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Further extension of an order preserving operator inequality and its application

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An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

The celebrated Löwner-Heinz inequality asserts that *if $A \geq B \geq 0$, then $A^{\alpha} \geq B^{\alpha}$ holds for any $\alpha \in [0, 1]$, but $A^p \geq B^p$ does not always hold for $p > 1$.* From this point of view, we shall show <u>*an order preserving operator inequality*</u> in Theorem 2.3 and we shall state an implication among Theorem 2.3 and the previous known results.

Theorem 2.3. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, ..., p_{2n} \geq 1$ for natural number n. Then the following inequality holds for $r \geq t$:*

$$A^{1-t+r} \geq$$

$$\left\{A^{\frac{r}{2}}\Big[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\cdots\cdots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}\ B^{p_1}\ \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\cdots\cdots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}\Big]^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{q[2n]+r-t}}$$

where $$q[2n] = \underbrace{\Big\{\cdots\cdots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\cdots\cdots-t\Big\}}_{\text{-t and t alternately } n \text{ times appear}}p_{2n}+t.$$

Remark 1. Consider the following operator function $f(\Box)$ defined by

$$f(\Box) = \left\{A^{\frac{r}{2}}\Big[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\cdots\cdots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}\ \Box\ \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\cdots\cdots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}\Big]^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{q[2n]+r-t}}.$$

Theorem 2.3 asserts that $f(A^{p_1}) \geq f(B^{p_1})$ holds for any $p_1 \geq 1$ whenever $A \geq B \geq 0$ with $A > 0$ under some suitable conditions, that is, $f(\Box)$ can be considered as order preserving operator function. Therorem 2.3 is further extension of the following previous one: *if $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$, $A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for $r \geq t$ and $s \geq 1$, in particular, $A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$.* Finally an application of Theorem 2.3 is shown.

§1 Introduction

An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by (LH) briefly).

$$\text{If } A \geq B \geq 0 \text{ holds, then } A^{\alpha} \geq B^{\alpha} \text{ for any } \alpha \in [0,1]. \tag{LH}$$

Although (LH) asserts that $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0,1]$, unfortunately $A^p \geq B^p$ *does not always hold for* $p > 1$. The following result has been obtained from this point of view.

Theorem F.
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

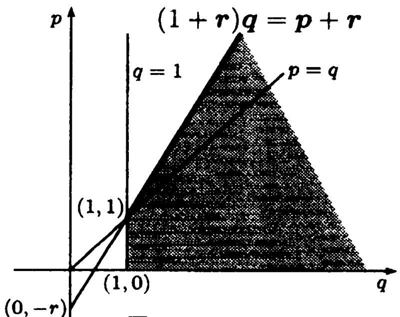

FIGURE 1

The original proof of Theorem F is shown in [F1], an elementary one-page proof is in [F2] and alternative ones are in [MF],[K1]. It is shown in [T1] that the conditions p, q and r in FIGURE 1 are best possible.

Theorem G. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $r \geq t$ *and* $s \geq 1$.

The original proof of Theorem G is in [F3], and an alternative one is in [MF-K]. An elementary one-page proof of (1.1) is in [F4]. We mention that further extensions of Theorem G and related results to Theorem F are in [MF-K-N],[F5],[F-H-I],[F-Y-Y], [K2] and etc. It is shown in [T2] that the exponent value $\dfrac{1-t+r}{(p-t)s+r}$ of the right hand of (1.1) is best possible and alternative ones are in [MF-M-N],[Y]. It is known that the operator inequality (1.1) interpolates Theorem F and an inequality (see Theorem AH under below at

4 page) equivalent to the main result of Ando-Hiai log majorization [A-H] by the parameter $t \in [0,1]$.

§2 **Further extension of Theorem G**

Theorem 2.1 [F7]. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geq 1$. Then the following inequality holds,*

$$A \geq \Bigg[\underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}[A^{\frac{t}{2}}..........[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}....A^{\frac{t}{2}}]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \Bigg]^{\frac{1}{q[2n]}} \tag{2.1}$$

where $q[2n] = q[2n; p_1, p_2, .., p_n, .., p_{2(n-1)}, p_{2n-1}, p_{2n}]$

$$= \underbrace{\Big\{.....[\{[(p_1 - t)p_2 + t]p_3 - t\}p_4 + t]p_5 - ...... - t\Big\}}_{\text{-t and t alternately } n \text{ times appear}} p_{2n} + t. \tag{2.2}$$

Corollary 2.2 [F6]. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$, then the following inequality holds,*

$$A \geq \Big\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\Big\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4+t}}.$$

Theorem 2.3 [F7]. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geq 1$ for natural number n. Then the following inequality holds for $r \geq t$,*

$$A^{1-t+r} \geq$$

$$\Big\{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}..................[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}.............A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}]^{p_{2n}}A^{\frac{r}{2}}\Big\}^{\frac{1-t+r}{q[2n]+r-t}} \tag{2.3}$$

where $q[2n]$ is defined in (2.2).

Corollary 2.4 [F6]. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$,*

$$A^{1-t+r} \geq \Big\{A^{\frac{r}{2}}\Big[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\Big]^{p_4}A^{\frac{r}{2}}\Big\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geq t$.

Corollary 2.4 obtained by Theorem 2.3 and previous known results

Corollary 2.4. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$,*

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geq t$.

$p_2 = p_3 = 1$ ↓

Theorem G. *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

$t = 0$ and $s = 1$ ↙ $t = 1$ and $r = s$ ↘

Theorem F. $A \geq B \geq 0 \Longrightarrow$

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

for $p \geq 1$ and $r \geq 0$.

Theorem AH. *If $A \geq B \geq 0$ with $A > 0 \Longrightarrow$*

$$A^r \geq \{ A^{\frac{r}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}} \}^{\frac{1}{p}}$$

for $r, p \geq 1$.

⇕

Theorem AH-L. *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(A \#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r \qquad \text{for } r \geq 1.$$

⇕

Theorem F.
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

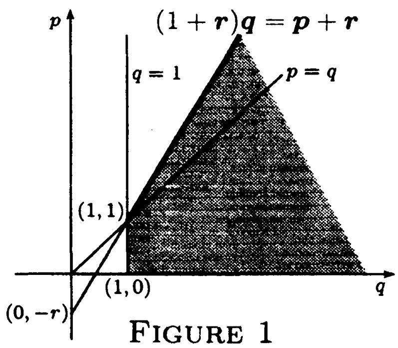

FIGURE 1

§3 Further extension of an operator function implying Theorem G

Theorem H. *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$F_{A,B}(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geq t$ and $s \geq 1$, and $F_{A,A}(r,s) \geq F_{A,B}(r,s)$ holds, that is,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$.

$F_{A,B}(r,s)$ in Theorem H is an operator function implying (1.1) in Theorem G. The original proof of Theorem H is in [F3], and an alternative one is in [MF-K]. Further extensions of Theorem H and related results are in [MF-K-N],[F5],[F-H-I],[F-Y-Y], [K2] and etc.

We shall state further extension of Theorem H as an application of Theorem 2.3.

Theorem 3.1 [F8]. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n} \geq 1$ for natural number n. Then*

$$G_{A,B}[r, p_{2n}] \tag{3.1}$$

$$= A^{\frac{-r}{2}}\left\{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\ldots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}]^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{1+r-t}{q[2n]+r-t}}A^{\frac{-r}{2}}$$

is a decreasing function of $p_{2n} \geq 1$ and $r \geq t$, and the following inequality holds

$$G_{A,A}[r, p_{2n}] \geq G_{A,B}[r, p_{2n}],$$

that is,

$$A^{1-t+r} \geq$$

$$\left\{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\ldots\}A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}]^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{q[2n]+r-t}} \tag{3.2}$$

where $q[2n]$ is defined by (2.2).

Corollary 3.2 [F8]. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$,*

$$G_{A,B}[r,p_4] = A^{\frac{-r}{2}}\left\{A^{\frac{r}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}A^{\frac{-r}{2}}$$

is a decreasing function of $p_4 \geq 1$ and $r \geq t$, and the following inequality holds $G_{A,A}[r,p_4] \geq G_{A,B}[r,p_4]$, that is,

$$A^{1-t+r} \geq \left\{A^{\frac{r}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $t \in [0,1]$, $r \geq t$ and $p_1, p_2, p_3, p_4 \geq 1$.

Remark 3.1. Theorem 3.1 yields Corollary 3.2 by putting $n = 2$ and also Corollary 3.2 yields Theorem H by putting $p_2 = p_3 = 1$.

§4 Satellite inequalities as an application of Theorem 3.1

Theorem 4.1 [F8]. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geq 1$, then the following inequality holds*:

$$A \geq B$$
$$\geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2+t}}$$
$$\geq \left\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4+t}}$$

.....

$\geq$

......

$$\geq \Bigg[\underbrace{A^{\frac{t}{2}}\Big\{A^{\frac{-t}{2}}\Big[A^{\frac{t}{2}}..........\Big[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\Big]^{p_4}....A^{\frac{t}{2}}\Big]^{p_{2n-1}}A^{\frac{-t}{2}}\Big\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}}\Bigg]^{\frac{1}{q[2n]}} \quad (4.1)$$

where $q[2n]$ is defined in (2.2).

Remark 4.1. Corollary 2 in § 3.2.5 of [F5] states that *if $A \geq B > 0$, then*

$A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$ *holds for each $t \in [0,1]$ and $p, s \geq 1$* (4.2)

and Theorem 4.1 is further extension of (4.2).

References

[A-H] T.Ando and F.Hiai, *Log majorization and complementary Golden-Thompson type inequalities,* Linear Alg. and Its Appl.,**197, 198** (1994),113-131.

[MF] M. Fujii, Furuta's inequality and its mean theoretic approach, J. Operator Theory, **23**(1990), 67-72.

[MF-K] M. Fujii and E.Kamei, Mean theoretic approach to the grand Furuta inequality, Proc. Amer. Math. Soc., **124**(1996), 2751-2756.

[MF-K-N] M. Fujii, E.Kamei and R.Nakamoto, Grand Furuta inequality and its variant, J. Math. Inequal., **1**(2007), 437-441.

[MF-M-N] M. Fujii, A.Matsumoto and R.Nakamoto, A short proof of the best possibility for the grand Furuta inequality, J. of Inequal. and Appl., **4**(1999), 339-344.

[F1] T.Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc.,**101** (1987),85-88.

[F2] T.Furuta, Elementary proof of an order preserving inequality, Proc. Japan Acad.,**65**(1989),126.

[F3] T.Furuta, An extension of the Furuta inequality and Ando-Hiai log majorization, Linear Alg. and Its Appl.,**219** (1995), 139-155.

[F4] T.Furuta, Simplified proof of an order preserving operator inequality, Proc, Japan Acad., **74**(1998), 114.

[F5] T.Furuta, Invitation to Linear Operators, Taylor & and Francis, **2001**, London.

[F6] T.Furuta, An order preserving operator inequality, to appear in Math. Inequal. Appl.

[F7] T.Furuta, Further extension of an order preserving operator inequality, J. Math. Inequal., **2** (2008), 465-472.

[F8] T.Furuta, Operator function associated with an order preserving operator inequality, preprint.

[F-H-I] T.Furuta, M.Hashimoto and M.Ito, Equivalence relation between generalized Furuta inequality and related operator functions, Scientiae Mathematicae, **1**(1998), 257-259.

[F-Y-Y] T.Furuta, M.Yanagida and T.Yamazaki, Operator functions implying Furuta inequality, Math. Inequal. Appl., **1**(1998), 123-130.

[K1] E.Kamei, A sattelite to Furuta's inequality, Math. Japon., **33**(1988), 883-886.

[K2] E.Kamei, Parametrized grand Furuta inequality, Math. Japon., **50**(1999), 79-83.

[T1] K.Tanahashi, Best possibility of the Furuta inequality, Proc. Amer. Math. Soc., **124**(1996),141-146.

[T2] K.Tanahashi, The best possibility of the grand Furuta inequality,Proc. Amer. Math. Soc., **128**(2000),511-519.

[Y] T.Yamazaki, Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality, Math. Inequal. Appl., **2**(1999),, 473-477.

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