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Equivalent characterizations of partial randomness for a recursively enumerable real

中央大学・研究開発機構, JST・CREST 只木孝太郎 (Kohtaro Tadaki)*
Research and Development Initiative, Chuo University CREST, JST

Abstract. A real number α is called recursively enumerable if there exists a computable, increasing sequence of rational numbers which converges to α. The randomness of a recursively enumerable real α can be characterized in various ways using each of the notions; program-size complexity, Martin-Löf test, Chaitin's Ω number, the domination and Ω-likeness of α, the universality of a computable, increasing sequence of rational numbers which converges to α, and universal probability. In this paper, we generalize these characterizations of randomness over the notion of partial randomness by parameterizing each of the notions above by a real number $T \in (0, 1]$. We thus present several equivalent characterizations of partial randomness for a recursively enumerable real number.

Key words: algorithmic randomness, recursively enumerable real number, partial randomness, Chaitin's Ω number, program-size complexity, universal probability

1 Introduction

A real number α is called *recursively enumerable* ("r.e." for short) if there exists a computable, increasing sequence of rational numbers which converges to α. The randomness of an r.e. real α can be characterized in various ways using each of the notions; *program-size complexity*, *Martin-Löf test*, *Chaitin's* Ω *number*, the *domination* and Ω*-likeness* of α, the *universality* of a computable, increasing sequence of rational numbers which converges to α, and *universal probability*. These equivalent characterizations of randomness for an r.e. real number are summarized in Theorem 3.4 (see Section 3), where the equivalences are established by a series of works of Schnorr [13], Chaitin [4], Solovay [14], Calude, Hertling, Khoussainov and Wang [1], Kučera and Slaman [8], and Tadaki [17]. In this paper, we generalize these characterizations of randomness over the notion of *partial randomness*, which was introduced by Tadaki [15, 16]. We introduce several characterizations of partial randomness for an r.e. real number by parameterizing each of the notions above on randomness by a real number $T \in (0, 1]$. We prove the equivalence of all these characterizations of partial randomness in Theorem 4.6, our main result, in Section 4.

The paper is organized as follows. We begin in Section 2 with some preliminaries to algorithmic information theory and partial randomness. In Section 3, we review the previous results on the equivalent characterizations of randomness for an r.e. real number. Our main result on

*E-mail: tadaki@kc.chuo-u.ac.jp

partial randomness of an r.e. real number is presented in Section 4, and its proof is completed in Section 5. In Section 6, we investigate some properties of the notion of *T-convergence* for an increasing sequence of real numbers, which plays a crucial role in our characterizations of partial randomness. We conclude this paper with a mention of the future direction of this work in Section 7.

2 Preliminaries

2.1 Basic notation

We start with some notation about numbers and strings which will be used in this paper. $\#S$ is the cardinality of S for any set S. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{R}$ is the set of real numbers. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of numbers (rational numbers or real numbers) is called *increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

$\{0,1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \dots\}$ is the set of finite binary strings where λ denotes the *empty string*, and $\{0,1\}^*$ is ordered as indicated. We identify any string in $\{0,1\}^*$ with a natural number in this order, i.e., we consider $\varphi\colon \{0,1\}^* \to \mathbb{N}$ such that $\varphi(s) = 1s - 1$ where the concatenation $1s$ of strings 1 and s is regarded as a dyadic integer, and then we identify s with $\varphi(s)$. For any $s \in \{0,1\}^*$, $|s|$ is the *length* of s. A subset S of $\{0,1\}^*$ is called a *prefix-free set* if no string in S is a prefix of another string in S. For any partial function f, the domain of definition of f is denoted by $\operatorname{dom} f$. We write "r.e." instead of "recursively enumerable."

Normally, $o(n)$ denotes any function $f\colon \mathbb{N}^+ \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)/n = 0$. On the other hand, $O(1)$ denotes any function $g\colon \mathbb{N}^+ \to \mathbb{R}$ such that there is $C \in \mathbb{R}$ with the property that $|g(n)| \le C$ for all $n \in \mathbb{N}^+$.

Let α be an arbitrary real number. We denote $\alpha - \lfloor\alpha\rfloor$ by $\alpha \bmod 1$, where $\lfloor\alpha\rfloor$ is the greatest integer less than or equal to α. Hence, $\alpha \bmod 1 \in [0, 1)$. Normally, $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α. We denote by $\alpha_n \in \{0,1\}^*$ the first n bits of the base-two expansion of $\alpha \bmod 1$ with infinitely many zeros. Thus, in particular, if $\alpha \in [0, 1)$, then α_n denotes the first n bits of the base-two expansion of α with infinitely many zeros. For example, in the case of $\alpha = 5/8$, $\alpha_6 = 101000$.

A real number α is called *r.e.* if there exists a computable, increasing sequence of rational numbers which converges to α. An r.e. real number is also called a *left-computable* real number. On the other hand, a real number α is called *right-computable* if $-\alpha$ is left-computable. We say that a real number α is *computable* if there exists a computable sequence $\{a_n\}_{n\in\mathbb{N}}$ of rational numbers such that $|\alpha - a_n| < 2^{-n}$ for all $n \in \mathbb{N}$. It is then easy to see that, for every $\alpha \in \mathbb{R}$, α is computable if and only if α is both left-computable and right-computable. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is called *computable* if there exists a total recursive function $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{Q}$ such that $|a_n - f(n,m)| < 2^{-m}$ for all $n, m \in \mathbb{N}$. See e.g. Pour-El and Richards [11] and Weihrauch [20] for the detail of the treatment of the computability of real numbers and sequences of real numbers.

2.2 Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [4, 5]. A *computer* is a partial recursive function $C\colon \{0,1\}^* \to \{0,1\}^*$ such that $\operatorname{dom} C$ is a prefix-free set. For each computer C and each $s \in \{0,1\}^*$, $H_C(s)$ is defined by $H_C(s) = \min\{\, |p| \mid p \in \{0,1\}^* \;\&\; C(p) = s \,\}$. A computer U is said to be *optimal* if for each computer C there exists a constant $\mathrm{sim}(C)$ with the following property; if $C(p)$ is defined, then there is a p' for which $U(p') = C(p)$ and $|p'| \le |p| + \mathrm{sim}(C)$. It is easy to see that there exists an optimal computer. We choose a particular optimal computer U as the standard one for use, and define $H(s)$ as $H_U(s)$, which is referred to as the *program-size complexity* of s, the *information content* of s, or the *Kolmogorov complexity* of s [7, 9, 4]. Thus, $H(s) \le H_C(s) + \mathrm{sim}(C)$ for every computer C.

Let V be an arbitrary optimal computer. For each $s \in \{0,1\}^*$, $P_V(s)$ is defined as $\sum_{V(p)=s} 2^{-|p|}$. *Chaitin's halting probability* Ω_V of V is defined by

$$\Omega_V = \sum_{p \in \mathrm{dom}\, V} 2^{-|p|}.$$

Thus, $\Omega_V = \sum_{s \in \{0,1\}^*} P_V(s)$.

Definition 2.1 (weak Chaitin randomness, Chaitin [4, 5]). *For any $\alpha \in \mathbb{R}$, we say that α is weakly Chaitin random if there exists $c \in \mathbb{N}$ such that $n - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$.* □

Chaitin [4] showed that, for every optimal computer V, Ω_V is weakly Chaitin random.

Definition 2.2 (Martin-Löf randomness, Martin-Löf [10]). *A subset C of $\mathbb{N}^+ \times \{0,1\}^*$ is called a Martin-Löf test if C is an r.e. set and*

$$\forall n \in \mathbb{N}^+ \quad \sum_{s \in C_n} 2^{-|s|} \le 2^{-n},$$

where $C_n = \{\, s \mid (n, s) \in C \,\}$. For any $\alpha \in \mathbb{R}$, we say that α is Martin-Löf random if for every Martin-Löf test C, there exists $n \in \mathbb{N}^+$ such that, for every $k \in \mathbb{N}^+$, $\alpha_k \notin C_n$. □

Theorem 2.3 (Schnorr [13]). *For every $\alpha \in \mathbb{R}$, α is weakly Chaitin random if and only if α is Martin-Löf random.* □

It follows from Theorem 2.3 that Ω_V is Martin-Löf random for every optimal computer V.

The program-size complexity $H(s)$ is originally defined using the concept of program-size, as stated above. However, it is possible to define $H(s)$ without referring to such a concept, i.e., as in the following, we first introduce a *universal probability* m, and then define $H(s)$ as $-\log_2 m(s)$. A universal probability is defined as follows [21].

Definition 2.4 (universal probability). *A function $r\colon \{0,1\}^* \to [0,1]$ is called a lower-computable semi-measure if $\sum_{s \in \{0,1\}^*} r(s) \le 1$ and the set $\{(a,s) \in \mathbb{Q} \times \{0,1\}^* \mid a < r(s)\}$ is r.e. We say that a lower-computable semi-measure m is a universal probability if for every lower-computable semi-measure r, there exists $c \in \mathbb{N}^+$ such that, for all $s \in \{0,1\}^*$, $r(s) \le cm(s)$.* □

The following theorem can be then shown (see e.g. Theorem 3.4 of Chaitin [4] for its proof).

Theorem 2.5. *For every optimal computer V, both $2^{-H_V(s)}$ and $P_V(s)$ are universal probabilities.* □

By Theorem 2.5, we see that $H(s) = -\log_2 m(s) + O(1)$ for every universal probability m. Thus it is possible to define $H(s)$ as $-\log_2 m(s)$ with a particular universal probability m instead of as $H_U(s)$. Note that the difference up to an additive constant is nonessential to algorithmic information theory. Any universal probability is not computable, as corresponds to the uncomputability of $H(s)$. As a result, we see that $0 < \sum_{s \in \{0,1\}^*} m(s) < 1$ for every universal probability m.

2.3 Partial randomness

In the works [15, 16], we generalized the notion of the randomness of a real number so that *the degree of the randomness*, which is often referred to as *the partial randomness* recently [2, 12, 3], can be characterized by a real number T with $0 < T \leq 1$ as follows.

Definition 2.6 (weak Chaitin T-randomness). *Let $T \in \mathbb{R}$ with $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that α is weakly Chaitin T-random if there exists $c \in \mathbb{N}$ such that $Tn - c \leq H(\alpha_n)$ for all $n \in \mathbb{N}^+$.* □

Definition 2.7 (Martin-Löf T-randomness). *Let $T \in \mathbb{R}$ with $T \geq 0$. A subset C of $\mathbb{N}^+ \times \{0,1\}^*$ is called a Martin-Löf T-test if C is an r.e. set and*

$$\forall n \in \mathbb{N}^+ \quad \sum_{s \in C_n} 2^{-T|s|} \leq 2^{-n}.$$

For any $\alpha \in \mathbb{R}$, we say that α is Martin-Löf T-random if for every Martin-Löf T-test C, there exists $n \in \mathbb{N}^+$ such that, for every $k \in \mathbb{N}^+$, $\alpha_k \notin C_n$. □

In the case where $T = 1$, the weak Chaitin T-randomness and Martin-Löf T-randomness result in weak Chaitin randomness and Martin-Löf randomness, respectively. Tadaki [16] generalized Theorem 2.3 over the notion of T-randomness as follows.

Theorem 2.8 (Tadaki [16]). *Let T be a computable real number with $T \geq 0$. Then, for every $\alpha \in \mathbb{R}$, α is weakly Chaitin T-random if and only if α is Martin-Löf T-random.* □

Definition 2.9 (T-compressibility). *Let $T \in \mathbb{R}$ with $T \geq 0$. For any $\alpha \in \mathbb{R}$, we say that α is T-compressible if $H(\alpha_n) \leq Tn + o(n)$, which is equivalent to*

$$\varlimsup_{n \to \infty} \frac{H(\alpha_n)}{n} \leq T.$$

□

For every $T \in [0, 1]$ and every $\alpha \in \mathbb{R}$, if α is weakly Chaitin T-random and T-compressible, then

$$\lim_{n \to \infty} \frac{H(\alpha_n)}{n} = T, \tag{1}$$

and therefore the *compression rate* of α by the program-size complexity H is equal to T. Note, however, that (1) does not necessarily imply that α is weakly Chaitin T-random.

In the works [15, 16], we generalized Chaitin's halting probability Ω to $\Omega(T)$ as follows. For each optimal computer V and each real number $T > 0$, the *generalized halting probability* $\Omega_V(T)$ of V is defined by

$$\Omega_V(T) = \sum_{p \in \operatorname{dom} V} 2^{-\frac{|p|}{T}}.$$

Thus, $\Omega_V(1) = \Omega_V$. If $0 < T \leq 1$, then $\Omega_V(T)$ converges and $0 < \Omega_V(T) < 1$, since $\Omega_V(T) \leq \Omega_V < 1$. The following theorem holds for $\Omega_V(T)$.

Theorem 2.10 (Tadaki [15, 16]). *Let V be an optimal computer and let $T \in \mathbb{R}$.*

(i) If $0 < T \leq 1$ and T is computable, then $\Omega_V(T)$ is weakly Chaitin T-random and T-compressible.

(ii) If $1 < T$, then $\Omega_V(T)$ diverges to ∞. □

Note also that the computability of $\Omega_V(T)$ gives a sufficient condition for a real number $T \in (0,1)$ to be a *fixed point on partial randomness* as follows.

Theorem 2.11 (Tadaki [18]). *Let V be an optimal computer. For every $T \in (0,1)$, if $\Omega_V(T)$ is computable, then T is weakly Chaitin T-random and T-compressible, and therefore*

$$\lim_{n\to\infty} \frac{H(T_n)}{n} = T.$$

□

3 Previous results on the randomness of an r.e. real

In this section, we review the previous results on the randomness of an r.e. real number. First we review some notions on r.e. real numbers.

Definition 3.1 (Ω-likeness). *For any r.e. real numbers α and β, we say that α dominates β if there are computable, increasing sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers and $c \in \mathbb{N}^+$ such that $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \beta$, and $c(\alpha - a_n) \geq \beta - b_n$ for all $n \in \mathbb{N}$. An r.e. real number α is called Ω-like if it dominates all r.e. real numbers.* □

Solovay [14] showed the following theorem. For its proof, see also Theorem 4.9 of [1].

Theorem 3.2 (Solovay [14]). *For every r.e. real numbers α and β, if α dominates β then $H(\beta_n) \leq H(\alpha_n) + O(1)$.* □

Definition 3.3 (universality). *A computable, increasing and converging sequence $\{a_n\}$ of rational numbers is called universal if for every computable, increasing and converging sequence $\{b_n\}$ of rational numbers there exists $c \in \mathbb{N}^+$ such that $c(\alpha - a_n) \geq \beta - b_n$ for all $n \in \mathbb{N}$, where $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty} b_n$.* □

The previous results on the equivalent characterizations of randomness for an r.e. real number are summarized in the following theorem.

Theorem 3.4 ([13, 4, 14, 1, 8, 17]). *Let α be an r.e. real number with $0 < \alpha < 1$. Then the following conditions are equivalent:*

(i) The real number α is weakly Chaitin random.

(ii) The real number α is Martin-Löf random.

(iii) The real number α is Ω-like.

(iv) $H(\beta_n) \leq H(\alpha_n) + O(1)$ for every r.e. real number β.

(v) There exists an optimal computer V such that $\alpha = \Omega_V$.

(vi) There exists a universal probability m such that $\alpha = \sum_{s\in\{0,1\}^*} m(s)$.

(vii) Every computable, increasing sequence of rational numbers which converges to α is universal.

(viii) There exists a universal computable, increasing sequence of rational numbers which converges to α. □

The historical remark on the proofs of equivalences in Theorem 3.4 is as follows. Schnorr [13] showed that (i) and (ii) are equivalent to each other. Chaitin [4] showed that (v) implies (i). Solovay [14] showed that (v) implies (iii), (iii) implies (iv), and (iii) implies (i). Calude, Hertling, Khoussainov, and Wang [1] showed that (iii) implies (v), and (v) implies (vii). Kučera and Slaman [8] showed that (ii) implies (vii). Finally, (vi) was inserted in the course of the derivation from (v) to (viii) by Tadaki [17].

4 New results on the partial randomness of an r.e. real

In this section, we generalize Theorem 3.4 above over the notion of partial randomness. For that purpose, we first introduce some new notions. Let T be an arbitrary real number with $0 < T \leq 1$ throughout the rest of this paper. These notions are parametrized by the real number T.

Definition 4.1 (T-convergence). *An increasing sequence $\{a_n\}$ of real numbers is called T-convergent if $\sum_{n=0}^{\infty}(a_{n+1} - a_n)^T < \infty$. An r.e. real number α is called T-convergent if there exists a T-convergent computable, increasing sequence of rational numbers which converges to α.* □

Note that every increasing and converging sequence of real numbers is 1-convergent, and thus every r.e. real number is 1-convergent. In general, based on the following lemma, we can freely switch from "T-convergent computable, increasing sequence of real numbers" to "T-convergent computable, increasing sequence of rational numbers."

Lemma 4.2. *For every $\alpha \in \mathbb{R}$, α is an r.e. T-convergent real number if and only if there exists a T-convergent computable, increasing sequence of real numbers which converges to α.*

Proof. The "only if" part is obvious. We show the "if" part. Suppose that $\{a_n\}$ is a T-convergent computable, increasing sequence of real numbers which converges to α. Then, we first see that there exists a computable sequence $\{b_n\}$ of rational numbers such that $a_n < b_n < a_{n+1}$ for all $n \in \mathbb{N}$. Obviously, $\{b_n\}$ is an increasing sequence of rational numbers which converges to α. On the other hand, using the inequality $(x + y)^t \leq x^t + y^t$ for real numbers $x, y > 0$ and $t \in (0, 1]$, we see that $(b_{n+1} - b_n)^T < (a_{n+2} - a_n)^T \leq (a_{n+2} - a_{n+1})^T + (a_{n+1} - a_n)^T$. Thus, since $\sum_{n=0}^{\infty}(a_{n+2} - a_{n+1})^T$ and $\sum_{n=0}^{\infty}(a_{n+1} - a_n)^T$ both converge, the increasing sequence $\{b_n\}$ of rational numbers is T-convergent. □

The following argument illustrates the way of using Lemma 4.2: Let V be an optimal computer, and let $p_0, p_1, p_2, \dots$ be a recursive enumeration of the r.e. set $\operatorname{dom} V$. Then $\Omega_V(T) = \sum_{i=0}^{\infty} 2^{-|p_i|/T}$, and the increasing sequence $\{\sum_{i=0}^{n} 2^{-|p_i|/T}\}_{n\in\mathbb{N}}$ of real numbers is T-convergent since $\Omega_V = \sum_{i=0}^{\infty} 2^{-|p_i|} < 1$. If T is computable, then this sequence of real numbers is computable. Thus, by Lemma 4.2 we have Theorem 4.3 below.

Theorem 4.3. *Let V be an optimal computer. If T is computable, then $\Omega_V(T)$ is an r.e. T-convergent real number.* □

Definition 4.4 ($\Omega(T)$-likeness). *An r.e. real number α is called $\Omega(T)$-like if it dominates all r.e. T-convergent real numbers.* □

Note that an r.e. real number α is $\Omega(1)$-like if and only if α is Ω-like.

Definition 4.5 (T-universality). *A computable, increasing and converging sequence $\{a_n\}$ of rational numbers is called T-universal if for every T-convergent computable, increasing and converging sequence $\{b_n\}$ of rational numbers there exists $c \in \mathbb{N}^+$ such that $c(\alpha - a_n) \geq \beta - b_n$ for all $n \in \mathbb{N}$, where $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty} b_n$.* □

Note that a computable, increasing and converging sequence $\{a_n\}$ of rational numbers is 1-universal if and only if $\{a_n\}$ is universal.

Using the notions introduced above, Theorem 3.4 is generalized as follows.

Theorem 4.6 (main result). *Let α be an r.e. real number with $0 < \alpha < 1$. Suppose that T is computable. Then the following conditions are equivalent:*

(i) The real number α is weakly Chaitin T-random.

(ii) The real number α is Martin-Löf T-random.

(iii) The real number α is $\Omega(T)$-like.

(iv) $H(\beta_n) \leq H(\alpha_n) + O(1)$ for every r.e. T-convergent real number β.

(v) For every r.e. T-convergent real number $\gamma > 0$, there exist an r.e. real number $\beta \geq 0$ and a rational number $q > 0$ such that $\alpha = \beta + q\gamma$.

(vi) There exist an optimal computer V and an r.e. real number $\beta \geq 0$ such that $\alpha = \beta + \Omega_V(T)$.

(vii) There exists a universal probability m such that $\alpha = \sum_{s\in\{0,1\}^*} m(s)^{\frac{1}{T}}$.

(viii) Every computable, increasing sequence of rational numbers which converges to α is T-universal.

(ix) There exists a T-universal computable, increasing sequence of rational numbers which converges to α. □

The condition (vi) of Theorem 4.6 corresponds to the condition (v) of Theorem 3.4. Note, however, that, in the condition (vi) of Theorem 4.6, a non-negative r.e. real number β is needed. The reason is as follows: In the case of $\beta = 0$, the possibility that α is weakly Chaitin T'-random with a real number $T' > T$ is excluded by the T-compressibility of $\Omega_V(T)$ imposed by Theorem 2.10 (i). However, this exclusion is inconsistent with the condition (i) of Theorem 4.6.

Theorem 4.6 is proved as follows, partially based on Theorems 5.3, 5.4, 5.5, and 5.6, which will be proved in the next section.

Proof of Theorem 4.6. We prove the equivalences in Theorem 4.6 by showing the two paths [A] and [B] of implications below.

[A] The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i): First, by Theorem 2.8, (i) implies (ii) obviously. It follows from Theorem 5.3 below that (ii) implies (v), and also it follows from Theorem 5.4 below that (v) implies (vi). For the forth implication, let V be an optimal computer, and let β be an r.e. real number. It is then easy to show that $\beta + \Omega_V(T)$ dominates $\Omega_V(T)$ (see the condition 2 of Lemma 4.4 of [1]). It follows from Theorem 3.2 and Theorem 2.10 (i) that the condition (vi) results in the condition (i) of Theorem 4.6.

[B] The implications (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ix) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i): First, it follows from Theorem 5.5 below that (v) implies (vii), and also it follows from Theorem 5.6 below that (vii) implies (viii). Obviously, (viii) implies (ix) and (ix) implies (iii). It follows from Theorem 3.2 that (iii) implies (iv). Finally, note that $\Omega_U(T)$ is an r.e. T-convergent real number which is weakly Chaitin T-random by Theorem 2.10 (i) and Theorem 4.3. Thus, by setting β to $\Omega_U(T)$ in the condition (iv), the condition (iv) results in the condition (i). □

As a consequence of Theorem 4.6, we obtain the following corollary, for example.

Corollary 4.7. *Suppose that T is computable. Then, for every two optimal computers V and W,*

$$H((\Omega_V(T))_n) = H((\Omega_W(T))_n) + O(1).$$

Proof. Corollary 4.7 follows immediately from Theorem 4.3 and the implication (vi) $\Rightarrow$ (iv) of Theorem 4.6. $\square$

5 The completion of the proof of the main result

In this section, we prove several theorems needed to complete the proof of Theorem 4.6. For the sake of convenience, we first rephrase the definition of Martin-Löf T-randomness of a real number as follows. We denote by $\mathcal{I}$ the set $\{(n,q,r) \in \mathbb{N}^+ \times \mathbb{Q} \times \mathbb{Q} \mid q < r\}$. A subset $\mathcal{D}$ of $\mathcal{I}$ is called a *rational Martin-Löf T-test* if $\mathcal{D}$ is an r.e. set and

$$\forall n \in \mathbb{N}^+ \quad \sum_{(q,r)\in\mathcal{D}(n)} (r-q)^T \leq 2^{-n},$$

where $\mathcal{D}(n) = \{\,(q,r) \mid (n,q,r) \in \mathcal{D}\,\}$. We can then show the following lemma, which rephrases the definition of the Martin-Löf T-randomness of a real number to give it more flexibility.

Lemma 5.1. *For every $\alpha \in \mathbb{R}$, α is Martin-Löf T-random if and only if for every rational Martin-Löf T-test $\mathcal{D}$, there exists $n \in \mathbb{N}^+$ such that, for every $q,r \in \mathbb{Q}$, if $(q,r) \in \mathcal{D}(n)$ then $\alpha \notin [q,r]$, where $[q,r] = \{x \in \mathbb{R} \mid q \leq x \leq r\}$.*

Proof. First, we show the "if" part by showing its contraposition. Suppose that α is not Martin-Löf T-random. Then there exists a Martin-Löf T-test $\mathcal{C}$ such that

$$\forall n \in \mathbb{N}^+ \;\; \exists k \in \mathbb{N}^+ \;\; \alpha_k \in \mathcal{C}_n. \tag{2}$$

We define a set $\mathcal{D} \subset \mathcal{I}$ by

$$\mathcal{D} = \{(n, 0.s + \lfloor\alpha\rfloor, 0.s + 2^{-|s|} + \lfloor\alpha\rfloor) \mid s \in \mathcal{C}_n\}.$$

Since $\mathcal{C}$ is an r.e. set, $\mathcal{D}$ is also an r.e. set. We also see that, for each $n \in \mathbb{N}^+$,

$$\sum_{(q,r)\in\mathcal{D}(n)} (r-q)^T = \sum_{s\in\mathcal{C}_n} 2^{-T|s|} \leq 2^{-n}.$$

Thus, $\mathcal{D}$ is a rational Martin-Löf T-test. On the other hand, note that $\beta \in [0.\beta_k + \lfloor\beta\rfloor, 0.\beta_k + 2^{-|\beta_k|} + \lfloor\beta\rfloor]$ for every $\beta \in \mathbb{R}$ and every $k \in \mathbb{N}^+$. It follows from (2) that, for every $n \in \mathbb{N}^+$, there exist $q,r \in \mathbb{Q}$ such that $(q,r) \in \mathcal{D}(n)$ and $\alpha \in [q,r]$. This completes the proof of the "if" part.

Next, we show the "only if" part by showing its contraposition. Suppose that there exists a rational Martin-Löf T-test $\mathcal{D}$ such that

$$\forall n \in \mathbb{N}^+ \;\; \exists q,r \in \mathbb{Q} \;\; [(q,r) \in \mathcal{D}(n) \;\&\; \alpha \in [q,r]]. \tag{3}$$

In the case of $\alpha \in \mathbb{Q}$, α is not Martin-Löf T-random, obviously. This can be shown as follows. We choose any one $m \in \mathbb{N}^+$ with $Tm \geq 1$. We then define a set $\mathcal{C} \subset \mathbb{N}^+ \times \{0,1\}^*$ by $\mathcal{C} = \{(n, \alpha_{mn}) \mid n \in \mathbb{N}^+\}$. Recall here that $\alpha_{mn} \in \{0,1\}^*$ denotes the first mn bits of the base-two expansion of α mod 1 with infinitely many zeros. Obviously, $\mathcal{C}$ is an r.e. set. We also see that, for each $n \in \mathbb{N}^+$,

$$\sum_{s\in\mathcal{C}_n} 2^{-T|s|} = 2^{-Tmn} \leq 2^{-n}.$$

Therefore, $\mathcal{C}$ is a Martin-Löf T-test. On the other hand, $\alpha_{mn} \in \mathcal{C}_n$ for every $n \in \mathbb{N}^+$. Hence, α is not Martin-Löf T-random, as desired.

Thus, in what follows, we assume that $\alpha \notin \mathbb{Q}$. We choose any one $n_0 \in \mathbb{N}$ such that

$$2^{-\frac{n_0}{T}} < \min\{\alpha - \lfloor\alpha\rfloor, \lfloor\alpha\rfloor + 1 - \alpha\}.$$

We then define a set $\mathcal{D}^{(0)} \subset \mathcal{I}$ by

$$\mathcal{D}^{(0)} = \{(n, q - \lfloor\alpha\rfloor, r - \lfloor\alpha\rfloor) \mid n \in \mathbb{N}^+ \ \&\ (n + n_0, q, r) \in \mathcal{D} \ \&\ \lfloor\alpha\rfloor < q, r < \lfloor\alpha\rfloor + 1\}.$$

Obviously, $\mathcal{D}^{(0)}$ is an r.e. set. We also see that

$$\sum_{(q,r) \in \mathcal{D}^{(0)}(n)} (r-q)^T \le \sum_{(q,r) \in \mathcal{D}(n+n_0)} (r-q)^T \le 2^{-(n+n_0)} \le 2^{-n}$$

for each $n \in \mathbb{N}^+$. Thus, $\mathcal{D}^{(0)}$ is a rational Martin-Löf T-test, and also $\mathcal{D}^{(0)} \subset \mathbb{N}^+ \times (0,1) \times (0,1)$. On the other hand, by the choice of n_0, it is easy to see that, for every $(n,q,r) \in \mathcal{I}$, if $(q,r) \in \mathcal{D}(n+n_0)$ and $\alpha \in [q,r]$, then $r - q \le 2^{-n_0/T}$ and therefore $\lfloor\alpha\rfloor < q, r < \lfloor\alpha\rfloor + 1$. It follows from (3) that

$$\forall n \in \mathbb{N}^+ \ \exists q, r \in \mathbb{Q} \ \ [(q,r) \in \mathcal{D}^{(0)}(n) \ \&\ \alpha \bmod 1 \in [q,r]]. \tag{4}$$

For each $q, r \in \mathbb{Q}$ with $0 < q < r < 1$, let $v(q,r)$ and $w(q,r)$ be finite binary strings such that (i) $v(q,r) = q_k$ and $w(q,r) = r_k$ for some $k \in \mathbb{N}^+$, and (ii) $v(q,r)+1 = w(q,r)$ where $v(q,r)$ and $w(q,r)$ are regarded as a dyadic integer. Such a pair $(v(q,r), w(q,r))$ of finite binary strings exists uniquely since $0 < q < r < 1$. Then, for every $q, r \in \mathbb{Q}$ with $0 < q < r < 1$, it follows that (i) $2^{-|v(q,r)|} = 2^{-|w(q,r)|} \le r - q$, and (ii) for every $\beta \in \mathbb{R}$, if $\beta \bmod 1 \in [q,r]$ then there exists $k \in \mathbb{N}^+$ such that either $\beta_k = v(q,r)$ or $\beta_k = w(q,r)$. We define a set $\mathcal{C} \subset \mathbb{N}^+ \times \{0,1\}^*$ by

$$\mathcal{C} = \bigcup_{(n+1,q,r) \in \mathcal{D}^{(0)}} \{(n, v(q,r)), (n, w(q,r))\}.$$

Note that, given $q, r \in \mathbb{Q}$ with $0 < q < r < 1$, one can compute both $v(q,r)$ and $w(q,r)$. Thus, since $\mathcal{D}^{(0)}$ is an r.e. set, $\mathcal{C}$ is also an r.e. set. We also see that, for each $n \in \mathbb{N}^+$,

$$\sum_{s \in \mathcal{C}_n} 2^{-T|s|} = \sum_{(q,r) \in \mathcal{D}^{(0)}(n+1)} \left\{2^{-T|v(q,r)|} + 2^{-T|w(q,r)|}\right\} \le \sum_{(q,r) \in \mathcal{D}^{(0)}(n+1)} 2(r-q)^T \le 2^{-n}.$$

Thus, $\mathcal{C}$ is a Martin-Löf T-test. On the other hand, it is easy to see that, for every $(n,q,r) \in \mathcal{I}$, if $(q,r) \in \mathcal{D}^{(0)}(n+1)$ and $\alpha \bmod 1 \in [q,r]$, then there exists $k \in \mathbb{N}^+$ such that either $\alpha_k = v(q,r)$ or $\alpha_k = w(q,r)$, and therefore $(n, \alpha_k) \in \mathcal{C}$. It follows from (4) that, for every $n \in \mathbb{N}^+$, there exists $k \in \mathbb{N}^+$ such that $\alpha_k \in \mathcal{C}_n$. Thus, α is not Martin-Löf T-random. This completes the proof of the "only if" part. □

Lemma 5.2 and Theorem 5.7 below can be proved, based on the generalization of the techniques used in the proof of Theorem 2.1 of Kučera and Slaman [8] over partial randomness. We also use Lemma 5.1 to prove Lemma 5.2 below.

Lemma 5.2. *Let α be an r.e. real number, and let $\{d_n\}$ be a computable sequence of positive rational numbers such that $\sum_{n=0}^{\infty} {d_n}^T \le 1$. If α is Martin-Löf T-random, then for every $\varepsilon > 0$ there exist a computable, increasing sequence $\{a_n\}$ of rational numbers and a rational number $q > 0$ such that $a_{n+1} - a_n > q d_n$ for every $n \in \mathbb{N}$, $a_0 > \alpha - \varepsilon$, and $\alpha = \lim_{n \to \infty} a_n$.*

Proof. We choose any one rational number r with $2^{-1/T} \geq r > 0$. Since α is an r.e. real number, there exists a computable, increasing sequences $\{b_n\}$ of rational numbers such that $b_0 > \alpha - \varepsilon$ and $\alpha = \lim_{n\to\infty} b_n$. We construct a rational Martin-Löf T-test $\mathcal{D}$ by enumerating $\mathcal{D}(i)$ for each $i \in \mathbb{N}^+$ as follows. During the enumeration of $\mathcal{D}(i)$ we simultaneously construct a sequence $\{a(i)_n\}_n$ of rational numbers.

Initially, we set $\mathcal{D}(i) := \emptyset$ and then specify $a(i)_0$ by $a(i)_0 := b_0$. In general, whenever $a(i)_n$ is specified as $a(i)_n := b_m$, we update $\mathcal{D}(i)$ by $\mathcal{D}(i) := \mathcal{D}(i) \cup \{(i, a(i)_n, a(i)_n + r^i d_n)\}$, and calculate $b_{m+1}, b_{m+2}, b_{m+3}, \cdots$ one by one. During the calculation, if we find m_1 such that $m_1 > m$ and $b_{m_1} > a(i)_n + r^i d_n$, then we specify $a(i)_{n+1}$ by $a(i)_{n+1} := b_{m_1}$ and we repeat this procedure for $n + 1$.

For the completed $\mathcal{D}$ through the above procedure, we see that, for every $i \in \mathbb{N}^+$,

$$\sum_{(r_1,r_2)\in\mathcal{D}(i)} (r_2 - r_1)^T = \sum_n (r^i d_n)^T \leq 2^{-i} \sum_n {d_n}^T \leq 2^{-i}.$$

Here the second and third sums on n may be finite or infinite. Thus, $\mathcal{D}$ is a rational Martin-Löf T-test. Since α is Martin-Löf T-random, there exists $k \in \mathbb{N}^+$ such that, for every $r_1, r_2 \in \mathbb{Q}$, if $(r_1, r_2) \in \mathcal{D}(k)$ then $\alpha \notin [r_1, r_2]$. It follows from $\alpha = \lim_{n\to\infty} b_n$ that in the above procedure for enumerating $\mathcal{D}(k)$, for every $n \in \mathbb{N}$ we ever find m_1 such that $m_1 > m$ and $b_{m_1} > a(k)_n + r^k d_n$. Therefore, $\mathcal{D}(k)$ is constructed as an infinite set and also $\{a(k)_n\}_n$ is constructed as an infinite sequence of rational numbers. Thus, we have $a(k)_{n+1} > a(k)_n + r^k d_n$ for all $n \in \mathbb{N}$. Since $\{a(k)_n\}_n$ is a subsequence of $\{b_n\}$, it follows that the sequence $\{a(k)_n\}_n$ is increasing, $a(k)_0 > \alpha - \varepsilon$, and $\alpha = \lim_{n\to\infty} a(k)_n$. This completes the proof. □

Theorem 5.3. *Suppose that T is computable. For every r.e. real number $\alpha > 0$, if α is Martin-Löf T-random, then for every r.e. T-convergent real number $\gamma > 0$ there exist an r.e. real number $\beta > 0$ and a rational number $q > 0$ such that $\alpha = \beta + q\gamma$.*

Proof. Suppose that γ is an arbitrary r.e. T-convergent real number with $\gamma > 0$. Then there exists a T-convergent computable, increasing sequence $\{c_n\}$ of rational numbers which converges to γ. Since $\gamma > 0$, without loss of generality we can assume that $c_0 = 0$. We choose any one rational number $\varepsilon > 0$ such that

$$\sum_{n=0}^{\infty} (c_{n+1} - c_n)^T \leq \left(\frac{1}{\varepsilon}\right)^T.$$

Such ε exists since the sequence $\{c_n\}$ is T-convergent. It follows that

$$\sum_{n=0}^{\infty} [\varepsilon(c_{n+1} - c_n)]^T \leq 1.$$

Note that the sequence $\{\varepsilon(c_{n+1} - c_n)\}$ is a computable sequence of positive rational numbers. Thus, since α is r.e. and Martin-Löf T-random by the assumption, it follows from Lemma 5.2 that there exist a computable, increasing sequence $\{a_n\}$ of rational numbers and a rational number $r > 0$ such that $a_{n+1} - a_n > r\varepsilon(c_{n+1} - c_n)$ for every $n \in \mathbb{N}$, $a_0 > 0$, and $\alpha = \lim_{n\to\infty} a_n$. We then define a sequence $\{b_n\}$ of positive real numbers by $b_n = a_{n+1} - a_n - r\varepsilon(c_{n+1} - c_n)$. It follows that $\{b_n\}$ is a computable sequence of rational numbers and $\sum_{n=0}^{\infty} b_n$ converges to $\alpha - a_0 - r\varepsilon(\gamma - c_0)$. Thus we have $\alpha = a_0 + \sum_{n=0}^{\infty} b_n + r\varepsilon\gamma$, where $a_0 + \sum_{n=0}^{\infty} b_n$ is a positive r.e. real number. This completes the proof. □

Theorem 5.4. *Suppose that T is computable. For every real number α, if for every r.e. T-convergent real number $\gamma > 0$ there exist an r.e. real number $\beta \geq 0$ and a rational number $q > 0$*

such that $\alpha = \beta + q\gamma$, *then there exist an optimal computer* V *and an r.e. real number* $\beta \geq 0$ *such that* $\alpha = \beta + \Omega_V(T)$.

Proof. First, for the optimal computer U, it follows from Theorem 4.3 that $\Omega_U(T)$ is an r.e. T-convergent real number. Thus, by the assumption there exist an r.e. real number $\beta \geq 0$ and a rational number $q > 0$ such that $\alpha = \beta + q\Omega_U(T)$. We choose any one $n \in \mathbb{N}$ with $q > 2^{-n/T}$. We then define a partial function $V\colon \{0,1\}^* \to \{0,1\}^*$ by the conditions that (i) $\operatorname{dom} V = \{0^n p \mid p \in \operatorname{dom} U\}$ and (ii) for every $p \in \operatorname{dom} U$, $V(0^n p) = U(p)$. Since $\operatorname{dom} V$ is a prefix-free set, it follows that V is a computer. It is then easy to see that $H_V(s) = H_U(s) + n$ for every $s \in \{0,1\}^*$. Therefore, since U is an optimal computer, V is also an optimal computer. It follows that $\Omega_V(T) = 2^{-n/T}\Omega_U(T)$. Thus we have $\alpha = \beta + (q - 2^{-n/T})\Omega_U(T) + \Omega_V(T)$. On the other hand, since T is computable, $\beta + (q - 2^{-n/T})\Omega_U(T)$ is an r.e. real number. This completes the proof. □

Theorem 5.5. *Suppose that* T *is computable. For every real number* $\alpha \in (0,1)$, *if for every r.e.* T*-convergent real number* $\gamma > 0$ *there exist an r.e. real number* $\beta \geq 0$ *and a rational number* $q > 0$ *such that* $\alpha = \beta + q\gamma$, *then there exists a universal probability* m *such that* $\alpha = \sum_{s\in\{0,1\}^*} m(s)^{\frac{1}{T}}$.

Proof. First, based on the optimal computer U we define a computer $V\colon \{0,1\}^* \to \{0,1\}^*$ by the conditions that (i) $H_V(s) = H(s) + 1$ for every $s \in \{0,1\}^*$ and (ii) for every $s \in \{0,1\}^*$ and every $n \in \mathbb{N}$, if $n > H(s)$ then there exists a unique $p \in \{0,1\}^*$ such that $|p| = n$ and $V(p) = s$. The existence of such a computer V can be easily shown using Theorem 3.2 of [4], based on the fact that the set $\{(n,s) \in \mathbb{N} \times \{0,1\}^* \mid n > H(s)\}$ is r.e. and

$$\sum_{n>H(s)} 2^{-n} = \sum_{s\in\{0,1\}^*} \sum_{n=H(s)+1}^{\infty} 2^{-n} = \sum_{s\in\{0,1\}^*} 2^{-H(s)} < 1,$$

where the first sum is over all $(n,s) \in \mathbb{N} \times \{0,1\}^*$ with $n > H(s)$. It follows that V is optimal and

$$\Omega_V(T) = \sum_{s\in\{0,1\}^*} \sum_{n=H(s)+1}^{\infty} 2^{-n/T} = \frac{1}{2^{1/T}-1} \sum_{s\in\{0,1\}^*} 2^{-H(s)/T}. \tag{5}$$

By Theorem 4.3, we also see that $\Omega_V(T)$ is an r.e. T-convergent real number. Thus, by the assumption, there exist an r.e. real number $\beta \geq 0$ and a rational number $q > 0$ such that $\alpha = \beta + q\Omega_V(T)$. We choose any one rational number $\varepsilon > 0$ such that $\varepsilon \leq 1 - \alpha^T$ and $\varepsilon^{1/T} < q/(2^{1/T} - 1)$. It follows from (5) that

$$\begin{aligned}\alpha = \beta &+ \frac{q}{2^{1/T}-1} 2^{-H(\lambda)/T} + \left(\frac{q}{2^{1/T}-1} - \varepsilon^{1/T}\right) \sum_{s\neq\lambda} 2^{-H(s)/T} \\ &+ \sum_{s\neq\lambda} \left(\varepsilon 2^{-H(s)}\right)^{1/T}.\end{aligned} \tag{6}$$

Let γ be the sum of the first, second, and third terms on the right-hand side of (6). Then, since T is computable, γ is an r.e. real number. We define a function $m\colon \{0,1\}^* \to (0,\infty)$ by $m(s) = \gamma^T$ if $s = \lambda$; $m(s) = \varepsilon 2^{-H(s)}$ otherwise. Since $\gamma^T < \alpha^T \leq 1 - \varepsilon$, we see that $\sum_{s\in\{0,1\}^*} m(s) < \gamma^T + \varepsilon < 1$. Since T is right-computable, γ^T is an r.e. real number. Therefore, since $2^{-H(s)}$ is a lower-computable semi-measure by Theorem 2.5, m is also a lower-computable semi-measure. Thus, since $2^{-H(s)}$ is a universal probability by Theorem 2.5 again and $\gamma^T > 0$, it is easy to see that m is a universal probability. On the other hand, it follows from (6) that $\alpha = \sum_{s\in\{0,1\}^*} m(s)^{\frac{1}{T}}$. This completes the proof. □

Theorem 5.6 below is obtained by generalizing the proofs of Solovay [14] and Theorem 6.4 of Calude, Hertling, Khoussainov, and Wang [1].

Theorem 5.6. *Suppose that T is computable. For every $\alpha \in (0,1)$, if there exists a universal probability m such that $\alpha = \sum_{s\in\{0,1\}^*} m(s)^{\frac{1}{T}}$, then every computable, increasing sequence of rational numbers which converges to α is T-universal.*

Proof. Suppose that $\{a_n\}$ is an arbitrary computable, increasing sequence of rational numbers which converges to $\alpha = \sum_{s\in\{0,1\}^*} m(s)^{\frac{1}{T}}$. Since m is a lower-computable semi-measure and T is left-computable, there exists a total recursive function $f\colon \mathbb{N} \to \mathbb{N}^+$ such that, for every $n \in \mathbb{N}$, $f(n) < f(n+1)$ and

$$\sum_{k=0}^{f(n)-1} m(k)^{\frac{1}{T}} \geq a_n. \tag{7}$$

Recall here that we identify $\{0,1\}^*$ with $\mathbb{N}$. We then define a total recursive function $g\colon \mathbb{N} \to \mathbb{N}$ by $g(k) = \min\{n \in \mathbb{N} \mid k \leq f(n)\}$. It follows that $g(f(n)) = n$ for every $n \in \mathbb{N}$ and $\lim_{k\to\infty} g(k) = \infty$.

Suppose that $\{b_n\}$ is an arbitrary T-convergent computable, increasing and converging sequence of rational numbers. We then choose any one $d \in \mathbb{N}^+$ with $\sum_{n=0}^{\infty}(b_{n+1} - b_n)^T \leq d$. We then define a function $r\colon \mathbb{N} \to [0,\infty)$ by $r(k) = (b_{g(k+1)} - b_{g(k)})^T/d$. Since T is computable, $\{b_n\}$ is T-convergent, and $g(k+1) = g(k), g(k)+1$, we see that r is a lower-computable semi-measure. Thus, since m is a universal probability, there exists $c \in \mathbb{N}^+$ such that, for every $k \in \mathbb{N}$, $cm(k) \geq r(k)$. It follows from (7) that, for each $n \in \mathbb{N}$,

$$(cd)^{\frac{1}{T}}(\alpha - a_n) \geq d^{\frac{1}{T}} \sum_{k=f(n)}^{\infty} (cm(k))^{\frac{1}{T}} \geq \sum_{k=f(n)}^{\infty} (b_{g(k+1)} - b_{g(k)}) = \beta - b_n,$$

where $\beta = \lim_{n\to\infty} b_n$. This completes the proof. □

Note that, using Lemma 5.2, we can directly show that the condition (ii) implies the condition (iii) in Theorem 4.6 without assuming the computability of $T \in (0,1]$, as follows. Theorem 5.7 below holds for an arbitrary real number $T \in (0,1]$.

Theorem 5.7. *For every r.e. real number α, if α is Martin-Löf T-random, then α is $\Omega(T)$-like.*

Proof. Suppose that β is an arbitrary r.e. T-convergent real numbers. Then there is a T-convergent computable, increasing sequence $\{b_n\}$ of rational numbers which converges to β. Since $\{b_n\}$ is T-convergent, without loss of generality we can assume that $\sum_{n=0}^{\infty}(b_{n+1}-b_n)^T \leq 1$. Since α is r.e. and Martin-Löf T-random by the assumption, it follows from Lemma 5.2 that there exist a computable, increasing sequence $\{a_n\}$ of rational numbers and a rational number $q > 0$ such that $a_{n+1} - a_n > q(b_{n+1} - b_n)$ for every $n \in \mathbb{N}$ and $\alpha = \lim_{n\to\infty} a_n$. It is then easy to see that $\alpha - a_n > q(\beta - b_n)$ for every $n \in \mathbb{N}$. Therefore α dominates β. This completes the proof. □

6 Some results on T-convergence

In this section, we investigate some properties of the notion of T-convergence. As one of the applications of Theorem 4.6, the following theorem can be obtained first.

Theorem 6.1. *Suppose that T is computable. For every r.e. real number α, if α is T-convergent, then α is T-compressible.*

Proof. Using (vi) $\Rightarrow$ (iv) of Theorem 4.6, we see that $H(\alpha_n) \le H((\Omega_U(T))_n) + O(1)$ for every r.e. T-convergent real number α. It follows from Theorem 2.10 (i) that α is T-compressible for every r.e. T-convergent real number α. □

In the case of $T < 1$, the converse of Theorem 6.1 does not hold, as seen in the following theorem in a sharper form.

Theorem 6.2. *Suppose that T is computable and $T < 1$. Then there exists an r.e. real number η such that (i) η is weakly Chaitin T-random and T-compressible, and (ii) η is not T-convergent.* □

In order to prove Theorem 6.2, the following lemma is useful.

Lemma 6.3.

(i) If $\{a_n\}$ is a T-convergent increasing sequence of real numbers, then every subsequence of the sequence $\{a_n\}$ is also T-convergent.

(ii) Let α be a T-convergent r.e. real number. If $\{a_n\}$ is a computable, increasing sequence of rational numbers converging to α, then there exists a subsequence $\{a'_n\}$ of the sequence $\{a_n\}$ such that $\{a'_n\}$ is a T-convergent computable, increasing sequence of rational numbers converging to α.

Proof. (i) Let $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) < f(n+1)$ for all $n \in \mathbb{N}$. Then, using repeatedly the inequality $(x+y)^t \le x^t + y^t$ for real numbers $x, y > 0$ and $t \in (0,1]$, we have

$$\left(a_{f(n+1)} - a_{f(n)}\right)^T = \left[\sum_{k=f(n)}^{f(n+1)-1} (a_{k+1} - a_k)\right]^T \le \sum_{k=f(n)}^{f(n+1)-1} (a_{k+1} - a_k)^T.$$

It follows that

$$\sum_{n=0}^{m} \left(a_{f(n+1)} - a_{f(n)}\right)^T \le \sum_{k=f(0)}^{f(m+1)-1} (a_{k+1} - a_k)^T.$$

Since $\{a_n\}$ is T-convergent, we see that the subsequence $\{a_{f(n)}\}$ of $\{a_n\}$ is also T-convergent.

(ii) We choose any one T-convergent computable, increasing sequence $\{b_n\}$ of rational numbers converging to α. It is then easy to show that there exist total recursive functions $g\colon \mathbb{N} \to \mathbb{N}$ and $h\colon \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$, (i) $g(n) < g(n+1)$, (ii) $h(n) < h(n+1)$, and (iii) $b_{g(n)} < a_{h(n)} < b_{g(n+1)}$. It follows from Lemma 6.3 (i) that the subsequence $\{b_{g(n)}\}$ of $\{b_n\}$ is T-convergent. Using the inequality $(x+y)^t \le x^t + y^t$ for real numbers $x, y > 0$ and $t \in (0,1]$, we see that

$$\left(a_{h(n+1)} - a_{h(n)}\right)^T < \left(b_{g(n+2)} - b_{g(n)}\right)^T \le \left(b_{g(n+2)} - b_{g(n+1)}\right)^T + \left(b_{g(n+1)} - b_{g(n)}\right)^T.$$

Thus, we see that the subsequence $\{a_{h(n)}\}$ of $\{a_n\}$ is a T-convergent computable, increasing sequence of rational numbers converging to α. □

The proof Theorem 6.2 is given as follows.

Proof of Theorem 6.2. We choose any one recursive enumeration $p_0, p_1, p_2, \ldots$ of the r.e. set $\operatorname{dom} U$, and define η by

$$\eta = \sum_{i=0}^{\infty} |p_i|\, 2^{-|p_i|/T}.$$

Then, since T is computable and $T < 1$, by Theorem 3 of Tadaki [18] we see that η is an r.e. real number which is weakly Chaitin T-random and T-compressible.[1] Since T is computable, it is easy to show that there exists a computable, increasing sequence $\{a_n\}$ of rational numbers such that

$$\sum_{i=0}^{n-1} |p_i|\, 2^{-|p_i|/T} < a_n < \sum_{i=0}^{n} |p_i|\, 2^{-|p_i|/T} \tag{8}$$

for all $n \in \mathbb{N}^+$. Obviously, $\{a_n\}$ is an increasing sequence of rational numbers converging to η.

To show that η satisfies the condition (ii) of Theorem 6.2, let us assume contrarily that η is T-convergent. Then it follows from Lemma 6.3 (ii) that there exists a total recursive function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) < f(n+1)$ for all $n \in \mathbb{N}$, and $\{a_{f(n)}\}$ is a T-convergent computable, increasing sequence of rational numbers converging to η. On the other hand, since T is computable, it is easy to show that there exists a computable, increasing sequence $\{b_n\}$ of rational numbers such that

$$\sum_{i=0}^{f(n)} 2^{-|p_i|/T} < b_n < \sum_{i=0}^{f(n+1)} 2^{-|p_i|/T} \tag{9}$$

for all $n \in \mathbb{N}$. Obviously, $\{b_n\}$ is an increasing sequence of rational numbers converging to $\Omega_U(T)$. Since U is an optimal computer, using (vi) $\Rightarrow$ (viii) of Theorem 4.6, we see that there exists $c \in \mathbb{N}^+$ such that $c(\Omega_U(T) - b_n) \geq \eta - a_{f(n)}$ for all $n \in \mathbb{N}$. It follows from (8) and (9) that

$$c \sum_{i=f(n)+1}^{\infty} 2^{-|p_i|/T} > \sum_{i=f(n)+1}^{\infty} |p_i|\, 2^{-|p_i|/T}$$

for all $n \in \mathbb{N}^+$. Therefore, we have

$$\sum_{i=f(n)+1}^{\infty} (c - |p_i|) 2^{-|p_i|/T} > 0 \tag{10}$$

for all $n \in \mathbb{N}^+$. On the other hand, it is easy to show that $\lim_{i\to\infty} |p_i| = \infty$. Therefore, since $\lim_{n\to\infty} f(n) = \infty$, there exists $n_0 \in \mathbb{N}^+$ such that, for all $i \in \mathbb{N}$, if $i \geq f(n_0) + 1$ then $|p_i| \geq c$. Thus, by setting n to n_0 in (10), we have a contradiction. This completes the proof. $\square$

Let T_1 and T_2 be arbitrary computable real numbers with $0 < T_1 < T_2 < 1$, and let V be an arbitrary optimal computer. By Theorem 2.10 (i) and Theorem 6.1, we see that the r.e. real number $\Omega_V(T_2)$ is not T_1-convergent and therefore every computable, increasing sequence $\{a_n\}$ of rational numbers which converges to $\Omega_V(T_2)$ is not T_1-convergent. Thus, conversely, the following question naturally arises: Is there any computable, increasing sequence of rational numbers which converges to $\Omega_V(T_1)$ and which is not T_2-convergent ? We can answer this question affirmatively in the following form.

Theorem 6.4. *Let T_1 and T_2 be arbitrary computable real numbers with $0 < T_1 < T_2 < 1$. Then there exist an optimal computer V and a computable, increasing sequence $\{a_n\}$ of rational numbers such that (i) $\Omega_V(T_1) = \lim_{n\to\infty} a_n$, (ii) $\{a_n\}$ is T-convergent for every $T \in (T_2, \infty)$, and (iii) $\{a_n\}$ is not T-convergent for every $T \in (0, T_2]$.*

[1]In Theorem 3 of Tadaki [18], η is furthermore shown to be Chaitin T-random, i.e., $\lim_{n\to\infty} H(\eta_n) - Tn = \infty$ holds.

Proof. First, we choose any one computable, increasing sequence $\{c_n\}$ of real numbers such that (i) $\{c_n\}$ converges to a computable real number $\gamma > 0$, (ii) $\{c_n\}$ is T-convergent for every $T \in (T_2, \infty)$, and (iii) $\{c_n\}$ is not T-convergent for every $T \in (0, T_2]$. Such $\{c_n\}$ can be obtained, for example, in the following manner.

Let $\{c_n\}$ be an increasing sequence of real numbers with

$$c_n = \sum_{k=1}^{n+1} \left(\frac{1}{k}\right)^{\frac{1}{T_2}}.$$

Since $T_2 > 0$, we first see that $\{c_n\}$ is T-convergent for every $T \in (T_2, \infty)$, and $\{c_n\}$ is not T-convergent for every $T \in (0, T_2]$. Since T_2 is a computable real number with $0 < T_2 < 1$, it is easy to see that $\{c_n\}$ is a computable sequence of real numbers which converges to a computable real number $\gamma > 0$. Thus, this sequence $\{c_n\}$ has the properties (i), (ii), and (iii) desired above.

We choose any one rational number r with $0 < r < 1/\gamma$, and let $\beta = r\gamma$. Obviously, β is a computable real number with $0 < \beta < 1$. Let $b = 2^{\frac{1}{T_1}}$. Then $1 < b$. We can then effectively expand β to the base-b, i.e., Property 1 below holds for the pair of β and b.

Property 1. *There exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{N}$ such that $f(k) \leq \lceil b \rceil - 1$ for all $k \in \mathbb{N}^+$ and $\beta = \sum_{k=1}^{\infty} f(k) b^{-k}$.*

This can be possible since both β and b are computable. The detail is as follows. In the case where Property 2 below holds for the pair of β and b, Property 1 holds, obviously.

Property 2. *There exist $m \in \mathbb{N}^+$ and a function $g\colon \{1, 2, \ldots, m\} \to \mathbb{N}$ such that $g(k) \leq \lceil b \rceil - 1$ for all $k \in \{1, 2, \ldots, m\}$ and $\beta = \sum_{k=1}^{m} g(k) b^{-k}$.*

Thus, in what follows, we assume that Property 2 does not hold. In this case, we construct the total recursive function $f\colon \mathbb{N}^+ \to \mathbb{N}$ by calculating $f(1), f(2), f(3), \ldots, f(m), \ldots$ one by one in this order, based on recursion on stages m. We start with stage 1 and follow the instructions below. Note there that the sum $\sum_{k=1}^{m-1} f(k) b^{-k}$ is regarded as 0 in the case of $m = 1$.

At the beginning of stage m, assume that $f(1), f(2), f(3), \ldots, f(m-1)$ are calculated already. We approximate the real number $\beta - \sum_{k=1}^{m-1} f(k) b^{-k}$ and the $\lceil b \rceil - 1$ real numbers

$$b^{-m},\ 2b^{-m},\ \ldots,\ (\lceil b \rceil - 2) b^{-m},\ (\lceil b \rceil - 1) b^{-m}$$

by rational numbers with increasing precision. During the approximation, if we find $l \in \{0, 1, 2, \ldots, \lceil b \rceil - 1\}$ such that

$$l b^{-m} < \beta - \sum_{k=1}^{m-1} f(k) b^{-k} < (l+1) b^{-m}, \tag{11}$$

then we set $f(m) := l$ and begin stage $m + 1$.

We can check that our recursion works properly, as follows. Since $0 < \beta < 1 \leq \lceil b \rceil b^{-1}$, we see that $0 < \beta - \sum_{k=1}^{m-1} f(k) b^{-k} < \lceil b \rceil b^{-m}$ at the beginning of stage $m = 1$. Thus, in general, we assume that $0 < \beta - \sum_{k=1}^{m-1} f(k) b^{-k} < \lceil b \rceil b^{-m}$ at the beginning of stage m. Then, since β and b are computable and Property 2 does not hold, we can eventually find $l \in \{0, 1, 2, \ldots, \lceil b \rceil - 1\}$ which satisfies (11). Since $b^{-m} \leq \lceil b \rceil b^{-(m+1)}$, we have $0 < \beta - \sum_{k=1}^{m} f(k) b^{-k} < \lceil b \rceil b^{-(m+1)}$ at the beginning of stage $m + 1$.

Thus, Property 1 holds in any case. We choose any one $L \in \mathbb{N}$ with $2^L \geq \lceil b \rceil - 1$. Then $\sum_{k=1}^{\infty} f(k) 2^{-(k+L)} \leq \sum_{k=1}^{\infty} (\lceil b \rceil - 1) 2^{-(k+L)} \leq 1$. Hence, by Theorem 3.2 of [4], it is easy to show that there exists a computer C such that (i) $\#\{p \mid |p| = k + L \;\&\; p \in \operatorname{dom} C\} = f(k)$ for every $k \in \mathbb{N}^+$, and (ii) $|p| \geq 1 + L$ for every $p \in \operatorname{dom} C$. We then define a partial function

$V: \{0,1\}^* \to \{0,1\}^*$ by the conditions that (i) $\operatorname{dom} V = \{ 0p \mid p \in \operatorname{dom} U \} \cup \{ 1p \mid p \in \operatorname{dom} C \}$, (ii) $V(0p) = U(p)$ for all $p \in \operatorname{dom} U$, and (iii) $V(1p) = C(p)$ for all $p \in \operatorname{dom} C$. Since $\operatorname{dom} V$ is a prefix-free set, it follows that V is a computer. It is then easy to check that $H_V(s) \leq H_U(s) + 1$ for every $s \in \{0,1\}^*$. Therefore, since U is an optimal computer, V is also an optimal computer. On the other hand, we see that

$$\begin{aligned} \Omega_V(T_1) &= \sum_{p \in \operatorname{dom} U} 2^{-(|p|+1)/T_1} + \sum_{p \in \operatorname{dom} C} 2^{-(|p|+1)/T_1} \\ &= 2^{-\frac{1}{T_1}} \Omega_U(T_1) + 2^{-\frac{L+1}{T_1}} \sum_{k=1}^{\infty} f(k) 2^{-k/T_1} \qquad (12) \\ &= 2^{-\frac{1}{T_1}} \Omega_U(T_1) + 2^{-\frac{L+1}{T_1}} \beta. \end{aligned}$$

Since T_1 is computable with $0 < T_1 < 1$, it follows from Theorem 4.3 that there exists a T_1-convergent computable, increasing sequence $\{w_n\}$ of rational numbers which converges to $\Omega_U(T_1)$. Then, since T_1 is computable, it is easy to show that there exists a computable, increasing sequence $\{a_n\}$ of rational numbers such that

$$\eta w_n + \xi c_n < a_n < \eta w_{n+1} + \xi c_{n+1}$$

for all $n \in \mathbb{N}$, where $\eta = 2^{-\frac{1}{T_1}}$ and $\xi = 2^{-\frac{L+1}{T_1}} r$. Obviously, by (12) we have $\lim_{n\to\infty} a_n = 2^{-\frac{1}{T_1}} \Omega_U(T_1) + 2^{-\frac{L+1}{T_1}} r\gamma = \Omega_V(T_1)$. Using the inequality $(x+y)^t \leq x^t + y^t$ for real numbers $x, y > 0$ and $t \in (0,1]$, we have

$$\begin{aligned} (a_{n+1} - a_n)^T &< [(\eta w_{n+2} + \xi c_{n+2}) - (\eta w_n + \xi c_n)]^T \\ &\leq \eta^T (w_{n+2} - w_n)^T + \xi^T (c_{n+2} - c_n)^T \\ &\leq \eta^T (w_{n+2} - w_{n+1})^T + \eta^T (w_{n+1} - w_n)^T \\ &\quad + \xi^T (c_{n+2} - c_{n+1})^T + \xi^T (c_{n+1} - c_n)^T. \end{aligned}$$

Thus, for each $T \in (T_2, \infty)$, since both $\{w_n\}$ and $\{c_n\}$ are T-convergent, $\{a_n\}$ is also T-convergent. We also have

$$\begin{aligned} (c_{n+2} - c_{n+1})^T &< [\eta(w_{n+2} - w_{n+1}) + \xi(c_{n+2} - c_{n+1})]^T / \xi^T \\ &= [(\eta w_{n+2} + \xi c_{n+2}) - (\eta w_{n+1} + \xi c_{n+1})]^T / \xi^T \\ &< (a_{n+2} - a_n)^T / \xi^T \\ &\leq (a_{n+2} - a_{n+1})^T / \xi^T + (a_{n+1} - a_n)^T / \xi^T. \end{aligned}$$

Thus, for each $T \in (0, T_2]$, since $\{c_n\}$ is not T-convergent, it is easy to see that $\{a_n\}$ is not T-convergent also. This completes the proof. □

7 Concluding remarks

In this paper, we have generalized the equivalent characterizations of randomness for a recursively enumerable real over the notion of partial randomness, so that the generalized characterizations are all equivalent to the weak Chaitin T-randomness. As a stronger notion of partial randomness of a real number α, Tadaki [15, 16] introduced the notion of the Chaitin T-randomness of α, which is defined as the condition on α that $\lim_{n\to\infty} H(\alpha_n) - Tn = \infty$.[2] Thus, future work may aim at modifying our equivalent characterizations of partial randomness so that they become equivalent to the Chaitin T-randomness.

[2]The actual separation of the Chaitin T-randomness from the weak Chaitin T-randomness is done by Reimann and Stephan [12].

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