrac{\partial}{\partial x_3}(\varrho^a w^a) = 0, \quad x \in \Omega^a(t), \ t > 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{\partial \mathbf{v}^s}{\partial t} + (\mathbf{v}^s \cdot \nabla)\mathbf{v}^s + w^s \dfrac{\partial \mathbf{v}^s}{\partial x_3} - \left[\mu_1^s \Delta \mathbf{v}^s + \mu_2^s \dfrac{\partial^2 \mathbf{v}^s}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v}^s = -\dfrac{1}{\varrho^s}\nabla p^s + \mathbf{F}_1^s, \\ \dfrac{\partial p^s}{\partial x_3} = -\varrho^s g, \\ \nabla \cdot \mathbf{v}^s + \dfrac{\partial w^s}{\partial x_3} = 0, \\ \dfrac{\partial \theta^s}{\partial t} + (\mathbf{v}^s \cdot \nabla)\theta^s + w^s \dfrac{\partial \theta^s}{\partial x_3} - \left[\mu_3^s \Delta \theta^s + \mu_4^s \dfrac{\partial^2 \theta^s}{\partial x_3^2}\right] = F_2^s, \\ \dfrac{\partial S}{\partial t} + (\mathbf{v}^s \cdot \nabla) S + w^s \dfrac{\partial S}{\partial x_3} - \left[\mu_5^s \Delta S + \mu_6^s \dfrac{\partial^2 S}{\partial x_3^2}\right] = F_3^s, \quad x \in \Omega^s(t), \ t > 0. \end{cases} \tag{2.2}$$

Here, $f\mathbf{A}\mathbf{v}$ is a Coriolis force with $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Coriolis parameter f; ∇ and Δ are 2 dimensional gradient and Laplacian, respectively; $\mathbf{F}_1^\tau$ $(\tau = a, s)$ are the horizontal components of external forces. $\mathbf{V}^\tau = (\mathbf{v}^\tau, w^\tau)$ $(\tau = a, s)$ is the 3 dimensional velocity vector filed of the ocean and the atmosphere with the horizontal component $\mathbf{v}^\tau$ and the vertical component w^τ; p^τ is the pressure, ϱ^τ is the density, θ^τ is the temperature, S is the salinity, and q is the moisture; F_2^τ and F_3^τ are the sources of heat and salinity or moisture, respectively; μ_1^τ and μ_2^τ are the coefficients of turbulent viscosity; (μ_3^τ, μ_4^τ) and (μ_5^τ, μ_6^τ) are, respectively, given by scaling sum of turbulent and molecular diffusivities of heat and salinity.

The conditions on the free surface $\Gamma_s(t)$, $t > 0$ are as follows:

$$\begin{cases} \mathbf{v}^a = \mathbf{v}^s, \\ [\mathbf{T}^a(\mathbf{v}^a) - \mathbf{T}^s(\mathbf{v}^s)]\,\mathbf{n} - ([\mathbf{T}^a(\mathbf{v}^a) - \mathbf{T}^s(\mathbf{v}^s)]\,\mathbf{n}\cdot\mathbf{n})\mathbf{n} \\ = |\mathbf{V}^a - \delta\mathbf{V}^s|^\alpha((\mathbf{V}^a - \delta\mathbf{V}^s) - (|\mathbf{V}^a - \delta\mathbf{V}^s|\cdot\mathbf{n})\mathbf{n}), \\ -\left\{\mu_3^a\left(\dfrac{\partial\theta^a}{\partial x_1}n_1 + \dfrac{\partial\theta^a}{\partial x_2}n_2\right) + \mu_4^a\dfrac{\partial\theta^a}{\partial x_3}n_3\right\} \\ +\left\{\mu_3^s\left(\dfrac{\partial\theta^s}{\partial x_1}n_1 + \dfrac{\partial\theta^s}{\partial x_2}n_2\right) + \mu_4^s\dfrac{\partial\theta^s}{\partial x_3}n_3\right\} \\ = -l(\theta_e)\mathcal{V} + g_1|\mathbf{V}^a - \delta\mathbf{V}^s|^\alpha\theta_e + \sigma L H, \\ (\theta^a, \theta^s, q, S, p^a, p^s) = (\theta_e, \theta_e, q_e, S_e, p_0, p_0), \quad x \in \Gamma_s(t),\ t > 0, \end{cases} \tag{2.3}$$

where

$$\mathbf{T}^\tau(\mathbf{v}^\tau) = \begin{pmatrix} \mu_1^\tau\dfrac{\partial v_1^\tau}{\partial x_1} & \mu_1^\tau\dfrac{\partial v_1^\tau}{\partial x_2} & \mu_2^\tau\dfrac{\partial v_1^\tau}{\partial x_3} \\ \mu_1^\tau\dfrac{\partial v_2^\tau}{\partial x_1} & \mu_1^\tau\dfrac{\partial v_2^\tau}{\partial x_2} & \mu_2^\tau\dfrac{\partial v_2^\tau}{\partial x_3} \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

are stress tensors, $\mathcal{V}$ is the normal velocity of the free surface, $l(\theta_e)$ is a latent heat with saturation temperature θ_e, σ is the surface tension coefficient, H is twice mean curvature, $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal vector to $\Gamma_s(t)$ at time t pointing to the atmospheric region, L is the heat capacity, g_1 is a given function representing the turbulent transition on the free surface including the albedo of the earth.

Boundary conditions at the bottom of the ocean $\Gamma_b = \{(x', -b(x'))|x' \in \mathbf{R}^2\}$ are:

$$\frac{\partial \mathbf{v}^a}{\partial x_3} = \alpha_v(\mathbf{v}_c - \mathbf{v}^a), \tag{2.5}$$

$$w^a = 0, \tag{2.6}$$

$$\frac{\partial \theta^a}{\partial x_3} = \alpha_\theta(\theta_c - \theta^a), \tag{2.7}$$

$$\frac{\partial q}{\partial x_3} = \alpha_q(q_c - q), \quad x \in \Gamma_u,\ t > 0, \tag{2.8}$$

$$(\mathbf{V}^s, \theta^s, S)(t,x) = (\mathbf{0}, \theta_b, S_b)(t,x), \quad x \in \Gamma_b,\ t > 0, \tag{2.9}$$

where α_v, α_θ, α_q are positive constants, $\mathbf{v}_c$, θ_c, q_c are velocity, temperature, moisture of the layer above the scale height.

Initial conditions are:

$$(\mathbf{V}^a, \theta^a, q)(0,x) = (\mathbf{V}^a_0, \theta^a_0, q_0)(x), \quad x \in \Omega^a_0. \tag{2.10}$$

$$(\mathbf{V}^s, \theta^s, S)(0,x) = (\mathbf{V}^s_0, \theta^s_0, S_0)(x), \quad x \in \Omega^s_0. \tag{2.11}$$

Now, let us introduce the "p-coordinate system". From $(2.2)_2$ and $(2.1)_2$, p^a and p^s can be represented as

$$p^a = p_0 \exp(-\int_d^{x_3} \frac{g}{R\theta^a} dx_3) \equiv p_0 \exp\{\Phi^a(t,x',x_3) - \Phi^a(t,x',d(t,x'))\}, \tag{2.12}$$

$$p^s = p_0 + \int_d^{x_3} \frac{\partial p^s}{\partial x_3}\, dx_3 = p_0 - (x_3 - d)\varrho^s g \tag{2.13}$$

where Φ^a is a primitive function of the integral term of (2.12), and we used the relationship $p^a = \varrho^a R\theta^a$. Since $\frac{\partial p^\tau}{\partial x_3}$ is negative, we can take the pressure as an independent variable in place of x_3.

Now we define

$$h^a(t,x') = p_0 \exp\{\Phi^a(t,x',u) - \Phi^a(t,x',d(t,x'))\},$$

$$h^s(t,x') = p_0 + (b+d)\varrho^s g,$$

being pressure at the bottom of the ocean and the upper surface of the atmosphere.

Obviously $h^\tau(t,x')$ is unknown since $d(t,x')$ is unknown. Let us consider a function

$$h^a_0(t,x') = p_0 \exp\{\Phi^a(t,x',u) - \Phi^a(t,x',d_0(t,x'))\},$$

$$h_0^s(t,x') = p_0 + (b+d_0)\varrho^s g,$$

with $d_0(x') = d(0,x')$. Now we assume that $u > d_0(x') > -b(x')$ holds for any $x' \in \mathbf{R}^2$.

Then d can be represented as an implicit function : $d = \Psi^s(t,x',h^s(t,x'),b(x')) = \Psi^a(t,x',h^a(t,x'),u)$.

From (2.12) and (2.13), we get

$$\frac{\partial p^a}{\partial x_i} = p^a\Big(\left(\Psi^a_{x_i} + \Psi^a_h h^a_{x_i}\right)\Phi^a_{x_3}(t,x) - \left(\Phi^a_{x_i}(t,x',x_3) - \Phi^a_{x_i}(t,x',\Psi^a)\right)\Big)$$
$$=: F^a_{5i}(t,x,\Psi^a(t,x_1,x_2),\theta^a,h^a) \quad (i=1,2),$$

$$\frac{\partial p^a}{\partial t} = p^a\Big(\left(\Psi^a_t + \Psi^a_h h_t\right)\Phi^a_{x_3}(t,x) - \left(\Phi^a_t(t,x',x_3) - \Phi^a_t(t,x',\Psi^a)\right)\Big)$$
$$=: F^a_6(t,x,\Psi^a(t,x_1,x_2),\theta^a,h^a),$$
$$\nabla p^s = \varrho^s g \nabla\left(-b + \frac{h^s}{\varrho^s g}\right) =: \mathbf{F}^s_5(t,x',h^s),$$
$$\frac{\partial p^s}{\partial t} = \varrho^s g \frac{\partial}{\partial t}\left(\frac{h^s}{\varrho^s g}\right) =: F^s_6(t,x',h^s).$$

For simplicity, instead of p^τ, $-\varrho^s g w^s$ and $\dfrac{dp^a}{dt}$, we use x_3, w^s and w^a, respectively. We define the following notations:

$$\tilde{\Omega}^a(t) = \left\{(x_1,x_2,x_3) | x' \in \mathbf{R}^2, h^a(t,x') \le x_3 \le p_0\right\},$$

$$\tilde{\Omega}^s(t) = \{(x_1,x_2,x_3) | x' \in \mathbf{R}^2, p_0 \le x_3 \le h^s(t,x')\},$$
$$\tilde{\Gamma}_s = \{(x_1,x_2,x_3) | x_3 = p_0,\ x' \in \mathbf{R}^2\},$$
$$\tilde{\Gamma}_u(t) = \{(x_1,x_2,x_3) | x_3 = h^a(t,x'), x' \in \mathbf{R}^2\}.$$
$$\tilde{\Gamma}_b(t) = \{(x_1,x_2,x_3) | x_3 = h^s(t,x'), x' \in \mathbf{R}^2\}.$$

Note that after introducing p-coordinates, the free surface is represented by the equation $x_3 = p_0$ (constant), where p_0 is atmospheric pressure at the free surface.

Hereafter we will use the notation $F^{(h^\tau)}$ to denote a function F on the transformed coordinates that includes h^τ:

$$F^{(h^\tau)}(t,x',p) = F(t,x',X_3^\tau(t,x',p^\tau,h^\tau(t,x'))),$$

where $x_3 = X_3^\tau(t,x',p,h^\tau)$ with X_3^τ being an inverse function of (2.12), (2.13).

Then, we can rewrite (2.1)-(2.11) as follows.

$$\begin{cases} \dfrac{\partial \mathbf{v}^a}{\partial t} = L_1^a \mathbf{v}^a + \mathbf{G}_{1,h^a}^a(x,t,\mathbf{v}^a,w^a), \\ \dfrac{\partial w^a}{\partial x_3} = G_{3,h^a}^a(x,t,\mathbf{v}^a) \\ \dfrac{\partial \theta^a}{\partial t} = L_2^a \theta^a + G_{4,h^a}^a(x,t,\mathbf{v}^a,w^a,\theta^a), \\ \dfrac{\partial q}{\partial t} = L_3^a q + G_{5,h^a}^a(x,t,\mathbf{v}^a,w^a,q), \quad x \in \tilde{\Omega}^a(t),\ t>0, \end{cases} \tag{2.14}$$

$$\begin{cases} \dfrac{\partial \mathbf{v}^s}{\partial t} = L_1^s \mathbf{v}^s + \mathbf{G}_{1,h^s}^s(x,t,\mathbf{v}^s,w^s), \\ \dfrac{\partial w^s}{\partial x_3} = G_{3,h^s}^s(x,t,\mathbf{v}^s) \\ \dfrac{\partial \theta^s}{\partial t} = L_2^s \theta^s + G_{4,h^s}^s(x,t,\mathbf{v}^s,w^s,\theta^s), \\ \dfrac{\partial S}{\partial t} = L_3^s S + G_{5,h^s}^s(x,t,\mathbf{v}^s,w^s,S), \quad x \in \tilde{\Omega}^s(t),\ t>0, \end{cases} \tag{2.15}$$

$$\begin{cases} \mathbf{v}^a = \mathbf{v}^s, \\ B(\mathbf{v}^s,\mathbf{v}^a) = \mathbf{G}_{2,h^s}(\mathbf{v}^s,\mathbf{v}^a,w^s,w^a), \quad x \in \tilde{\Gamma}_s,\ t>0, \\ -\left\{\mu_3^a\left(\nabla\theta^a\cdot\mathbf{n} + \mathbf{F}_5^{a(h^a)}\cdot\mathbf{n}\dfrac{\partial\theta^a}{\partial x_3}\right) - \mu_4^a \dfrac{g x_3}{R\theta^a}\dfrac{\partial \theta^s}{\partial x_3} n_3\right\} \\ +\left\{\mu_3^s\left(\nabla\theta^s\cdot\mathbf{n} + \mathbf{F}_5^{s(h^s)}\cdot\mathbf{n}\dfrac{\partial\theta^s}{\partial x_3}\right) - \varrho^s g \mu_4^s \dfrac{\partial \theta^s}{\partial x_3} n_3\right\} \\ \qquad = -l(\theta_e)\mathcal{V} + g_1\left(|\mathbf{v}^a - \delta\mathbf{v}^s|^2 + \left(\dfrac{w^a-\delta w^s}{\varrho^s g}\right)^2\right)^{\alpha/2}\theta_e + \sigma L H, \\ \quad x \in \tilde{\Gamma}_s,\ t>0, \\ (\theta^a,q)(t,x) = (\theta_e^{(h^a)}, q_e^{(h^a)})(t,x), \quad x\in\tilde{\Gamma}_s,\ t>0, \\ (\theta^s,S)(t,x) = (\theta_e^{(h^s)}, S_e^{(h^s)})(t,x), \quad x\in\tilde{\Gamma}_s,\ t>0, \end{cases} \tag{2.16}$$

$$
\begin{cases}
\left(\frac{\partial \mathbf{v}^a}{\partial x_3}, \frac{\partial \theta^a}{\partial x_3}, \frac{\partial q}{\partial x_3}\right) = \left(\alpha_v(\mathbf{v}_c - \mathbf{v}^a), \alpha_\theta(\theta_c - \theta^a), \alpha_q(q_c - q)\right), \quad x \in \tilde{\Gamma}_u(t),\ t > 0, \\
w^a = 0, \quad x \in \tilde{\Gamma}_u(t),\ t > 0, \\
(\mathbf{v}^s, w^s, \theta^s, S)(t,x) = (\mathbf{0}, 0, \theta_b, S_b)(t,x), \quad x \in \tilde{\Gamma}_b(t),\ t > 0, \\
(\mathbf{v}^a, \theta^a, q)(0,x) = (\mathbf{v}_0^{a(h_0^a)}, \theta_0^{a(h_0^a)}, q_0^{(h_0^a)})(x), \quad x \in \Omega_0^a, \\
(\mathbf{v}^s, \theta^s, S)(0,x) = (\mathbf{v}_0^{s(h_0^s)}, \theta_0^{s(h_0^s)}, S_0^{(h_0^s)})(x), \quad x \in \Omega_0^s,
\end{cases}
\tag{2.17}
$$

where

$$
L_1^a \mathbf{v}^a := \frac{\mu_1^a R\theta^a}{x_3}\left[\triangle \mathbf{v}^a + 2\mathbf{F}_5^{a(h^a)} \cdot \nabla \frac{\partial \mathbf{v}^a}{\partial x_3} + |\mathbf{F}_5^{a(h^a)}|^2 \frac{\partial^2 \mathbf{v}^a}{\partial x_3^2}\right] + \frac{\mu_2^a g^2 x_3}{R\theta^a}\frac{\partial^2 \mathbf{v}^a}{\partial x_3^2}
$$

$$
L_2^a \theta^a := \frac{\mu_3^a R\theta^a}{C_p x_3}\left[\triangle \theta^a + 2\mathbf{F}_5^{a(h^a)} \cdot \nabla \frac{\partial \theta^a}{\partial x_3} + |\mathbf{F}_5^{a(h^a)}|^2 \frac{\partial^2 \theta^a}{\partial x_3^2}\right] + \frac{\mu_4^a g^2 x_3}{R\theta^a}\frac{\partial^2 \theta^a}{\partial x_3^2}
$$

$$
L_3^a q := \frac{\mu_5^a R\theta^a}{C_q x_3}\left[\triangle q + 2\mathbf{F}_5^{a(h^a)} \cdot \nabla \frac{\partial q}{\partial x_3} + |\mathbf{F}_5^{a(h^a)}|^2 \frac{\partial^2 q}{\partial x_3^2}\right] + \frac{\mu_6^a g^2 x_3}{R\theta^a}\frac{\partial^2 q}{\partial x_3^2}
$$

$$
\mathbf{G}_{1,h^a}^a(\mathbf{v}^a, \theta) := -(\mathbf{v}^a \cdot \nabla)\mathbf{v}^a - w^a \frac{\partial \mathbf{v}^a}{\partial x_3} - F_6^{a(h^a)} \frac{\partial \mathbf{v}^a}{\partial x_3} - (\mathbf{F}_5^{a(h^a)} \cdot \mathbf{v}^a)\frac{\partial \mathbf{v}^a}{\partial x_3}
$$

$$
+\mu_1^a \frac{\partial \mathbf{v}^a}{\partial x_3}\left(\nabla \cdot \mathbf{F}_5^{a(h^a)} + \mathbf{F}_5^{a(h^a)} \cdot \frac{\partial \mathbf{F}_5^{a(h^a)}}{\partial x_3}\right) + \mu_2^a \frac{g^2 x_3(\theta^a - x_3 \frac{\partial \theta^a}{\partial x_3})}{R^2 \theta^{a3}} + f\mathbf{A}\mathbf{v}^a - \mathbf{F}_5^{a(h^a)} + \mathbf{F}_1^{a(h^a)},
$$

$$
G_{3,h^a}^a(\mathbf{v}^a) := \nabla \cdot \mathbf{v}^a + \frac{\partial \mathbf{v}^a}{\partial x_3} \cdot \mathbf{F}_5^{a(h^a)},
$$

$$
\mathbf{G}_{4,h^a}^a(\mathbf{v}^a, \theta) := -(\mathbf{v}^a \cdot \nabla)\theta^a - w^a \frac{\partial \theta^a}{\partial x_3} - F_6^{a(h^a)} \frac{\partial \theta^a}{\partial x_3} - (\mathbf{F}_5^{a(h^a)} \cdot \mathbf{v}^a)\frac{\partial \theta^a}{\partial x_3}
$$

$$
+\mu_3^a \frac{\partial \theta^a}{\partial x_3}\left(\nabla \cdot \mathbf{F}_5^{a(h^a)} + \mathbf{F}_5^{a(h^a)} \cdot \nabla \frac{\partial \mathbf{F}_5^{a(h^a)}}{\partial x_3}\right) + \mu_4^a \frac{g^2 x_3(\theta^a - x_3 \frac{\partial \theta^a}{\partial x_3})}{R^2 (\theta^a)^3} + F_2^{a(h^a)},
$$

$$
\mathbf{G}_{5,h^a}^a(\mathbf{v}^a, \theta, q) := -(\mathbf{v}^a \cdot \nabla)q - w^a \frac{\partial q}{\partial x_3} - F_6^{a(h^a)} \frac{\partial q}{\partial x_3} - (\mathbf{F}_5^{a(h^a)} \cdot \mathbf{v}^a)\frac{\partial q}{\partial x_3}
$$

$$
+\mu_5^a \frac{\partial q}{\partial x_3}\left(\nabla \cdot \mathbf{F}_5^{a(h^a)} + \mathbf{F}_5^{a(h^a)} \cdot \nabla \frac{\partial \mathbf{F}_5^{a(h^a)}}{\partial x_3}\right) + \mu_6^a \frac{g^2 x_3(\theta^a - x_3 \frac{\partial \theta^a}{\partial x_3})}{R^2 (\theta^a)^3} + F_3^{a(h^a)},
$$

$$L_1^s\mathbf{v}^s := \mu_1\Delta\mathbf{v}^s + \mu_2^s(\varrho^s g)^2\frac{\partial^2\mathbf{v}^s}{\partial x_3^2} + 2\mu_1^s\mathbf{F}_5^{s(h^s)}\cdot\nabla\frac{\partial\mathbf{v}^s}{\partial x_3} + \mu_1^s|\mathbf{F}_5^{s(h^s)}|^2\frac{\partial^2\mathbf{v}^s}{\partial x_3^2},$$

$$\begin{aligned}\mathbf{G}_{1,h^s}^s(\mathbf{v}^s, w^s) := {} & -(\mathbf{v}^s\cdot\nabla)\mathbf{v}^s - w^s\tfrac{\partial\mathbf{v}^s}{\partial x_3} - F_6^{s(h^s)}\tfrac{\partial\mathbf{v}^s}{\partial x_3} - (\mathbf{F}_5^{s(h^s)}\cdot\mathbf{v}^s)\tfrac{\partial\mathbf{v}^s}{\partial x_3} \\ & +\mu_1^s\left(\nabla\cdot\mathbf{F}_5^{s(h^s)} + \mathbf{F}_5^{s(h^s)}\cdot\tfrac{\partial\mathbf{F}_5^{s(h^s)}}{\partial x_3}\right)\tfrac{\partial\mathbf{v}^s}{\partial x_3} \\ & -f\mathbf{A}\mathbf{v}^s - \mathbf{F}_5^{s(h^s)} + \mathbf{F}_1^{s(h^s)},\end{aligned}$$

$$G_{3,h^s}^s(\mathbf{v}^s) := \nabla\cdot\mathbf{v}^s - \tfrac{\partial\mathbf{v}^s}{\partial x_3}\cdot\mathbf{F}_5^{s(h^s)},$$

$$L_2^s\theta^s := \left[\mu_3^s\Delta\theta^s + 2\mu_3^s\mathbf{F}_5^{s(h^s)}\cdot\nabla\tfrac{\partial\theta^s}{\partial x_3} + \mu_3^s|\mathbf{F}_5^{s(h^s)}|^2\tfrac{\partial^2\theta^s}{\partial x_3^2} + \mu_4^s(\varrho^s g)^2\tfrac{\partial^2\theta^s}{\partial x_3^2}\right],$$

$$\begin{aligned}G_{4,h^s}^s(\mathbf{v}^s, w^s, \theta^s) := {} & -(\mathbf{v}^s\cdot\nabla)\theta^s - w^s\tfrac{\partial\theta^s}{\partial x_3} - F_6^{s(h^s)}\tfrac{\partial\theta^s}{\partial x_3} - (\mathbf{F}_5^{s(h^s)}\cdot\mathbf{v}^s)\tfrac{\partial\theta^s}{\partial x_3} \\ & +\mu_3^s\tfrac{\partial\theta^s}{\partial x_3}\left(\nabla\cdot\mathbf{F}_5^{s(h^s)} + \mathbf{F}_5^{s(h^s)}\cdot\tfrac{\partial\mathbf{F}_5^{s(h^s)}}{\partial x_3}\right) + F_2^{s(h^s)},\end{aligned}$$

$$L_3^s S := \left[\mu_5^s\Delta S + 2\mu_5^s\mathbf{F}_5^{s(h^s)}\cdot\nabla\tfrac{\partial S}{\partial x_3} + \mu_5^s|\mathbf{F}_5^{s(h^s)}|^2\tfrac{\partial^2 S}{\partial x_3^2} + \mu_6^s(\varrho^s g)^2\tfrac{\partial^2 S}{\partial x_3^2}\right],$$

$$\begin{aligned}G_{5,h^s}^s(\mathbf{v}^s, w^s, S) := {} & -(\mathbf{v}^s\cdot\nabla)S - w^s\tfrac{\partial S}{\partial x_3} - F_6^{s(h^s)}\tfrac{\partial S}{\partial x_3} - (\mathbf{F}_5^{s(h^s)}\cdot\mathbf{v}^s)\tfrac{\partial S}{\partial x_3} \\ & +\mu_5^s\tfrac{\partial S}{\partial x_3}\left(\nabla\cdot\mathbf{F}_5^{s(h^s)} + \mathbf{F}_5^{s(h)}\cdot\tfrac{\partial\mathbf{F}_5^{s(h^s)}}{\partial x_3}\right) + F_3^{s(h^s)},\end{aligned}$$

$$\mathbf{G}_{2,h^s}(\mathbf{v}^a, \mathbf{v}^s, w^a, w^s) := |\mathbf{V}^a - \delta\mathbf{V}^s|^\alpha((\mathbf{V}^a - \delta\mathbf{V}^s) - (|\mathbf{V}^a - \delta\mathbf{V}^s|\cdot\mathbf{n})\mathbf{n}),$$

$$\tilde{\mathbf{V}}_0^\tau = (\mathbf{v}_0, w_0)\,, \tilde{\mathbf{V}}^\tau = (\mathbf{v}, w)\,,$$

$\mathbf{n} = \mathbf{n}(h^s) = (\mathbf{n}'(h^s), n_3(h^s)) = \frac{\mathbf{a}}{|\mathbf{a}|}$, $\mathbf{a} = \left[\ \mathbf{F}_{51}^{s(h^s)},\ \ \mathbf{F}_{52}^{s(h^s)},\ \ -\varrho^s g\ \right]$. If d varies only nearby d_0, then we may consider

$$|h^\tau(t, x') - h_0^\tau(x')| < \lambda_0,$$

with some positive constant λ_0, and we define a stripe $\Gamma_0^\tau = (h_0^\tau - 3\lambda_0, h_0^\tau + 3\lambda_0)$, $N_0^\tau := \{\mathbf{x}\in\mathbf{R}^3|\mathbf{x} = \delta^\tau + \mathbf{n}^\tau(\delta)\lambda^\tau, \lambda^\tau\in\Gamma_0^\tau\}$. Then, we can define 1-to-1 mappings [4]:

$x = \delta^\tau + \mathbf{n}^\tau(\delta)\lambda^\tau \in N_0^\tau \longmapsto (\delta^\tau, \lambda^\tau) \in \mathbf{R}^2 \times \Gamma_0^\tau$, where $\mathbf{n}^\tau$ ($\tau = a, s$) are normal vectors of $\tilde{\Gamma}_u(t)$, $\tilde{\Gamma}_b(t)$, respectively. We also define the coordinates y such that $y = (\delta^\tau(y), \eta^\tau(y))$ such that

$$\delta^\tau(x) = \delta^\tau(y), \quad \lambda^\tau(x) = \eta^\tau(y) + \chi(\eta^\tau(y)) \quad \text{if } (x,t) \in N_0^\tau \times [0,T], \tag{2.18}$$

$$\delta^\tau(x) = \delta^\tau(y), \quad \lambda^\tau(x) = \eta^\tau(y) \quad \text{if } (x,t) \in N_0^{\tau c} \times [0,T], \tag{2.19}$$

where $\chi(\eta) \in C^\infty(-\infty,\infty)$ is a smooth cut-off function satisfying

$$\chi(\eta) = 1 \quad \text{if} \quad |\eta| \le \lambda_0,$$
$$\chi(\eta) = 0 \quad \text{if} \quad |\eta| \ge 3\lambda_0,$$
$$|\chi'(\eta)| < \tfrac{3}{4\lambda_0}.$$

Then we define a mapping e_d^τ:

$$e_d^\tau(t, y(\delta^\tau, \eta^\tau)) = (t, x(\delta^\tau, \eta^\tau + \chi(\eta^\tau)(-h^\tau(t,\delta^\tau) + h_0^\tau))), \quad \text{if } (x,t) \in N_0^\tau \times [0,T],$$
$$e_d^\tau(t, y(\delta^\tau, \eta^\tau)) = (t, x(\delta^\tau, \eta^\tau)), \quad \text{if } (x,t) \in N_0^{\tau c} \times [0,T].$$

Clearly, the region $\tilde{\Omega}_T^\tau$ is transformed onto the region $\bigcup_t(\tilde{\Omega}^\tau(t) \times \{t\})$. We denote the inverse and transposed matrix of the Jacobian matrix by $J^\tau[x/y]^{-T} = \left(\frac{\partial e_d^\tau(t,y)}{\partial x}\right)^{-T} = (a^{ij}(h^\tau))$. In the following, we denote $\nabla_{h^\tau} := J^\tau[x/y]^{-T}\nabla = (\nabla_{h^\tau,i})$ ($i = 1,2,3$). After this transformation, by using x in place of the new coordinates y and the same letters as before, for example, $\mathbf{v}^\tau$ instead of $\mathbf{v}\circ e_d^\tau$, the problem (2.14)-(2.17) becomes as follows:

$$\begin{cases} \dfrac{\partial \mathbf{v}^a}{\partial t} = L_{1,h^a}^a \mathbf{v}^a + \tilde{\mathbf{G}}_{1,h^a}^a(\mathbf{v}^a, w^a), \\ \nabla_{h^a,3} w^a = \tilde{G}_{3,h^a}^a(\mathbf{v}^a) \\ \dfrac{\partial \theta^a}{\partial t} = L_{2,h^a}^a \theta^a + \tilde{G}_{4,h^a}^a(\mathbf{v}^a, w^a, \theta^a), \\ \dfrac{\partial q}{\partial t} = L_{3,h^a}^a q + \tilde{G}_{5,h^a}^a(\mathbf{v}^a, w^a, q), \quad (t,x) \in \tilde{\Omega}_T^a, \\ \dfrac{\partial h^a}{\partial t} = L_{4,h^a}^a h^a + \tilde{G}_{6,h^a,h^s}^a(\mathbf{v}^a, w^a, \theta^a), \quad (t,x) \in \mathbf{R}_T^2, \end{cases} \tag{2.20}$$

$$\begin{cases} \dfrac{\partial \mathbf{v}^s}{\partial t} = L^s_{1,h^s}\mathbf{v}^s + \tilde{\mathbf{G}}^s_{1,h^s}(\mathbf{v}^s, w^s), \\ \nabla_{h^s,3} w^s = \tilde{G}^s_{3,h^s}(\mathbf{v}^s), \\ \dfrac{\partial \theta^s}{\partial t} = L^s_{2,h^s}\theta^s + \tilde{G}^s_{4,h^s}(\mathbf{v}^s, w^s, \theta^s), \\ \dfrac{\partial S}{\partial t} = L^s_{3,h^s} S + \tilde{G}^s_{5,h^s}(\mathbf{v}^s, w^s, S), \quad (t,x) \in \tilde{\Omega}_T, \\ \dfrac{\partial h^s}{\partial t} = L^s_{4,h^s} h^s + \tilde{G}^s_{6,h^a,h^s}(\mathbf{v}^s, w^s, \theta^s), \quad (t,x) \in \mathbf{R}^2_T, \end{cases} \tag{2.21}$$

$$\begin{cases} \mathbf{v}^a = \mathbf{v}^s, \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ B_{h^a,h^s}(\mathbf{v}^s, \mathbf{v}^a) = \tilde{\mathbf{G}}_{2,h^s}(\mathbf{v}^a, \mathbf{v}^s, w^a, w^s), \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ (\theta^a, q)(t,x) = (\theta_e^{(h^a)}, q_e^{(h^a)})(t,x), \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ (\theta^s, S)(t,x) = (\theta_e^{(h^s)}, S_e^{(h^s)})(t,x), \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ (\mathbf{v}^s, w^s, \theta, S)(t,x) = (\mathbf{0}, 0, \theta_b, S_b)(t,x), \;\; (t,x) \in \tilde{\Gamma}_{bT}. \\ a^{33}(h^a)\left(\dfrac{\partial \mathbf{v}^a}{\partial x_3}, \dfrac{\partial \theta^a}{\partial x_3}, \dfrac{\partial q}{\partial x_3}\right) = (\alpha_v(\mathbf{v}_c - \mathbf{v}^a), \alpha_\theta(\theta_c - \theta^a), \alpha_q(q_c - q)), \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ w^a = 0, \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ w^s = 0, \quad (t,x) \in \tilde{\Gamma}_{bT}, \\ (\mathbf{v}^\tau, \theta^\tau, S, q)(0,x) = (\mathbf{v}_0^\tau, \theta_0^\tau, S_0, q_0)(x), \;\; x \in \tilde{\Omega}_0^\tau. \end{cases} \tag{2.22}$$

Here L^τ_{i,h^τ} $(i = 1,2,3)$, B_{h^a,h^s} and $\tilde{G}^\tau_{i,h^\tau}$ $(i = 1,2,\ldots,5)$ are obtained from L^τ_i, B, G^τ_{i,h^τ} by replacing ∇ with ∇_{h^τ}. In B_{h^a,h^s}, we replace $\mathbf{T}$ by $\mathbf{T}_{h^\tau}$, in which ∇ is replaced by ∇_{h^τ}.

In $(2.21)_5$, $(2.20)_5$,

$$L_{4,h^a}h^a = \frac{\sigma LR\theta_e\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}{l(\theta_e)gp_0\sqrt{1+|\mathbf{F}_5^{a(h^a)}|^2}}\nabla^2 h^a,$$

$$L_{4,h^s}h^s = \frac{\varrho^s g\sigma L\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}{2l(\theta_e)\sqrt{1+|\mathbf{F}_5^{s(h^s)}|^2}}\nabla^2 h^s,$$

$$\begin{aligned}\tilde{G}^a_{6,h^a,h^s} = {}& \frac{gp_0\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}{l(\theta_e)R\theta^a}\Bigg[\mu_3^a\sum_{i=1}^2\frac{F_{5i}^{s(h^s)}}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}\left(\frac{\partial\theta^a}{\partial x_i}+a^{3i}(h^a)\frac{\partial\theta^a}{\partial x_3}\right)\\
&-\mu_4^a\frac{\varrho^s g}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}a^{33}(h^a)\frac{\partial\theta^s}{\partial x_3}+g_1^{(h^a)}|\mathbf{V}^a-\delta\mathbf{V}^s|^\alpha\theta_e\\
&-\mu_3^s\sum_{i=1}^2\frac{F_{5i}^{s(h^s)}}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}\left(\frac{\partial\theta^a}{\partial x_i}+a^{3i}(h^a)\frac{\partial\theta^a}{\partial x_3}\right)\\
&+\mu_4^s\frac{\varrho^s g}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}a^{33}(h^a)\frac{\partial\theta^a}{\partial x_3}+\sigma LR^a(\Psi^a,h^a,h^s,b)\Bigg]-\frac{gp_0\Psi_t^a}{R\theta_e},\end{aligned}$$

$$\begin{aligned}\tilde{G}^s_{6,h^a,h^s} = {}& -\frac{\varrho^s g\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}{2l(\theta_e)}\Bigg[\mu_3^s\sum_{i=1}^2\frac{F_{5i}^{s(h^s)}}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}\left(\frac{\partial\theta^s}{\partial x_i}+a^{3i}(h^s)\frac{\partial\theta^s}{\partial x_3}\right)\\
&-\mu_4^s\frac{\varrho^s g}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}a^{33}(h^s)\frac{\partial\theta^s}{\partial x_3}+g_1^{(h^s)}|\mathbf{V}^a-\delta\mathbf{V}^s|^\alpha\theta_e\\
&-\mu_3^a\sum_{i=1}^2\frac{F_{5i}^{s(h^s)}}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}\left(\frac{\partial\theta^a}{\partial x_i}+a^{3i}(h^s)\frac{\partial\theta^a}{\partial x_3}\right)\\
&+\mu_4^a\frac{\varrho^s g}{\sqrt{|\mathbf{F}_5^{s(h^s)}|^2+(\varrho^s g)^2}}a^{33}(h^s)\frac{\partial\theta^a}{\partial x_3}+\sigma LR^s(\Psi^s,h^a,h^s,b)\Bigg],\end{aligned}$$

where

$$\begin{aligned}R^a(\Psi^a,h^a,h^s,u) = {}& -\frac{1}{\sqrt{1+|\mathbf{F}_5^{a(h^a)}|^2}}\Bigg[\frac{g}{R\theta^a}(\mathbf{F}_5^a\cdot\nabla\Psi^a+\Psi_{h^a}^a\mathbf{F}_5^a\cdot\nabla h^a)\\
&-\frac{gp_0}{R(\theta^a)^2}(\nabla\theta^a\cdot\nabla\Psi^a+\Psi_{h_a}^a\nabla\theta^a\cdot\nabla h^a)+p_0(\nabla^2\Psi^a+2\nabla\Psi_{h_a}^a\cdot\nabla h^a+\Psi_{h^ah^a}^a\nabla h^a)\Bigg],\end{aligned}$$

$$\begin{aligned}R^s(\Psi^s,h^a,h^s,b) = {}& -\frac{\varrho^s g}{\sqrt{1+|\mathbf{F}_5^{s(h^s)}|^2}}\Bigg[\nabla^2\Psi^s+2(\nabla\Psi_{h^s}^s\cdot\nabla h^s+\nabla\Psi_b^s\cdot\nabla b+\Psi_{h^sb}^s\nabla h^s\cdot\nabla b)\\
&+\Psi_{h^sh^s}^s|\nabla h^s|^2+\Psi_{bb}^s|\nabla b|^2+\Psi_b^s\nabla^2 b\Bigg],\end{aligned}$$

Since h and Ψ does not depend on x_3, ∇_h is equal to ∇ here.

3 Main Theorem

In this section, we introduce function spaces used throughout this paper. Let $\mathcal{G}$ be a domain in $\mathbf{R}^n$ and l is a non-negative number.

By $W_2^l(\mathcal{G})$ we mean a space of functions $u(x), x \in \mathcal{G}$, equipped with the norm $\|u\|^2_{W_2^l(\mathcal{G})} = \sum_{|\alpha|<l} \|D^\alpha u\|^2_{L_2(\mathcal{G})} + \|u\|^2_{\dot{W}_2^l(\mathcal{G})}$, where

$$\begin{cases} \|u\|^2_{\dot{W}_2^l(\mathcal{G})} = \sum_{|\alpha|=l} \|D^\alpha u\|^2_{L_2(\mathcal{G})} = \sum_{|\alpha|=l} \int_{\mathcal{G}} |D^\alpha u(x)|^2 dx \quad \text{if } l \text{ is an integer}, \\ \|u\|^2_{\dot{W}_2^l(\mathcal{G})} = \sum_{|\alpha|=[l]} \int_{\mathcal{G}} \int_{\mathcal{G}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} dxdy \quad \text{if } l \text{ is a non-integer, } l = [l] + \{l\}, \\ 0 < \{l\} < 1. \end{cases}$$

Now, we introduce anisotropic Sobolev-Slobodetskiĭ spaces $W_2^{l,\frac{l}{2}}(\mathcal{G}_T)$, $\mathcal{G}_T := \mathcal{G} \times [0,T]$.

For $x \in \mathcal{G}, t \in (0,T)$, we denote $W_2^{l,\frac{l}{2}}(\mathcal{G}_T) = L_2(0,T; W_2^l(\mathcal{G})) \cap L_2(\mathcal{G}; W_2^{\frac{l}{2}}(0,T))$ and introduce in this space the norm

$$\|u\|^2_{W_2^{l,\frac{l}{2}}(\mathcal{G}_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\mathcal{G})} dt + \int_{\mathcal{G}} \|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)} dx.$$

We also define a function space $\tilde{W}_2^{l,\frac{l}{2}}(\mathcal{G}_T) = \left\{ w \in W_2^{l,\frac{l}{2}}(\mathcal{G}_T) \,\middle|\, \frac{\partial w}{\partial x_3} \in W_2^{l,\frac{l}{2}}(\mathcal{G}_T) \right\}$. The norm of the vector space and the product space are defined by the usual vector norm and the sum of the norm of each space, respectively.
The following is our main result.

Theorem 3.1 *Let $l \in (\frac{1}{2}, 1)$, and T be an arbitrary positive number. Assume that $\mathbf{v}_0^a$, $q_0 \in W_2^{1+l}(\Omega_0^a)$, $\mathbf{v}_0^s$, $S_0 \in W_2^{1+l}(\Omega_0^s)$, $\theta_0^\tau \in W_2^{2+l}(\Omega_0^\tau)$, h_0, $b \in W_2^{\frac{5}{2}+l}(\mathbf{R}^2)$, θ_e, q_e, $S_e \in W_2^{3+l,\frac{3}{2}+\frac{l}{2}}(\mathbf{R}_T^3)$, $\mathbf{v}_c$, $q_c \in W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Gamma}_{uT})$, $\theta_c \in W_2^{2+l,1+\frac{l}{2}}(\tilde{\Gamma}_{uT})$, θ_b, $S_b \in W_2^{2+l,1+\frac{l}{2}}(\tilde{\Gamma}_{bT})$, $g_1 \in W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\mathbf{R}_T^3)$, $\mathbf{F}_1^\tau$, $F_3^\tau \in W_2^{l,\frac{l}{2}}(\mathbf{R}_T^3)$ and $F_2^\tau \in W_2^{1+l,\frac{1+l}{2}}(\mathbf{R}_T^3)$ satisfy the usual compatibility conditions.*

It is also assumed that $\mathbf{F}_1^\tau$, F_2^τ, F_3^τ, θ_e, S_e, their space derivatives with respect to time and space up to the 3rd order, with time derivative less than 1 order, satisfy Hölder condition with exponent equal to or greater than $\frac{1}{2}$.

Then, there exists T^* such that the problem (2.20)-(2.22) has a unique solution $(\mathbf{v}^a, w^a, \theta^a, q, h^a, \mathbf{v}^s, w^s, \theta^s, S, h^s) \in Z(T^*) := W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}^a) \times \tilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_{T^*}^a) \times W_2^{3+l,\frac{3}{2}+\frac{l}{2}}(\tilde{\Omega}_{T^*}^a) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}^a) \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_{T^*}^2) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}^s) \times \tilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_{T^*}^s) \times$

$W_2^{3+l,\frac{3}{2}+\frac{l}{2}}(\tilde{\Omega}^s_{T^*}) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}^s_{T^*}) \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_{T^*})$ *and satisfy*

$$\|(\mathbf{v}^\tau, w^\tau, \theta^\tau, S, q, h^\tau)\|_{Z(T^*)} \le C\Big[\sum_{i=1,3} \|\mathbf{F}_i^\tau\|_{W_2^{l,\frac{l}{2}}(\mathbf{R}^3_T)} + \|F_2^\tau\|_{W_2^{1+l,\frac{1+l}{2}}(\mathbf{R}^3_T)} + \|\mathbf{v}_0^\tau\|_{W_2^{1+l}(\Omega_0^\tau)}$$

$$+ \|\theta_0^\tau\|_{W_2^{2+l}(\Omega_0^\tau)} + \|S_0\|_{W_2^{1+l}(\Omega_0^s)} + \|q_0\|_{W_2^{1+l}(\Omega_0^s)} + \|h_0^\tau\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)} + \|\theta_e\|_{W_2^{3+l,\frac{3}{2}+\frac{l}{2}}(\mathbf{R}^3_T)}$$

$$+ \|S_e\|_{W_2^{2+l,1+\frac{l}{2}}(\mathbf{R}^3_T)} + \|g_1\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\mathbf{R}^3_T)} + \|\theta_b^s\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Gamma}_{bT})} + \|S_b\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Gamma}_{bT})}$$

$$+ \|\mathbf{v}_c\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|\theta_c\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|q_c\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|q_e\|_{W_2^{2+l,1+\frac{l}{2}}(\mathbf{R}^3_T)}\Big].$$

4 Linear Problems

In this section, we consider linear problems. Introduce linear operators that replace h^τ by $h^{\tau*}$. Note that $h^{\tau*}$, θ^{a*} and $\mathbf{F}_5^{a*} = \mathbf{F}_5(h^{a*}, \theta^{a*})$, $\mathbf{F}_5^{s*} = \mathbf{F}_5(h^{s*})$ are known functions. From the assumptions of Theorem 3.1, it is easily seen that their coefficients belong to $W_2^{2+l,1+\frac{l}{2}}(\Omega_T^\tau)$. Linear problems to be considered are as follows.

$$\begin{cases} \dfrac{\partial \mathbf{v}^\tau}{\partial t} - L_{1,h^{\tau*}}\mathbf{v} = \mathbf{l}_1^\tau, \quad (t,x) \in \tilde{\Omega}_T^\tau, \tau = a, s \\ \mathbf{v}^a = \mathbf{v}^s, \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ B_{h^{a*},h^{s*}}(\mathbf{v}^a, \mathbf{v}^s) = \mathbf{l}_2, \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ \mathbf{v}^s = \mathbf{0} \quad (t,x) \in \tilde{\Gamma}_{bT}, \\ a^{33}(h^{a*})\dfrac{\partial \mathbf{v}^a}{\partial x_3} = \alpha_v(\mathbf{v}_c - \mathbf{v}^a) \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ \mathbf{v}^\tau(0,x) = \mathbf{v}_0^\tau \quad x \in \tilde{\Omega}_0^\tau, \end{cases} \tag{4.1}$$

$$\begin{cases} \nabla_{h^{\tau*},3} w^\tau = l_3^\tau, \quad (t,x) \in \tilde{\Omega}_T^\tau, \\ w^a = 0, \quad x \in \tilde{\Gamma}_{uT}, \\ w^s = 0, \quad x \in \tilde{\Gamma}_{bT}, \end{cases} \tag{4.2}$$

$$\begin{cases} \dfrac{\partial \theta^\tau}{\partial t} - L_{2,h^{\tau *}}\theta^\tau = l_4^\tau, \quad (t,x) \in \tilde{\Omega}_T, \\ \theta^\tau = \theta_e^{(h^\tau)} \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ \theta^s = \theta_b \quad (t,x) \in \tilde{\Gamma}_{bT}, \\ a^{33}(h^{a*})\dfrac{\partial \theta^a}{\partial x_3} = \alpha_\theta(\theta_c - \theta^a), \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ \theta^\tau(0,x) = \theta_0^\tau \quad x \in \tilde{\Omega}_0; \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{\partial q}{\partial t} - L_{3,h^{\tau *}}q = l_5^a, \quad (t,x) \in \tilde{\Omega^a}_T, \\ q = q_e \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ a^{33}(h^{a*})\dfrac{\partial q}{\partial x_3} = \alpha_q(q_c - q) \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ q(0,x) = q_0 \quad x \in \tilde{\Omega}_0^a; \end{cases} \tag{4.4}$$

$$\begin{cases} \dfrac{\partial S}{\partial t} - L_{3,h^{\tau *}}S = l_5^s, \quad (t,x) \in \tilde{\Omega}_T^s, \\ S = S_e \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ S = S_b \quad (t,x) \in \tilde{\Gamma}_{bT}, \\ S(0,x) = S_0 \quad x \in \tilde{\Omega}_0^s; \end{cases} \tag{4.5}$$

$$\begin{cases} \dfrac{\partial h^\tau}{\partial t} - L_{4,h^{\tau *}}h^\tau = l_6^\tau, \quad (t,x') \in \mathbf{R}_T^2, \\ h^\tau(0,x') = h_0^\tau(x'), \quad x' \in \mathbf{R}^2. \end{cases} \tag{4.6}$$

where $h^{\tau *} \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $b \in W_2^{\frac{5}{2}+l}(\mathbf{R}^2)$ and n_i are some known functions; For problems (4.1)-(4.6), we have the following result:

Lemma 4.1 *For* $\mathbf{l}_1^\tau \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T^\tau)$, $\mathbf{l}_2 \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $l_i^\tau \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T^\tau)$ $(i = 4,5)$, $l_6 \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ $\mathbf{v}_0^\tau \in W_2^{1+l}(\tilde{\Omega}_0^\tau)$, $\theta_0^\tau \in W_2^{2+l}(\tilde{\Omega}_0^\tau)$, $S_0 \in W_2^{1+l}(\tilde{\Omega}_0^s)$, $q_0 \in W_2^{1+l}(\tilde{\Omega}_0^a)$, $h_0^\tau \in W_2^{\frac{3}{2}+l}(\mathbf{R}_T^2)$, $\mathbf{v}_c \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})$, $\theta_b \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})$, $\theta_e \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $\theta_c \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})$, $q_c \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})$, $q_e \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $S_b \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})$,

$S_e \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, *the problem* (4.1)-(4.5) *has a unique solution* $(\mathbf{v}^\tau, w^\tau, \theta^\tau, S, q, h^\tau) \in Z_T$ *satisfying*

$$
\begin{aligned}
\|(\mathbf{v}^\tau, w^\tau, \theta^\tau, S, q, h^\tau)\|_{Z(T)} \le C_4 \Big[&\|\mathbf{l}_1^\tau\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T^\tau)} + \|\mathbf{l}_2\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|\mathbf{v}_0^\tau\|_{W_2^{1+l}(\tilde{\Omega}_0^\tau)} \\
&+\|l_3^\tau\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T^\tau)} + \|l_4^\tau\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T^\tau)} + \|l_5^\tau\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T^\tau)} \\
&+\|l_6^\tau\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} + \|\theta_e\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} \\
&+\|S_e\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|\theta_b\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\Gamma_{bT})} + \|S_b\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{bT})} \\
&+\|\mathbf{v}_c\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|\theta_c\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|q_c\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{uT})} + \|q_e\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} \\
&+\|\theta_0^\tau\|_{W_2^{2+l}(\Omega_0^\tau)} + \|S_0\|_{W_2^{1+l}(\tilde{\Omega}_0^s)} + \|q_0\|_{W_2^{1+l}(\tilde{\Omega}_0^a)} + \|h_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)} \Big].
\end{aligned}
\tag{4.7}
$$

5 Nonlinear Problem

In this section, we shall prove Theorem 3.1 by an iteration method. The problem that we solve here is in the following. Note that in the following, without loss of generality, we consider the unknown variables from which the initial data are subtracted. Hence, initial data are assumed to be zero.

$$
\begin{cases}
\dfrac{\partial \mathbf{v}_{m+1}^\tau}{\partial t} - L_{1,h_m^\tau}^\tau \mathbf{v}_{m+1}^\tau = \tilde{\mathbf{G}}_{1,h_m^\tau}^\tau(v_m^\tau, w_m^\tau), \quad (t,x) \in \tilde{\Omega}_T^\tau, \\
\mathbf{v}^a = \mathbf{v}^s, \quad (t,x) \in \tilde{\Gamma}_{sT}, \\
B_{h_m^s, h_m^a}(\mathbf{v}_{m+1}^s, \mathbf{v}_{m+1}^s) =: \tilde{\mathbf{G}}_{2,h_m^s}(\mathbf{v}_m^a, \mathbf{v}_m^s, w_m^a, w_m^s), \quad (t,x) \in \tilde{\Gamma}_{sT}, \\
\mathbf{v}_{m+1}^s = 0, \quad (t,x) \in \tilde{\Gamma}_{bT}, \\
a^{33}(h_m^a)\dfrac{\partial \mathbf{v}_{m+1}^a}{\partial x_3} = \alpha_v(\mathbf{v}_c - \mathbf{v}_{m+1}^a) \quad (t,x) \in \tilde{\Gamma}_{uT}, \\
\mathbf{v}_{m+1}^\tau(0,x) = 0, \quad x \in \tilde{\Omega}_0^\tau,
\end{cases}
\tag{5.1}
$$

$$
\begin{cases}
\dfrac{\partial w_{m+1}^\tau}{\partial x_3} = G_{3,h_m^\tau}^\tau(x,t,\mathbf{v}_m^\tau), \quad (t,x) \in \tilde{\Omega}_T^\tau, \\
w_{m+1}^s = 0, \quad (t,x) \in \tilde{\Gamma}_{bT}, \\
w_{m+1}^a = 0, \quad (t,x) \in \tilde{\Gamma}_{uT},
\end{cases}
\tag{5.2}
$$

$$\begin{cases} \dfrac{\partial \theta^\tau_{m+1}}{\partial t} - L^\tau_{2,h^\tau_m}\theta^\tau_{m+1} = \tilde{G}^\tau_{4,h^\tau_m}(\mathbf{v}^\tau_m, w^\tau_m, \theta^\tau_m), \quad (t,x) \in \tilde{\Omega}^\tau_T, \\ \theta^s_{m+1} = \theta_e^{(h^s_m)} \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ \theta^s_{m+1} = \theta_b \quad (t,x) \in \tilde{\Gamma}_{bT}, \\ a^{33}(h^a_m)\dfrac{\partial \theta^a_{m+1}}{\partial x_3} = \alpha_\theta(\theta_c - \theta^a_{m+1}) \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ \theta^\tau_{m+1}(0,x) = 0, \quad x \in \tilde{\Omega}^\tau_0, \end{cases} \tag{5.3}$$

$$\begin{cases} \dfrac{\partial q_{m+1}}{\partial t} - L^a_{3,h^a_m} q_{m+1} = \tilde{G}^a_{5,h^a_m}(\mathbf{v}^a_m, w^a_m, \theta^a_m), \quad (t,x) \in \tilde{\Omega}^a_T, \\ q_{m+1} = q_e^{(h^a_m)}, \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ a^{33}(h^a_m)\dfrac{\partial q_{m+1}}{\partial x_3} = \alpha_q(q_c - q_{m+1}) \quad (t,x) \in \tilde{\Gamma}_{uT}, \\ q_{m+1}(0,x) = 0, \quad x \in \tilde{\Omega}^a_0. \end{cases} \tag{5.4}$$

$$\begin{cases} \dfrac{\partial S_{m+1}}{\partial t} - L^s_{3,h^s_m} S_{m+1} = \tilde{G}^s_{5,h^s_m}(\mathbf{v}^s_m, w^s_m, \theta^s_m), \quad (t,x) \in \tilde{\Omega}^s_T, \\ S_{m+1} = S_e^{(h^s_m)}, \quad (t,x) \in \tilde{\Gamma}_{sT}, \\ S_{m+1} = S_b, \quad (t,x) \in \tilde{\Gamma}_{bT}, \\ S_{m+1}(0,x) = 0, \quad x \in \tilde{\Omega}^s_0. \end{cases} \tag{5.5}$$

$$\begin{cases} \dfrac{\partial h^\tau_{m+1}}{\partial t} - L^\tau_{4,h^\tau_m} h^\tau_{m+1} = \tilde{G}^\tau_{6,h^\tau_m}(\mathbf{v}^\tau_m, w^\tau_m, \theta^\tau_m), \quad (t,x') \in \mathbf{R}^2_T, \\ h^\tau_{m+1}(0,x') = 0, \quad x' \in \mathbf{R}^2, \end{cases} \tag{5.6}$$

where $(\mathbf{v}^\tau_m, w^\tau_m, \theta^\tau_m, S_m, q_m, h^\tau_m) \in Z(T)$ is given and its product norm is bounded by a positive constant M from above.

By lemma 4.1, $(\mathbf{v}^\tau_{m+1}, w^\tau_{m+1}, \theta^\tau_{m+1}, S_{m+1}, q_m, h^\tau_{m+1})$ uniquely exists in Z_T and satisfies the estimates

$$\begin{aligned} &\|(\mathbf{v}^\tau_m, w^\tau_m, \theta^\tau_m, S_m, q_m, h^\tau_m)\|_{Z(T)} \\ &\le C_4\Big[\|\tilde{\mathbf{G}}^\tau_{1,h^\tau_m}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}^\tau_T)} + \|\tilde{\mathbf{G}}^\tau_{2,h^\tau_m}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|\tilde{G}^\tau_{3,h^\tau_m}\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}^\tau_T)} \\ &\quad + \|\tilde{G}^\tau_{4,h^\tau_m}\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}^\tau_T)} + \|\tilde{G}^\tau_{5,h^\tau_m}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}^\tau_T)} \end{aligned}$$

$$+\|\tilde{G}^{\tau}_{6,h^{\tau}_{m}}\|_{W_2^{\frac12+l,\frac14+\frac l2}(\mathbf{R}^2_T)} + \|\theta_e\|_{W_2^{\frac52+l,\frac54+\frac l2}(\tilde{\Gamma}_{sT})} + \|q_e\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{sT})} + \|\mathbf{v}_c\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{uT})}$$

$$+\|\theta_c\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{uT})} + \|q_c\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{uT})} + \|S_e\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{sT})} + \|g_1\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{sT})}$$

$$+\|\theta_b\|_{W_2^{\frac52+l,\frac54+\frac l2}(\tilde{\Gamma}_{bT})} + \|S_b\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{bT})}\Big].$$

Making use of these multiplicative inequalities and Young's inequalities, by taking M large enough so that

$$M > C_4\Big[\sum_{i=1,3}\|\mathbf{F}^{\tau}_i\|_{W_2^{l,\frac l2}(\mathbf{R}^3_T)} + \|F^{\tau}_2\|_{W_2^{1+l,\frac12+\frac l2}(\mathbf{R}^3_T)}$$

$$+\|\theta_e\|_{W_2^{\frac52+l,\frac54+\frac l2}(\tilde{\Gamma}_{sT})} + \|q_e\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{sT})} + \|S_e\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{sT})} + \|g_1\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{sT^*})}$$

$$+\|\theta_b\|_{W_2^{\frac52+l,\frac54+\frac l2}(\tilde{\Gamma}_{bT})} + \|\mathbf{v}_c\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{uT})} + \|\theta_c\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{uT})}$$

$$+\|q_c\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{uT})} + \|S_b\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{bT})}\Big]$$

holds, by taking $\epsilon = \epsilon_0$ small first and then T_1 small, we can show that

$$\|(\mathbf{v}^{\tau}_{m+1}, w^{\tau}_{m+1}, \theta^{\tau}_{m+1}, S_{m+1}, q_{m+1}, h^{\tau}_{m+1})\|_{Z(T_1)} < (\epsilon_0 + C_{\epsilon_0}T_1)F(M)$$

$$+C_4\Big(\sum_{i=1,3}\|\mathbf{F}^{\tau}_i\|_{W_2^{l,\frac l2}(\mathbf{R}^3_T)} + \|F^{\tau}_2\|_{W_2^{1+l,\frac12+\frac l2}(\mathbf{R}^3_T)} + \|\theta_e\|_{W_2^{\frac52+l,\frac54+\frac l2}(\tilde{\Gamma}_{sT})}$$

$$+\|q_e\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{sT})} + \|S_e\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{sT})} + \|g_1\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{sT^*})}$$

$$+\|\mathbf{v}_c\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{uT})} + \|\theta_c\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{uT})} + \|q_c\|_{W_2^{\frac12+l,\frac14+\frac l2}(\tilde{\Gamma}_{uT})}$$

$$+\|\theta_b\|_{W_2^{\frac52+l,\frac54+\frac l2}(\tilde{\Gamma}_{bT})} + \|S_b\|_{W_2^{\frac32+l,\frac34+\frac l2}(\tilde{\Gamma}_{bT})}\Big) < M.$$

Now let us prove the convergence of the successive approximations. Subtracting from (5.1) to (5.6) the similar equations with m replaced by $m-1$ and setting $(\tilde{\mathbf{v}}^{\tau}_{m+1}, \tilde{w}^{\tau}_{m+1}, \tilde{\theta}^{\tau}_{m+1}, \tilde{S}_{m+1}, \tilde{q}_{m+1}, \tilde{h}^{\tau}_{m+1}) = (\mathbf{v}^{\tau}_{m+1}-\mathbf{v}^{\tau}_{m}, w^{\tau}_{m+1}-w^{\tau}_{m}, \theta^{\tau}_{m+1}-\theta^{\tau}_{m}, S_{m+1}-S_m, q^{\tau}_{m+1}-q^{\tau}_{m}, h^{\tau}_{m+1}-h^{\tau}_{m})$, we obtain

$$\frac{\partial\tilde{\mathbf{v}}^{\tau}_{m+1}}{\partial t} - L^{\tau}_{1,h^{\tau}_{m-1}}\tilde{\mathbf{v}}^{\tau}_{m+1} = [L^{\tau}_{1,h^{\tau}_{m}} - L^{\tau}_{1,h^{\tau}_{m-1}}]\mathbf{v}^{\tau}_{m+1}$$

$$+[\mathbf{G}^{\tau}_{1,h^{\tau}_{m}}(\mathbf{v}^{\tau}_{m}, w^{\tau}_{m}) - \mathbf{G}^{\tau}_{1,h^{\tau}_{m-1}}(\mathbf{v}^{\tau}_{m-1}, w^{\tau}_{m-1})], \quad (t,x)\in\tilde{\Omega}^{\tau}_{T_1},$$

$$B_{h^a_m,h^s_m}(\tilde{\mathbf{v}}^a_{m+1}, \tilde{\mathbf{v}}^s_{m+1}) = -[B_{h^a_m,h^s_m}(\mathbf{v}^a_{m}, \tilde{\mathbf{v}}^s_{m+1}) + B_{h^a_m,h^s_m}(\tilde{\mathbf{v}}^a_{m+1}, \mathbf{v}^s_{m})]$$

$$-[B_{h_m^a,h_m^s}(\mathbf{v}_m^a,\mathbf{v}_m^s)-B_{h_{m-1}^a,h_m^s}(\mathbf{v}_m^a,\mathbf{v}_m^s)]-[B_{h_{m-1}^a,h_m^s}(\mathbf{v}_m^a,\mathbf{v}_m^s)-B_{h_{m-1}^a,h_{m-1}^s}(\mathbf{v}_m^a,\mathbf{v}_m^s)]$$
$$+[\mathbf{G}_{2,h_m^s}(\mathbf{v}_m^a,\mathbf{v}_m^s,w_m^a,w_m^s)-\mathbf{G}_{2,h_{m-1}^s}(\mathbf{v}_{m-1}^a,\mathbf{v}_{m-1}^s,w_{m-1}^a,w_{m-1}^s)],\quad (t,x)\in\tilde{\Gamma}_{sT_1},$$
$$a^{33}(h_{m-1}^a)\frac{\partial\tilde{\mathbf{v}}_{m+1}^a}{\partial x_3}=\alpha_{\mathbf{v}}\tilde{\mathbf{v}}_{m+1}^a-(a^{33}(h_m^a)-a^{33}(h_{m-1}^a))\frac{\partial\mathbf{v}_{m+1}^a}{\partial x_3},\quad (t,x)\in\tilde{\Gamma}_{uT_1},$$
$$\tilde{\mathbf{v}}_{m+1}^s=\mathbf{0},\quad (t,x)\in\tilde{\Gamma}_{bT_1},$$
$$\tilde{\mathbf{v}}_{m+1}^\tau(0,x)=\mathbf{0},\quad x\in\tilde{\Omega}_0^\tau,$$

$$\nabla_{h_m^\tau,3}\tilde{w}_{m+1}^\tau=[G_{3,h_m^\tau}^\tau(\mathbf{v}_m^\tau)-G_{3,h_{m-1}^\tau}^\tau(\mathbf{v}_{m-1}^\tau)]-(\nabla_{h_m^\tau,3}-\nabla_{h_{m-1}^\tau,3})w_m^\tau,\quad (t,x)\in\tilde{\Omega}_{T_1}^\tau,$$
$$\tilde{w}_{m+1}^a=0,\quad (t,x)\in\tilde{\Gamma}_{uT_1},$$
$$\tilde{w}_{m+1}^s=0,\quad (t,x)\in\tilde{\Gamma}_{bT_1},$$

$$\frac{\partial\tilde{\theta}_{m+1}^\tau}{\partial t}-L_{2,h_{m-1}^\tau}^\tau\tilde{\theta}_{m+1}^\tau=[L_{2,h_m^\tau}^\tau-L_{2,h_{m-1}^\tau}^\tau]\theta_{m+1}^\tau$$
$$+[G_{4,h_m^\tau}^\tau(\mathbf{v}_m^\tau,w_m^\tau,\theta_m^\tau)-G_{4,h_{m-1}^\tau}^\tau(\mathbf{v}_{m-1}^\tau,w_{m-1}^\tau,\theta_{m-1}^\tau)],\quad (t,x)\in\tilde{\Omega}_{T_1}^\tau,$$

$$\tilde{\theta}_{m+1}^a=\theta_e^{(h_{m+1}^a)}-\theta_e^{(h_m^a)},\quad (t,x)\in\tilde{\Gamma}_{uT_1},$$
$$\tilde{\theta}_{m+1}^s=\theta_e^{(h_{m+1}^s)}-\theta_e^{(h_m^s)},\quad (t,x)\in\tilde{\Gamma}_{sT_1},$$
$$\tilde{\theta}_{m+1}^s=0,\quad (t,x)\in\tilde{\Gamma}_{bT_1},$$
$$a^{33}(h_{m-1}^a)\frac{\partial\tilde{\theta}_{m+1}^a}{\partial x_3}=\alpha_\theta\tilde{\theta}_{m+1}^a-(a^{33}(h_m^a)-a^{33}(h_{m-1}^a))\frac{\partial\theta_{m+1}^a}{\partial x_3},\quad (t,x)\in\tilde{\Gamma}_{uT_1},$$
$$\tilde{\theta}_{m+1}^\tau(0,x)=0,\quad x\in\tilde{\Omega}_0^\tau,$$

$$\frac{\partial\tilde{q}_{m+1}}{\partial t}-L_{3,h_{m-1}^a}^a\tilde{q}_{m+1}=[L_{3,h_m^a}^a-L_{3,h_{m-1}^a}^a]q_{m+1}$$
$$+[G_{5,h_m^a}^a(\mathbf{v}_m^a,w_m^a,q_m)-G_{5,h_{m-1}^a}^a(\mathbf{v}_{m-1}^a,w_{m-1}^a,q_{m-1})],\quad (t,x)\in\tilde{\Omega}_{T_1}^a,$$
$$\tilde{q}_{m+1}=q_e^{(h_{m+1}^a)}-q_e^{(h_m^a)},\quad (t,x)\in\tilde{\Gamma}_{sT_1},$$
$$a^{33}(h_{m-1}^a)\frac{\partial\tilde{q}_{m+1}}{\partial x_3}=\alpha_q\tilde{q}_{m+1}-(a^{33}(h_m^a)-a^{33}(h_{m-1}^a))\frac{\partial q_{m+1}}{\partial x_3}\tilde{q}_{m+1},\quad (t,x)\in\tilde{\Gamma}_{uT_1},$$
$$\tilde{q}_{m+1}(0,x)=0,\quad x\in\tilde{\Omega}_0^a.$$

$$\frac{\partial \tilde{S}_{m+1}}{\partial t} - L^s_{3,h^s_{m-1}} \tilde{S}_{m+1} = [L_{3,h^s_m} - L_{3,h^s_{m-1}}] S_{m+1}$$

$$+[G^s_{5,h^s_m}(\mathbf{v}^s_m, w^s_m, S_m) - G^s_{5,h^s_{m-1}}(\mathbf{v}^s_{m-1}, w^s_{m-1}, S_{m-1})], \quad (t,x) \in \tilde{\Omega}^s_{T_1},$$

$$\tilde{S}_{m+1} = S_e^{(h^s_{m+1})} - S_e^{(h^s_m)}, \quad (t,x) \in \tilde{\Gamma}_{sT_1},$$

$$\tilde{S}_{m+1} = 0, \quad (t,x) \in \tilde{\Gamma}_{bT_1},$$

$$\tilde{S}_{m+1}(0,x) = 0, \quad x \in \tilde{\Omega}^s_0.$$

$$\frac{\partial \tilde{h}^\tau_{m+1}}{\partial t} - L^\tau_{4,h^\tau_{m-1}} \tilde{h}^\tau_{m+1} = [L^\tau_{4,h^\tau_m} - L^\tau_{4,h^\tau_{m-1}}] \tilde{h}^\tau_{m+1}$$

$$+[G^\tau_{6,h^\tau_m}(\mathbf{v}^\tau_m, w^\tau_m, \theta^\tau_m) - G^\tau_{6,h^\tau_{m-1}}(\mathbf{v}^\tau_{m-1}, w^\tau_{m-1}, \theta^\tau_{m-1})], \quad (t,x) \in \mathbf{R}^2_{T_1},$$

$$\tilde{h}_{m+1}(0,x') = 0, \quad x' \in \mathbf{R}^2.$$

Then, in the similar manner as above, we can show that

$$\|(\tilde{\mathbf{v}}^\tau_{m+1}, \tilde{w}^\tau_{m+1}, \tilde{\theta}^\tau_{m+1}, \tilde{q}_{m+1}, \tilde{S}_{m+1}, \tilde{h}^\tau_{m+1})\|_{Z(T_2)}$$
$$< (\epsilon_0 + C_{\epsilon_0} T_2) F(M) \|(\tilde{\mathbf{v}}^\tau_m, \tilde{w}^\tau_m, \tilde{\theta}^\tau_m, \tilde{q}_m, \tilde{S}_m, \tilde{h}^\tau_m)\|_{Z(T_2)},$$

wherer $F(M)$ is a polynomial of M again.

After taking ϵ_0 and T_2 small enough again, and replace the noation of T_2 by T^*, we can verify that $(\mathbf{v}^\tau_m, w^\tau_m, \theta^\tau_m, q_m, S_m, h^\tau_m)$ make a Cauchy sequence. Uniqueness of the solution can be proved in the same way. This completes the proof of the main theorem.

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