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正作用素の微分幾何と計量
Differential geometries and their metrics on the positive operators

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1 Introduction

To discuss geometries with metrics induced by unitarily invariant norms, let $\mathcal{A}$ be a unital C*-algebra on a finite dimensional Hilbert space H. The manifold in discourse is $\mathcal{A}^+$, the positive invertible elements. In 1990's, Corach-Porta-Lecht [7, 8, 9] discussed Finsler geometry on it, which we call the *CPR geometry*: The tangent space (and bundle) is the selfadjoint elements $\mathcal{A}^h$. The invertible elements $\mathcal{G}$ in $\mathcal{A}$ defines the principal fiber (frame) bundle: For a fixed $A \in \mathcal{A}$, the projection $\pi_A(G) = GAG^*$, the structure group $\mathcal{U}_A = \{V \in \mathcal{G} \mid VAV^* = A\} = A^{1/2}\mathcal{U}A^{-1/2}$ with the action $L_V A = VAV^*$, which shows $\mathcal{A}^+$ is homogeneous and hence A can be assumed the identity element I. In this case $\mathcal{U}_I$ is nothing but the unitary group $\mathcal{U}$. $\mathcal{A}$ itself is considered as the tangent space of $\mathcal{G}$ and it has the connection induced by the *horizontal space* $H_G = G\mathcal{A}^h$. Then, the *parallel displacement* of a tangent vector X along γ from 0 to t is

$$P_t X = \Gamma(t)\Gamma(0)^{-1}X(\Gamma(0)^*)^{-1}\Gamma(t)^*.$$

So the *covariant derivative* D_t of a tangent field $X(t)$ along the curve $\gamma(t)$ in $\mathcal{A}^+$ is given by

$$D_t X = \dot{X} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}).$$

Then the geodesic equation $O = D_t\dot{\gamma} = \ddot{\gamma} - \dot{\gamma}\gamma^{-1}\dot{\gamma}$ implies that the geodesic from A to B is the path of geometric Kubo-Ando means [14]:

$$\gamma(t) = A\#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}.$$

Moreover the above manifold $\mathcal{A}^+$ is the Finsler space with a Finsler metric

$$L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\|$$

at each point $A \in \mathcal{A}$ and the distance between two points defined as the shortest length of a path is so-called Thompson metric $d(A, B) = \|\log A^{-1/2}BA^{-1/2}\|$.

The CPR geometry does not always determine a unique Finsler metric. In fact, we show each unitarily invariant norm $|||\ |||$ also gives a Finsler metric for the CPR geometry:

Theorem 1. *For a unitarily invariant norm* $|||\ |||$ *on* $\mathcal{A}$, *a function*

$$L_{|||\ |||}(X;A) = |||X|||_A = |||A^{-1/2}XA^{-1/2}|||$$

determines a Finsler metric on $\mathcal{A}^+$ *for the CPR geometry.*

Proof. Since $U_t = \gamma(t)^{-1/2}\Gamma(t)$ defines a unitary for each t by $\gamma = \Gamma\Gamma^*$, we show the Finsler condition by

$$|||P_tX|||_{\gamma(t)} = |||U_tU_0^*\gamma(0)^{-1/2}X\gamma(0)^{-1/2}U_0U_t^*||| = |||\gamma(0)^{-1/2}X\gamma(0)^{-1/2}||| = |||X|||_{\gamma(0)}.$$

□

A Finsler metric $L_{|||\ |||}(X;A)$, which is called a *unitarily invariant Finsler one*, is homogeneous like $L(X;A)$:

Theorem 2. *For any invertible operator* Y,

$$L_{|||\ |||}(Y^*XY;Y^*AY) = L_{|||\ |||}(X;A).$$

Proof. Since $|||Z||| = |||\,|Z|\,||| = |||\sqrt{Z^*Z}||| = |||\sqrt{ZZ^*}|||$, we have

$$\begin{aligned}
L_{|||\ |||}(Y^*XY;Y^*AY) &= \left|\left|\left|(Y^*AY)^{-1/2}Y^*XY(Y^*AY)^{-1/2}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{(Y^*AY)^{-1/2}Y^*XY(Y^*AY)^{-1}Y^*XY(Y^*AY)^{-1/2}}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{(Y^*AY)^{-1/2}Y^*XA^{-1}XY(Y^*AY)^{-1/2}}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{A^{-1/2}XY(Y^*AY)^{-1}Y^*XA^{-1/2}}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{A^{-1/2}XA^{-1}XA^{-1/2}}\right|\right|\right| = \left|\left|\left|A^{-1/2}XA^{-1/2}\right|\right|\right| = L_{|||\ |||}(X;A).
\end{aligned}$$

□

2 Metric space of Thompson's type

The geodesic $A\#_tB$ is one of the shortest paths with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t);\gamma(t))dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|dt.$$

If $\gamma(t)$ is a path from A to B, then

$$\begin{aligned}
d(A,B) \equiv \inf_\gamma \ell(\gamma) &= \ell(A\#_tB) = \|\log(A^{-1/2}BA^{-1/2})\| \\
&= \log\left(\max\{\|A^{-1/2}BA^{-1/2}\|,\ \|B^{-1/2}AB^{-1/2}\|\}\right) \\
&= \log\left(\max\{r(A^{-1}B),\ r(B^{-1}A)\}\right).
\end{aligned}$$

The homogeneity of $\mathcal{A}^+$ implies

$$d(A,B) = d(Y^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})$$

for invertible X. The metric d makes $\mathcal{A}^+$ a complete metric space and it is called the *Thompson (part) one* [19].

Also, Corach et.al. [8, 3] showed the convexity for the metric: For geodesics γ and δ, the followings are equivalent:

(i) $F(t) = d(\gamma(t), \delta(t)) = \log \|\gamma(t)^{-1/2}\delta(t)\gamma(t)^{-1/2}\|$: convex:

(ii) $d(\gamma(t), \delta(t)) \leqq (1-t)d(\gamma(0), \delta(0)) + td(\gamma(1), \delta(1))$.

The above equivalence is guaranteed by the interpolationality for the path $A\#_t B$. This convexity suggests that the curvature of $\mathcal{A}^+$ is negative. In Riemannian geometry, the above convexity implies exactly the negativity of the curvature. But, in Finsler geometry, the notion of it has not been completely established yet.

Now we show the above properties also hold for the metric

$$d_{|||\ |||}(A,B) = \inf_{\gamma \in P(A,B)} \int_0^1 L_{|||\ |||}(\gamma'(t); \gamma(t))\ dt$$

for a unitarily invariant norm $|||\ |||$ where $P(A,B)$ denotes the (differentiable) paths from A to B. To see the path $A\#_t B$ is one of the shortest paths directly, recall the following 'logarithmic-geometric mean inequality' due to Hiai-Kosaki [13]:

Hiai-Kosaki inequality. $$\left|\left|\left| \int_0^1 H^t X K^{1-t} dt \right|\right|\right| \geqq |||H^{1/2} X K^{1/2}|||.$$

Theorem 2 implies the homogeneity of $d_{|||\ |||}$:

Lemma 3. *$d_{|||\ |||}(A,B) = d_{|||\ |||}(X^*AX, X^*BX)$ holds for invertible X.*

Proof. Note that $X^*P(A,B)X = P(X^*AX, X^*BX)$ holds for all invertible X. Since $L_{|||\ |||}((X^*\gamma'(t)X; X^*\gamma(t)X) = L_{|||\ |||}(\gamma'(t); \gamma(t))$, we have

$$\begin{aligned} d_{|||\ |||}(X^*AX, X^*BX) &= \inf_{\gamma \in P(X^*AX, X^*BX)} \int_0^1 L_{|||\ |||}(\gamma'(t); \gamma(t))\ dt \\ &= \inf_{\gamma \in P(A,B)} \int_0^1 L_{|||\ |||}(X^*\gamma'(t)X; X^*\gamma(t)X)\ dt \\ &= \inf_{\gamma \in P(A,B)} \int_0^1 L_{|||\ |||}(\gamma'(t); \gamma(t))\ dt = d_{|||\ |||}(A,B) \end{aligned}$$

by Theorem 2. □

The following formula is parallel to Bhatia's one [6, (6.42)].

Lemma 4. *Let $H(t) = \log\gamma(t)$ for $\gamma \in P(A,B)$. Then*

$$\frac{d}{dt}\gamma(t) = \frac{d}{dt}e^{H(t)} = \int_0^1 e^{uH(t)}H'(t)e^{(1-u)H(t)}du.$$

Proof. Since

$$-\frac{d}{du}e^{uH(t)}e^{(1-u)H(t+\varepsilon)} = e^{uH(t)}\big(H(t+\varepsilon)-H(t)\big)e^{(1-u)H(t+\varepsilon)},$$

we have

$$\int_0^1 e^{uH(t)}\big(H(t+\varepsilon)-H(t)\big)e^{(1-u)H(t+\varepsilon)}du = -\left[e^{uH(t)}e^{(1-u)H(t+\varepsilon)}\right]_0^1 = e^{H(t+\varepsilon)} - e^{H(t)}.$$

The require formula is obtained by multiplying $\frac{1}{\varepsilon}$ and $\varepsilon \to 0$. □

Now we show that it is a metric of Thompson type, which is mentioned also in [5], and moreover that the geodesic $A\#_tB$ is the shortest path in almost all cases. To see this, recall that the norm $|||\ |||$ is *strictly convex* if

$$|||(1-t)x+ty||| < 1 \qquad \text{for all } t \in (0,1) \text{ and all distinct unit vectors } x \text{ and } y.$$

Then the strict triangle inequality holds (cf. [17]):

$$|||x+y||| < |||x||| + |||y|||$$

unless one of the vectors is a nonnnegative multiple of the other.

Note that we identify $\gamma(f(t))$ with $\gamma(t)$ if f is increasing function with $f(0) = 1$ and $f(1) = 1$ since they are the same as sets; $\{\gamma(f(t))|t \in [0,1]\} = \{\gamma(t)|t \in [0,1]\}$ and give the same length: $\ell(\gamma(f(t))) = \ell(\gamma(t))$.

Theorem 5. *A function $d_{|||\ |||}$ is a metric on $\mathcal{A}^+$ and*

$$d_{|||\ |||}(A,B) = \int_0^1 L_{|||\ |||}\left(\frac{d}{dt}A\#_tB, A\#_tB\right)dt = \left|\left|\left|\log A^{-1/2}BA^{-1/2}\right|\right|\right|.$$

Moreover, if the norm $|||\ |||$ is strictly convex, then the geodesic $A\#_tB$ is the shortest path.

Proof. First we see $d_{|||\ |||}(A,B) \geqq \left|\left|\left|\log A^{-1/2}BA^{-1/2}\right|\right|\right|$. In fact, by Lemma 4 and the Hiai-Kosaki inequality,

$$\begin{aligned}|||\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}||| &= \left|\left|\left|e^{-H(t)/2}\left(\int_0^1 e^{uH(t)}H'(t)e^{(1-u)H(t)}du\right)e^{-H(t)/2}\right|\right|\right| \\ &= \left|\left|\left|\left(\int_0^1 \left(e^{H(t)}\right)^u e^{-H(t)/2}H'(t)e^{-H(t)/2}\left(e^{H(t)}\right)^{1-u}du\right)\right|\right|\right| \\ &\geqq \left|\left|\left|\left(e^{H(t)}\right)^{1/2}e^{-H(t)/2}H'(t)e^{-H(t)/2}\left(e^{H(t)}\right)^{1/2}\right|\right|\right| = |||H'(t)|||\end{aligned}$$

holds and hence the required inequality is obtained by

$$\begin{aligned}\ell_{|||\ |||}(\gamma) &= \int_0^1 |||\gamma^{-1/2}(t)\gamma'(t)\gamma^{-1/2}(t)|||dt \\ &\geqq \int_0^1 |||H'(t)|||dt \geqq \left|\left|\left|\int_0^1 H'(t)dt\right|\right|\right| = |||\log B - \log A|||.\end{aligned}$$

Let $C = A^{-1/2}BA^{-1/2}$ and $\Gamma(t) = C^t$. Then

$$\ell_{|||\ |||}(\Gamma) \equiv \int_0^1 \left|\left|\left|C^{-t/2}(C^t \log C)C^{-t/2}\right|\right|\right|dt = |||\log C||| = |||\log C - \log I|||$$

and hence Γ attains the shortest length and $d(I, C) = |||\log C|||$. By Lemma 3, we have the geodesic $\gamma(t) = A^{1/2}\Gamma(t)A^{1/2} = A^{1/2}C^tA^{1/2} = A\#_tB$. Moreover,

$$\begin{aligned}d_{|||\ |||}(A, B) = d_{|||\ |||}(I, C) = |||\log C||| &= \left|\left|\left|\log A^{-1/2}BA^{-1/2}\right|\right|\right| \\ &= \left|\left|\left|\log B^{-1/2}AB^{-1/2}\right|\right|\right| = d_{|||\ |||}(B, A).\end{aligned}$$

Thus the symmetry of $d_{|||\ |||}(A, B)$ holds. It is clear that $d_{|||\ |||}(A, B) = 0$ if and only if $A = B$ and that the triangle inequality for $d_{|||\ |||}$ holds, which shows $d_{|||\ |||}$ is a metric.

Next suppose the norm is strictly convex and γ attains the shortest. By homegeneity, we may assume $\gamma \in P(I, B)$. Then, for $H(t) = \log \gamma(t)$, it must satisfy

$$\int_0^1 |||H'(t)|||dt = \left|\left|\left|\int_0^1 H'(t)dt\right|\right|\right| = |||\log B|||.$$

Thereby we have $H'(t)$ is a nonnegative scalar multiple of $\log B$ for each t. In fact, we use the broken line approximation to obtain the length of $H(t)$:

$$\int_0^1 |||H'(t)|||dt = \lim_{|\Delta|\to 0} \sum_{t_n \in \Delta} |||H(t_{n+1}) - H(t_n)|||.$$

Take the following monotone increasing sequence converging to $\int_0^1 ||H'(t)||dt$:

$$\sum_{k=1}^{2^n} \left|\left|\left|H\left(\frac{k}{2^n}\right) - H\left(\frac{k-1}{2^n}\right)\right|\right|\right| \quad \uparrow \quad \int_0^1 |||H'(t)|||dt.$$

Then all the triangle inequalities are equal: e.g.,

$$\begin{aligned}&\left|\left|\left|H\left(\frac{k}{2^n}\right) - H\left(\frac{k-1}{2^n}\right)\right|\right|\right| \\ &\quad \leqq \left|\left|\left|H\left(\frac{2k}{2^{n+1}}\right) - H\left(\frac{2k-1}{2^{n+1}}\right)\right|\right|\right| + \left|\left|\left|H\left(\frac{2k-1}{2^{n+1}}\right) - H\left(\frac{2(k-1)}{2^{n+1}}\right)\right|\right|\right|.\end{aligned}$$

For $n = 0$ and $k = 1$, we obtain $H(1) - H(1/2) = s(H(1/2) - H(0)) = sH(1/2)$, that is, $H(1) = (s + 1)H(1/2)$. Putting $f(1/2) = 1/(1 + s)$, we have $H(1/2) =$

$f(1/2)H(1) = f(1/2)\log B$. Since H is continuous, we can define f for all $t \in [0,1]$ and

$$H(t) = f(t)\log B = \log B^{f(t)} \quad \text{i.e.,} \quad \gamma(t) = B^{f(t)}.$$

Then $f(1) = 0$ and $f(1) = 1$ and moreover f must be monotone, hence increasing, so that γ is the same as the geodesic B^t. $\square$

Remark 1. If the norm is the operator one or the trace one, then it is not strictly convex and indeed the shortest path is not uniquely determined. In fact, take a path $I\nabla_t B = (1-t)I + tB \in P(I,B)$ for $B \geq I$. Then, considering

$$L_{|||\ |||}(\dot{\gamma};\gamma) = |||(B-I)(I+t(B-I))|||$$

and a function $F(x) = \big(x/(1+tx)\big)$ is monotone increasing, we obtain $\ell(I\nabla_t B) = |||\log B|||$ in both cases.

Remark 2. In the above metric $d_{|||\ |||}$, it is easy to see that $\mathcal{A}^+$ is also complete. In particular, if a unitarily invariant norm is normalized; $|||P||| = 1$ for projections of rank one, then $\|X\| \leqq |||X|||$ holds and hence the convergence is reduced to the original Thompson metric.

Next we see the following equivalence:

Theorem 6. *For geodesics γ and δ, the followings hold and they are equivalent:*

(i) $F(t) = d_{|||\ |||}(\gamma(t),\delta(t)) = \left|\left|\left|\log \gamma(t)^{-1/2}\delta(t)\gamma(t)^{-1/2}\right|\right|\right|$ *is convex.*

(ii) $d_{|||\ |||}(A^t,B^t) \leqq t d_{|||\ |||}(A,B)$.

(iii) $d_{|||\ |||}(\gamma(t),\delta(t)) \leqq (1-t)d_{|||\ |||}(\gamma(0),\delta(0)) + t d_{|||\ |||}(\gamma(1),\delta(1))$.

Proof. It suffices to show the case of matrices. Then Araki [2] showed

$$\prod_{j=1}^{k} \lambda_j(B^{-t/2}A^tB^{-t/2}) \leqq \prod_{j=1}^{k} \lambda_j^t(B^{-1/2}AB^{-1/2})$$

for $0 \leqq t \leqq 1$ and $1 \leqq k \leqq n$ where λ_j is the j-th eigenvalue (singular value in this case) under the decreasing order. Thus we have the weak-majorization

$$\log B^{-t/2}A^tB^{-t/2} \underset{\mathrm{w}}{\prec} t\log B^{-1/2}AB^{-1/2}. \tag{1}$$

On the other hand, Araki's inequality also holds for A^{-1} and B^{-1};

$$\begin{aligned}\prod_{j=1}^{k} \lambda_j\left((B^{-t/2}A^tB^{-t/2})^{-1}\right) &= \prod_{j=1}^{k} \lambda_j(B^{t/2}A^{-t}B^{t/2}) \\ &\leqq \prod_{j=1}^{k} \lambda_j^t(B^{1/2}A^{-1}B^{1/2}) = \prod_{j=1}^{k} \lambda_j^t\left((B^{-1/2}AB^{-1/2})^{-1}\right),\end{aligned}$$

so that

$$(2) \qquad -\log B^{-t/2}A^tB^{-t/2} \underset{w}{\prec} -t\log B^{-1/2}AB^{-1/2}.$$

Combining (1) and (2), we have the majorization

$$\log B^{-t/2}A^tB^{-t/2} \prec t\log B^{-1/2}AB^{-1/2}$$

and consequently we have (ii) by the convexity of $f(x) = |x|$:

$$d_{|||\ |||}(A^t, B^t) = \left|\left|\left| \log B^{-t/2}A^tB^{-t/2} \right|\right|\right| \leqq t \left|\left|\left| \log B^{-1/2}AB^{-1/2} \right|\right|\right| = td_{|||\ |||}(A, B).$$

Then, the triangle inequality and the homogeneity (Lemma 3) and (ii) show (iii): For $\gamma(t) = A \text{ m }_t B$, $\delta(t) = C \text{ m }_t D$ and $\zeta(t) = C \text{ m }_t B$,

$$\begin{aligned} & d_{|||\ |||}(\gamma(t), \delta(t)) \leqq d_{|||\ |||}(\gamma(t), \zeta(t)) + d_{|||\ |||}(\zeta(t), \delta(t)) \\ & = d_{|||\ |||}((B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{1-t}, (B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{1-t}) + d_{|||\ |||}((C^{-\frac{1}{2}}BC^{-\frac{1}{2}})^t, (C^{-\frac{1}{2}}DC^{-\frac{1}{2}})^t) \\ & \leqq (1-t)d_{|||\ |||}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}, B^{-\frac{1}{2}}CB^{-\frac{1}{2}}) + td_{|||\ |||}(C^{-\frac{1}{2}}BC^{-\frac{1}{2}}, C^{-\frac{1}{2}}DC^{-\frac{1}{2}}) \\ & = (1-t)d_{|||\ |||}(A, C) + td_{|||\ |||}(B, D) = (1-t)d_{|||\ |||}(\gamma(0), \delta(0)) + td_{|||\ |||}(\gamma(1), \delta(1)). \end{aligned}$$

Let $\Gamma(t) = (A \text{ m }_p B) \text{ m }_t (A \text{ m }_q B)$ and $\Delta(t) = (C \text{ m }_p D) \text{ m }_t (C \text{ m }_q D)$. Then, by the interpolationality, we have

$$\Gamma(t) = A \text{ m }_{(1-t)p+tq} B \quad \text{and} \quad \Delta(t) = C \text{ m }_{(1-t)p+tq} D,$$

so that,

$$\Gamma(0) = A \text{ m }_p B, \quad \Gamma(1) = A \text{ m }_q B, \quad \Delta(0) = C \text{ m }_p D \quad \text{and} \quad \Delta(1) = C \text{ m }_q D.$$

Thus, we have (iii) for Γ and Δ implies (ii) for γ and δ. Considering geodesics $\gamma(t) = A^t \in P(I, A)$ and $\delta(t) = B^t \in P(I, A)$, we have (i) implies

$$\begin{aligned} d_{|||\ |||}(A^t, B^t) &= F(t) = F((1-t)0 + t) \leqq (1-t)F(0) + tF(1) \\ &= (1-t)d_{|||\ |||}(I, I) + td_{|||\ |||}(A, B) = td_{|||\ |||}(A, B). \end{aligned}$$

Thus all the relations hold and they are equivalent. □

3 Extreme Finsler geometry

Only two metric $d_{\pm 1}$ are derived from Finsler metrics since

$$A\nabla_t B = A \text{ m }_{1,t} B = A \mathfrak{m}_{1,t} B \quad \text{and} \quad A\,!_t B = A \text{ m }_{-1,t} B = A \mathfrak{m}_{-1,t} B.$$

Here we observe Finsler geometries and metrics for $r = \pm 1$. First we see the trivial fiber bundle $(\mathcal{A}^+ \times \mathcal{U}, \mathcal{A}^+, \mathcal{U}, \pi_1)$ where $\mathcal{U}$ is the unitary group of a unital C*-algebra

$\mathcal{A}$ where $\pi_1((A,U)) = A$ and the action is $L_{(A,U)}B = UBU^*$. Then the fiber is $\pi^{-1}(A) = A \times \mathcal{U}$ and the horizontal lift Γ of a curve γ is $\Gamma(t) = (\gamma(t), U)$ for a fixed $U \in \mathcal{U}$. So the parallel displacement P_t and the covariant derivative are trivial:

$$P_t X = X \quad \text{and} \quad D_t X(t) = \frac{dX}{dt}(t).$$

Thereby the geodesic equation is $0 = D_t\dot{\gamma} = \ddot{\gamma}(t)$. Thus we have the geodesic from A to B is the arithmetic mean $A\nabla_t B = (1-t)A + tB$. Here we define a trivial Finsler metric $L_1(X;A) = |||X|||$ and then

$$\int_0^1 L_1(\dot{\gamma}(t);\gamma(t)dt = \int_0^1 |||\dot{\gamma}(t)|||dt = \int_0^1 |||B-A|||dt = |||B-A|||.$$

We can verify $A\nabla_t B$ attains the shortest: For any $\gamma \in P(A,B)$, we have

$$\int_0^1 L_1(\dot{\gamma}(t);\gamma(t))dt \geqq \left|\left|\left| \int_0^1 \dot{\gamma}(t)dt \right|\right|\right| = |||[\gamma(t)]_0^1||| = |||B-A||| = d_1(A,B).$$

Remark 3. Like Theorem 5, if the norm is strictly convex, then the geodesic is a unique path attaining the shortest. Suppose the norm is the operator one or the trace one, a path B^t also attains the shortest length for $B \geq I$.

Next we consider the trivial bundle $(\mathcal{A}^+ \times \mathcal{U}, \mathcal{A}^+, \mathcal{U}, \pi_1)$ which is the same as the preceding case except the action $L_{(A,U)}B = AUBU^*A$. Then the parallel displacement is

$$P_t X = \gamma(t)U\left((\gamma(0)U)^{-1}X(U^*\gamma(0))^{-1}\right)U^*\gamma(t) = \gamma(t)\gamma(0)^{-1}X\gamma(0)^{-1}\gamma(t)$$

and hence the covariant derivative is obtained by

$$\begin{aligned} D_t(X(t)) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\gamma(t)\gamma(t+\varepsilon)^{-1}X(t+\varepsilon)\gamma(t+\varepsilon)^{-1}\gamma(t) - X(t)\right) \\ &= \gamma(\gamma^{-1}X\gamma^{-1})'\gamma(t) \\ &= \gamma\Bigg(-\gamma^{-1}\dot{\gamma}\gamma^{-1}X\gamma(t)^{-1} + \gamma^{-1}\dot{X}\gamma^{-1} - \gamma^{-1}X\gamma^{-1}\dot{\gamma}\gamma^{-1}\Bigg)\gamma(t) \\ &= \left(\dot{X} - \dot{\gamma}\gamma^{-1}X - X\gamma^{-1}\dot{\gamma}\right)(t). \end{aligned}$$

The geodesic equation is

$$0 = \ddot{\gamma}(t) - 2\dot{\gamma}(t)\gamma^{-1}(t)\dot{\gamma}(t)$$

and thereby

$$\begin{aligned} (-\gamma^{-1}\dot{\gamma}\gamma^{-1})' &= \gamma^{-1}\dot{\gamma}\gamma^{-1}\dot{\gamma}\gamma^{-1} - \gamma^{-1}\ddot{\gamma}\gamma^{-1} + \gamma^{-1}\dot{\gamma}\gamma^{-1}\dot{\gamma}\gamma^{-1} \\ &= 2\gamma^{-1}\dot{\gamma}\gamma^{-1}\dot{\gamma}\gamma^{-1} - \gamma^{-1}\ddot{\gamma}\gamma^{-1} = \gamma^{-1}\left(2\dot{\gamma}\gamma^{-1}\dot{\gamma} - \ddot{\gamma}\right)\gamma^{-1} = 0. \end{aligned}$$

So there exists a self-adjoint operator C with

$$C = -\gamma^{-1}(t)\dot{\gamma}(t)\gamma^{-1}(t) = (\gamma^{-1}(t))'$$

and consequently, $D + tC = \gamma^{-1}(t)$ for some positive operator D. Since $A = \gamma(0) = D^{-1}$, $B = \gamma(1) = (D+C)^{-1}$, we have $D = A^{-1}$, $C = B^{-1} - D = B^{-1} - A^{-1}$, that is,

$$\gamma(t) = \left(A^{-1} + t(B^{-1} - A^{-1})\right)^{-1} = \left((1-t)A^{-1} + tB^{-1}\right)^{-1} = A \,!_t B.$$

Define $L_{-1}(X; A) = |||A^{-1}XA^{-1}|||$. Then it is a Finsler metric. In fact,

$$L_{-1}(P_tX; \gamma(t)) = |||I\gamma(0)^{-1}X\gamma(0)^{-1}I||| = |||\gamma(0)^{-1}X\gamma(0)^{-1}||| = L_{-1}(X; \gamma(0)).$$

Now, for $f(t) = \gamma^{-1}$, we have

$$\dot{\gamma} = (f^{-1})' = -f^{-1}\dot{f}f^{-1} = -\gamma \cdot (B^{-1} - A^{-1}) \cdot \gamma.$$

Therefore

$$\begin{aligned}\int_0^1 L_{-1}(\dot{\gamma}(t); \gamma(t)dt &= \int_0^1 \left|\left|\left| -\gamma^{-1}\gamma \cdot (B^{-1} - A^{-1}) \cdot \gamma\gamma^{-1} \right|\right|\right| dt \\ &= \int_0^1 \left|\left|\left| B^{-1} - A^{-1} \right|\right|\right| dt = \left|\left|\left| B^{-1} - A^{-1} \right|\right|\right|,\end{aligned}$$

and moreover it attains the shortest:

$$\int_0^1 L_{-1}(\dot{\gamma}; \gamma)dt \geqq \left|\left|\left| \int_0^1 \gamma^{-1}\dot{\gamma}\gamma^{-1}dt \right|\right|\right| = ||| - [\gamma^{-1}]_0^1 ||| = |||B^{-1} - A^{-1}|||.$$

Remark 4. Like Theorem 5, if the norm is strictly convex, then the geodesic is a unique path attaining the shortest. Suppose the norm is the operator one or the trace one, a path B^t also attains the shortest length for $B \leq I$.

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