数理解析研究所講究録
第 1631 巻 2009 年 68-75

Asymptotic energy concentration in the phase space of the weak solutions to the Navier-Stokes equations

東北大学大学院理学研究科 岡部考宏 (Takahiro Okabe)
Mathematical Institute, Tohoku University

1 Introduction

We consider the asymptotic behavior of the energy of weak solutions to the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$;

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u\nabla u = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ \mathrm{div} u = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot, 0) = a, & \text{in } \mathbb{R}^n, \end{cases} \tag{N-S}$$

where $u = u(x,t) = (u_1(x,t),\dots,u_n(x,t))$ and $p = p(x,t)$ denote the unkown velocity vector and the pressure of the fluid at point $(x,t) \in \mathbb{R}^n \times (0,\infty)$, while $a = a(x) = (a_1(x),\dots a_n(x))$ is a given initial velocity vector field.

For existence of weak solutions of (N-S), Leray [4] constructed a *turbulent solution* on $\mathbb{R}^3$ which is a weak solution satisfying the strong energy inequality and he proposed the famous problem whether or not $\lim_{t\to\infty} \|u(t)\|_{L^2(\mathbb{R}^3)} = 0$.

There are many results on the decay property of the energy, that is, the L^2-norm of the solution of (N-S). Masuda [5] first gave a partial answer to Leray's problem and clarified that the strong energy inequality plays an important role for L^2-decay of weak solutions. Later, Miyakawa, Schonbek and et al. investigated the decay rate of the solution of (N-S) in various domains.

Kato [3] proved that the L^r-decay properties for strong solutions on $\mathbb{R}^n$ with small initial data. He also constructed a turbulent solution in $\mathbb{R}^4$. Using the uniqueness criterion for weak solutions given by serrin [8], we may identify the turbulent solution with the strong solution after some definite time.

Recently, another aspect of asymptotic behavior of the energy of the solution has been investigated. We consider the asymptotic behavior in the following sence:

$$\lim_{t\to\infty} \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2}, \tag{1}$$

where $\{E_\lambda\}_{\lambda\geq 0}$ is the spectral decomposition of the Stokes operator A, that is, the identity

$$A = \int_0^\infty \lambda \, dE_\lambda$$

holds. $E_\lambda u$ is regarded the lower frequency part of u. Indeed, with Fourier transformation, we obtain

$$\widehat{E_\lambda u}(\xi) = \chi_{\{|\xi|\leq\sqrt{\lambda}\}}\hat{u}(\xi),$$

where $\chi_{\{|\xi|\leq\sqrt{\lambda}\}}$ denotes the characteristic function on the set $\{\xi\,;\,|\xi|\leq\sqrt{\lambda}\}$. Hence, we emphasize that the equation (1) means the energy of the lower frequency part of $u(t)$ dominates the energy of $u(t)$ as $t\to\infty$.

Definition 1 (Energy concentration) *We say the energy concentration occurs in the phase λ, if the equation (1) holds for λ.*

Skalák proved the energy concentration for the strong solutions of (N-S) under the assumption that $\limsup_{t\to\infty}\|A^{1/2}u(t)\|_2/\|u(t)\|_2 < \infty$.

Our purpose of the present paper is to characterize the set of initial data that causes the equation (1). For this purpose, we consider the set of initial data that causes a lower bound of the decay rate of the energy of solutions. More precisely, we introduce the set

$$K^\delta_{m,\alpha} = \{\phi\in L^2\,;\,|\hat{\phi}|\geq\alpha|\xi|^m \text{ for } |\xi|\leq\delta\} \tag{2}$$

for α, $\delta>0$ and $m\geq 0$. The set $K^\delta_{m,\alpha}$ is a generalization of the set given by Schonbek [7]. We prove that if the initial data a belongs to $K^\delta_{m,\alpha}$, then the turbulent solution satisfies the energy concentration. Furthermore, we obtain the explicit convergence rate of $u(t)$ in (1).

2 Results

Before stating our results we introduce some function spaces and give our definition of *turbulent solutions* of (N-S). $C_{0,\sigma}^{\infty}$ denotes the set of all C^{∞}-real vector functions ϕ with compact support in $\mathbb{R}^n$ such that $\mathrm{div}\phi = 0$. L_{σ}^{r} is the closure of $C_{0,\sigma}^{\infty}$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ is the inner product in L^2. L^r stands for the usual (vector-valued) L^r-space over $\mathbb{R}^n$, $1 \le r \le \infty$. $H_{0,\sigma}^{1}$ is the closure of $C_{0,\sigma}^{\infty}$ with respect to the norm $\|\phi\|_{H^1} = \|\phi\|_2 + \|\nabla\phi\|_2$, where $\nabla\phi = (\partial\phi_i/\partial x_j)_{i,j=1,\dots,n}$. When X is a Banach space, we denote by $\|\cdot\|_X$ the norm on X. $C^m([t_1,\, t_2]\,;\, X)$ and $L^r(t_1,\, t_2\,;\, X)$ are the usual Banach spaces, where $m = 0, 1, \dots,$ and t_1 and t_2 are real numbers such that $t_1 < t_2$. In this paper we denote by C various constants.

Definition 2 *Let $a \in L_{\sigma}^{2}$. A measurable function u defined on $\mathbb{R}^n \times (0,\infty)$ is called a turbulent solution of (N-S) if*

(i) $u \in L^{\infty}(0,\, \infty\,;\, L_{\sigma}^{2}) \cap L^2(0,\, T\,;\, H_{0,\sigma}^{1})$ *for all* $0 < T < \infty$.

(ii) *The relation*

$$\int_0^T [\,-(u,\, \partial\phi/\partial t) + (\nabla u,\, \nabla\phi) + (u\cdot\nabla u,\, \phi)\,]\, dt = (a,\, \phi(0))$$

holds for almost all T and all $\phi \in C^1([0,\, T)\,;\, H_{0,\sigma}^{1} \cap L^n)$ such that $\phi(\cdot, T) = 0$.

(iii) *The strong enegy inequality*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2\, d\tau \le \|u(s)\|_2^2 \tag{3}$$

holds for almost all $s \ge 0$, includeing $s = 0$, and all $t > s$.

We call a function u satisfying the above conditions (i) and (ii) a weak solution of (N-S). We can redefine any weak solution $u(t)$ of (N-S) on a set of measure zero of the time interval $(0,\, \infty)$ so that $u(t)$ is weakly continuous in t with values in L_{σ}^{2}. Moreover, such a redefined weak solution u satisfies for each $0 \le s < t$,

$$\int_s^t [\,-(u,\, \partial\phi/\partial t) + (\nabla u,\, \nabla\phi) + (u\cdot\nabla u,\, \phi)\,]\, d\tau = -(u(t),\, \phi(t)) + (u(s),\, \phi(s)) \tag{4}$$

for all $\phi \in C^1([s,\, t]\,;\, H^1_{0,\sigma} \cap L^n)$, see Prodi [6]. The existence of turbulent solutions for $n = 3$ and $n = 4$ was given by Leray [4] and Kato [3], respectively.

Let us define the Stokes operator A_r in L^r_σ. We have the following Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G^r, \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r\,;\, p \in L^r_{loc}\}$. P_σ denotes the projection operator from L^r onto L^r_σ. The Stokes operator A_r is defined by $A_r = -P_\sigma \Delta$ with domain $D(A_r) = H^{2,r} \cap L^r_\sigma$. A_2 is nonnegative and self-adjoint operator on L^2_σ. For simplicity, A denotes the Stokes operator A_r if we have no possibility of confusion. $\{E_\lambda\}_{\lambda \geq 0}$ denotes the spectral decomposition of the nonnegative self-adjoint operator A .

Let us introduce the definition of strong solution of (N-S).

Definition 3 *Let $n < r < \infty$, $a \in L^n_\sigma$. A measurable function u defined on $\mathbb{R}^n \times (0, \infty)$ is called a global strong solution of (N-S) if*

$$u \in C([0,\, \infty)\,;\, L^n_\sigma) \cap C((0,\, \infty)\,;\, L^r),$$
$$\frac{\partial u}{\partial t},\ Au \in C((0,\, \infty)\,;\, L^n_\sigma),$$

and u satisfies

$$\frac{\partial u}{\partial t} + Au + P_\sigma(u \cdot \nabla u) = 0, \quad t > 0.$$

Now our results read:

Theorem 1 *Let $2 \leq n \leq 4$, and let $r > 1$ and $m \geq 0$ be*

(i) *for $n = 2$,*

$$1 < r < \frac{4}{3}, \quad 0 \leq m < \frac{4}{r} - 3,$$

(ii) *for $n = 3,\, 4$,*

$$1 < r < \frac{n}{n-1}, \quad 0 \leq m < \frac{n}{r} - (n-1).$$

Suppose that $K^{\delta}_{m,\alpha}$ is the same as (2). If $a \in L^r_\sigma \cap L^2_\sigma \cap K^{\delta}_{m,\alpha}$ for some $\alpha, \delta > 0$, then for every turbulent solution $u(t)$ there exist $T > 0$ and $C(n, r, m, \delta, \alpha, a) > 0$ such that

$$\left| \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} - 1 \right| \le \frac{C}{\lambda} t^{-(n/r-n+1-m)} \tag{5}$$

holds for all λ and for all $t > T$.

Theorem 2 *Let $n \ge 5$, and let $r > 1$ and $m \ge 0$ be*

$$1 < r < \frac{n}{n-1}, \quad 0 \le m < \frac{n}{r} - (n-1).$$

Then there exists $\gamma > 0$ such that if $a \in L^r_\sigma \cap L^n_\sigma \cap K^{\delta}_{m,\alpha}$ for some $\alpha, \delta > 0$ and if a satisfies $\|a\|_n \le \gamma$, then there exists a unique global strong solution $u(t)$ with the following property. There exist $T > 0$ and $C(n, r, m, \delta, \alpha, a) > 0$ such that

$$\left| \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} - 1 \right| \le \frac{C}{\lambda} t^{-(n/r-n+1-m)} \tag{6}$$

holds for all λ and for all $t > T$.

Remark 3 Skalák [10] proved the enegy concentration (1) under the assumption $\limsup_{t\to\infty} \|A^{1/2}u(t)\|_2/\|u(t)\|_2 < \infty$. From the assumption of Theorem 1 and of Theorem 2, we can show that $\lim_{t\to\infty} \|A^{1/2}u(t)\|_2/\|u(t)\|_2 = 0$. On the other hand, our advantage seems to characterize the set of initial data which causes an energy concentration. Moreover, we get the explicit convergence rate of (1). We introduce the set $K^{\delta}_{m,\alpha}$ of initial data that causes (1), especially, causes the lower bound of the L^2-decay of the solutions of (N-S). (See also Schonbek [7].)

3 Outline of the proof of Theorem 1

3.1 Key lemma

The following lemma plays an important role in the proof of the Theorem 1.

Lemma 4 *Let $2 \leq n \leq 4$. Let r and m be as*

(i) *for $n = 2$,*

$$1 < r < \frac{4}{3}, \quad 0 \leq m < \frac{4}{r} - 3$$

(ii) *for $n \geq 3$,*

$$1 < r < \frac{n}{n-1}, \quad 0 \leq m < \frac{n}{r} - (n-1).$$

If $a \in L^r_\sigma \cap L^2_\sigma \cap K^\delta_{m,\sigma}$, Then every turbulent solution $u(t)$ of (N-S) satisfies

$$\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2} \leq O(t^{-(n/r-n+1-m)}), \tag{7}$$

as $t \to \infty$.

To prove lemma 4, we need to compare the decay rate of $\|\nabla u(t)\|_2$ with the lower bound of the decay rate of $\|u(t)\|_2$. We obtain the lower bound of the decay rate of $\|u(t)\|_2$ due to the set $K^\delta_{m,\alpha}$.

3.2 Proof of Theorem 1

As we mentioned above, turbulent solutions of (N-S) become strong solutions after some definite time. So for the turbulent solution $u(t)$ of (N-S) there exists $T_* > 0$ such that $u(t)$ is strong solution of (N-S) on $[T_*, \infty)$. Hence we have the energy identity:

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|A^{1/2}u(t)\|_2^2 = 0 \tag{8}$$

for $t \geq T_*$. For any fixed $\lambda > 0$, the second term in (8) is estimated from below as

$$\begin{aligned} \|A^{1/2}u(t)\|_2^2 &= \int_0^\infty \rho\, d\|E_\rho u\|_2^2 \geq \int_\lambda^\infty \rho\, d\|E_\rho u\|_2^2 \\ &\geq \lambda \int_\lambda^\infty d\|E_\rho u\|_2^2 \geq \frac{\lambda}{2}(\|u(t)\|_2^2 - \|E_\lambda u(t)\|_2^2). \end{aligned} \tag{9}$$

From (8) and (9) we have

$$\frac{d}{dt}\|u(t)\|_2^2 + \lambda\|u(t)\|_2^2 \leq \lambda\|E_\lambda u(t)\|_2^2. \tag{10}$$

Dividing both sides of (10) by $\lambda\|u(t)\|_2^2$, we obtain

$$\frac{\frac{d}{dt}\|u(t)\|_2^2}{\lambda\|u(t)\|_2^2} + 1 \leq \frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2}. \tag{11}$$

On the other hand, by (8), we have $(d/dt)\|u(t)\|_2^2 = -2\|A^{1/2}u(t)\|_2^2 = -2\|\nabla u(t)\|_2^2$, from which and (11) it follows that

$$1 - \frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2} \leq \frac{2}{\lambda}\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2}.$$

Hence by Lemma 4, there exists T such that

$$\left|\frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2} - 1\right| \leq \frac{C}{\lambda}t^{-(n/r-n+1-m)}$$

for all $t \geq T$. This completes the proof of Theorem 1.

References

[1] W. Borchers and T. Miyakawa, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains*, Acta Math., **165** (1990), 189-227.

[2] R. Kajikiya and T. Miyakawa, *On L^2 decay of weak solutions of the Navier-Stokes Equations in $\mathbb{R}^n$*, Math. Z., **192** (1986), 315-148.

[3] T. Kato, *Strong L^p-solution of the Naier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*, Math. Z., **187** (1984), 471-480.

[4] J. Leray, *Sur le mouvement d'un liquids visqeux emplissant l'espace*, Acta. Math., **63** (1934), 193-248.

[5] K. Masuda, *Weak solutions of the Navier-Stokes equations*, Tohoku Math. J.(2), **36** (1984), 623-646.

[6] G. Prodi, *Un theorema di unicità per le equazioni di Navier-Stokes*, Annali di Mat. **48** (1959), 173-182.

[7] M. E. Schonbek, *Large time behaviour of solutions to the Navier-Stokes equations.* comm. Partial Differential Equations **11** (1986), no.7, 733-763.

[8] J. Serrin, *The initial value problem for the Navier-Stokes equations*, Nonlinear Problems, R. E. Langer ed., University of Wisconsin Press, Madison, 1963, 69-98.

[9] Z. Skalák, *Asymptotic behavior of modes in weak solutions to the homogeneous Navier-Stokes equations*, WSEAS Trans. Math., **3** No. 5 (2006) 280-288.

[10] Z. Skalák, *Some aspects of the asymptotic dynamics of solutions of the homogeneous Navier-Stokes equations in general domains*, preprint.

[11] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on* $\mathbb{R}^n$, J. London Math. Soc. (2), **35** (1987) 303-313.