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Unique existence of BV-entropy solutions for strongly degenerate convective diffusion equations

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Abstract. In this paper a new notion of generalized solution to the initial boundary value problem for a nonlinear strongly degenerate parabolic equation of the form $u_t + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(u)$ is treated; this type of solution is called a BV-entropy solution. Since equations of this form are linear conbinations of time-dependent conservation laws and porous medium type equation, it is interesting to investigate interactions between singularities of solutions provided by the two different kinds of nonlinearities. Therefore the part of conservation laws and that of porous medium type diffusion term have to be treated in a separate way. In fact, for the convective term, method for the existence and uniqueness of the entropy solutions in the sense of Kružkov is employed and so the initial-boundary-value problem is formulated in the space BV of functions of bounded variation. This observation leads us to the new notion of BV-entropy solution. Our objective here is to establish unique existence of such BV-entropy solutions under the homogeneous Neumann boundary conditions.

1. Initial boundary value problem

1.1. **Introduction.** We consider the initial boundary value problem for a nonlinear degenerate convective diffusion equations of the form

$$\text{(IBVP)} \qquad \begin{cases} u_t + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(u), \quad (x,t) \in \Omega \times (0,T), \\[4pt] \dfrac{\partial \beta(u)}{\partial \mathbf{n}}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\[4pt] u(x,0) = u_0(x), \quad u_0 \in L^\infty(\Omega) \cap BV(\Omega). \end{cases}$$

Here Ω is a bounded Lipschitz domain in $\mathbb{R}^N$. $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_N)$ and $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ are the spatial nabla and the Laplacian in $\mathbb{R}^N$, respectively. [0,T] is a fixed time interval. $A(x,t,\xi) = (A^1, ..., A^N)(x,t,\xi)$ is an $\mathbb{R}^N$-valued differentiable function on $\overline{\Omega} \times [0,T] \times \mathbb{R}$ and $B(x,t,\xi)$ is an $\mathbb{R}$-valued differentiable function on $\overline{\Omega} \times [0,T] \times \mathbb{R}$. The function β on the right-hand side is supposed to be monotone nondecreasing and locally Lipschitz continuous on $\mathbb{R}$. Moreover, $\mathbf{n}$ represents the unit normal to $\partial\Omega$. Since β is assumed to be monotone nondecreasing, the set of points where $\beta' = 0$ may have a positive measure. In this sense, we say that the equation posed in (IBVP) is a *strongly* degenerate parabolic equation.

This type of equation can be applied to the sedimentation-consolidation processes of particulate suspensions, filtration problems, Stefan problems and so on. On the other hand, this equation is also regarded as a linear combination of the time dependent conservation laws (quasilinear hyperbolic equation) :

$$u_t + \nabla \cdot A(x,t,u) + B(x,t,u) = 0$$

and the porous medium equation (nonlinear degenerate parabolic equation):

$$u_t = \Delta\beta(u).$$

That is, this equation has both properties of hyperbolic equations and parabolic equations. Moreover, by the assumption on β, this equation has the following properties :

(i) In the case that β is strictly increasing, "parabolicity" is majorant to "hyperbolicity".

(ii) In the case that β is monotone nondecreasing, "parabolicity" and "hyperbolicity" are not necessarily comparable.

In fact, in the case that β is strictly increasing, it is possible to prove uniqueness of generalized solution in the sense of distributions. However, in the case that β is simply monotone nondecreasing, it is impossible. Therefore, we employ generalized solutions in the sense of Kružkov which are called by entropy solutions.

1.2. **Previous work.** In 1967, A. I. Vol'pert [12] proposed a theory of functions of bounded variation in multi-dimentional cases and apply it to the first order quasi-linear equations. In 1969, A. I. Vol'pert and S. I. Hudjaev [14] considered second order equations and proved the existence and uniqueness of the BV-entropy solutions in a restricted sense, assuming the so-called discontinuity conditions. On the othe hand, in 1969 through 1970, S. N. Kružkov [7], [8] introduced a notion of entropy solution and proved the existence and uniqueness of a general class of first order quasi-linear equations. The idea employed in his proof for the uniqueness of entropy solutions is named Kružkov's doubling variable device. Our argument is relied essentially on Kružkov's device introduced in [8]. In 1979, C. Bardos, A. Y. Leroux and J. C. Nedelec [2] applied Kružkov's device to the first order quasi-linear equations and gave similar results on bounded domains under the Dirichlet boundary conditions.

Later, in 1999, J. Carrillo showed existence and uniqueness of entropy solutions to nonlinear strongly degenerate parabolic equations under the homogeneous Dirichlet boundary condition of the form $\beta(u) = 0$ on $\partial\Omega$. In particuler, his method for the proof of uniqueness is useful for the discussion of entropy solutions in strongly degenerate parabolic equations. On the other hand, in 2002, C. Mascia, A. Porretta and A. Terracina showed the uniqueness and consistency with parabolic regularizations of entropy solutions in nonhomogeneous Dirichlet boundary problems for strongly degenerate parabolic equations. It is striking that they use the notion of *boundary layer* to formulate the Dirichlet boundary condition.

As stated above, various results of this type of equations have been obtained. However, to the best of the author's knowlwdge, very little is known about the Neumann problems. Accordingly, we focus our attention on the Neumann problems for this type of equations. In fact, H. Watanabe [15] has proved the unique existence of the BV solutions in (P) under the assumption that β is strictly increasing.

1.3. **Basic hypotheses.** Throughout this paper, we put the following three conditions on the nonlinear functions A^i, $i = 1, \cdots N$, and B.

(H.1) Let $a^i = (\partial/\partial\xi)A^i$ for $i = 1, \ldots, N$, and $b = (\partial/\partial\xi)B$. For each $r > 0$, the functions A^i, $\partial_i A^i$, a^i, $\partial_{ij} A^i$, $\partial_i a^i$, B, $\partial_i B$, $i, j = 1, \ldots, N$ are all bounded and continuous on $Q_r = \overline{\Omega} \times [0, T] \times [-r, r]$.

(H.2) There exist constants α, α' such that

$$-\nabla \cdot a(x,t,\xi) - b(x,t,\xi) \leq \alpha \quad \text{and} \quad -b(x,t,\xi) \leq \alpha'$$

for all $(x,t,\xi) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$.

(H.3) For each $r > 0$ and $\tau > 0$,

$$\rho(\tau; r) = \max \left\{ \begin{array}{l} \sup |\partial_i A^i(x,t,\xi) - \partial_i A^i(x,s,\xi)|, \\ \sup |a^i(x,t,\xi) - a^i(x,s,\xi)|, \\ \sup |B(x,t,\xi) - B(x,s,\xi)| \end{array} \right\}$$

over $x \in \overline{\Omega}$, $s, t \in [0,T]$, $\xi \in \mathbb{R}$ such that $|s-t| \leq \tau$, $|\xi| \leq r$ and $i = 1, \ldots, N$. Then

$$\rho(\tau; r) \to 0 \quad \text{as } \tau \downarrow 0 \text{ for each } r > 0.$$

The above conditions are the assumptions for the cofficients of the main partial differential equation in (IBVP). Condition (H.1) is a smoothness condition for the coefficients, (H.2) is a growth conditions and (H.3) restricts the time dependence, respectively.

1.4. **The generalized solution to (IBVP).** In this section we introduce a notion of generalized solution to (IBVP). This notion derives the entropy solution introduced by S. N. Kružkov [7], [8]. He showed the existence and uniqueness of entropy solutions to the first order quasilinear equations. In view of the concept of entropy solution, we introduce a new notion of generalized solution to the second order equations:

Definition 1.1. *Let $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$. A function $u \in L^\infty(\Omega \times (0,T)) \cap BV(\Omega \times (0,T))$ is called a BV-entropy solution of the problem (IBVP), if it satisfies the two conditions:*

(1) u belongs to $C([0,T]; L^1(\Omega))$ and $L^1\text{-}\lim_{t\downarrow 0} u(\cdot, t) = u_0$;

(2) $\nabla\beta(u) \in L^2(0,T; L^2(\Omega)^N)$ and for $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))^+$ and $k \in \mathbb{R}$,

$$\int_0^T \int_\Omega |u-k|\varphi_t dx dt \geq \int_0^T \int_\Omega \operatorname{sgn}(u-k)(\nabla\beta(u) \cdot \nabla\varphi$$
$$-[A(x,t,u) - A(x,t,k)] \cdot \nabla\varphi + [B(x,t,u) + \nabla \cdot A(x,t,k)]\varphi) dx dt.$$

In the case of $\Omega = \mathbb{R}^N$, a BV-entropy solution is a distributional solution in $BV(\Omega)$. In this paper, we discuss unique existence of BV-entropy solutions assosiate with (IBVP).

Remark 1.1. *H. Watanabe* [15] *has proved unique existence of the BV solutions in (IBVP) below under the assumption that β is strictly increasing.*

Definition 1.2. *Let $u_0 \in L^\infty(\Omega)\cap BV(\Omega)$. A function $u \in L^\infty(\Omega\times(0,T))\cap BV(\Omega\times(0,T))$ is called a BV solution of the problem (IBVP), if it satisfies the two conditions below:*

(1) u belongs to $C([0,T];L^1(\Omega))$ and L^1-$\lim_{t\downarrow 0} u(\cdot,t) = u_0$;

(2) $\nabla\beta(u) \in L^2(0,T;L^2(\Omega)^N)$ and for $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))$,

$$\int_0^T \int_\Omega (u\varphi_t + A(x,t,u)\cdot\nabla\varphi - B(x,t,u)\varphi - \nabla\beta(u)\cdot\nabla\varphi)dxdt$$
$$-\int_0^T \int_{\partial\Omega} A(x,t,T_r u)\cdot \mathbf{n}(x)\ \varphi\ d\mathcal{H}^{N-1}dt = 0,$$

where $T_r : BV(\Omega) \to L^1(\partial\Omega;\mathcal{H}^{N-1})$ is the trace operator.

1.5. **Inflow-Outflow of the convective term.** We consider the probrem in the case that β is monotone nondecreasing. We then concentrate on the framework in such a way that a BV-entropy solution is a BV solution. To see this, we assume the following condition :

Inflow-Outflow condition :
Let $v \in L^\infty(\Omega)\cap BV(\Omega)$. For any $k \in \mathbb{R}$,

$$\int_{\partial\Omega} \mathrm{sgn}(T_r v - k)[A(x,t,T_r v) - A(x,t,k)]\cdot\mathbf{n}(x)\ d\mathcal{H}^{N-1} \geq 0. \tag{1.1}$$

Under this inflow-Outflow condition, a BV-entropy solution is a BV solution.

In C. Bardos, A. Y. Leroux and J. C. Nedelec (1979), a similar condition for the Dirichlet boundaty condition are introduced in the first order quasilinear equations.

1.6. **Neumann boundary condition.** In this section, we propose a natural way for formulating the Neumann boundary condition in the definition of BV-entropy solutions. When $\nabla\beta(u)$ is nondifferentiable in the classical sense, we usual consider the Neumann boundary condition in the following form :

$$\int_{\partial\Omega} \nabla\beta(u)\cdot\mathbf{n}\ d\mathcal{H}^{N-1} = 0. \tag{1.2}$$

However, we only have $\nabla\beta(u) \in [L^2(\Omega)]^N$ by definition, and so it is not straightforward to impose the Neumann boundary condition in the sense of (1.2).

In order to overcome this difficulty, we employ a natural way for approximating outward normal on the boundary. Let $\{\zeta_\delta\}$ be a sequence of $C^2(\Omega)\cap C^0(\overline{\Omega})$ functions such that

$$\lim_{\delta\to 0}\zeta_\delta = 1 \ \text{ in } \Omega, \quad 0 \leq \zeta_\delta \leq 1 \text{ in } \overline{\Omega}, \quad \zeta_\delta = 0 \ \text{ on } \partial\Omega.$$

Then we see that $-\nabla\zeta_\delta$ converges to the outward normal $\mathbf{n}$ of the boundary as $\delta \downarrow 0$ in the following sense :

$$\lim_{\delta\downarrow 0}\int_\Omega \varphi\cdot\nabla\zeta_\delta dx = -\lim_{\delta\downarrow 0}\int_\Omega \mathrm{div}(\varphi)\zeta_\delta dx = -\int_\Omega \mathrm{div}(\varphi)dx = -\int_{\partial\Omega}\varphi\cdot\mathbf{n}\ d\mathcal{H}^{N-1}$$

for $\varphi \in [C^\infty(\Omega)]^N$. In accordance with C. Mascia, A. Porretta and A. Terracina (2002), the sequence $\{\zeta_\delta\}$ is called a *boundary-layer sequence*. Here, we substitute $\varphi = (1-\zeta_\delta)\xi$, with $\xi \in C_0^\infty(\mathbb{R}^N)^+$ into the defintion of BV-entropy solution. Then we have

$$\int_0^T \int_\Omega [-[A(x,t,u) - A(x,t,k)] \cdot \nabla\zeta_\delta$$
$$-[B(x,t,u) + \nabla \cdot A(x,t,k)](1-\zeta_\delta) + \nabla\beta(u) \cdot \nabla\zeta_\delta]\xi dxdt = 0.$$

We here let $\delta \downarrow 0$ to get the follwing relation

$$\int_0^T \int_{\partial\Omega} [A(x,t,T_r u) - A(x,t,k)] \cdot \mathbf{n}(x)\xi d\mathcal{H}^{N-1}dt + \lim_{\delta\downarrow 0} \int_0^T \int_\Omega \nabla\beta(u) \cdot \nabla\zeta_\delta \xi dxdt = 0.$$

By means of the Inflow-Outflow condition, we get the following result:

$$\lim_{\delta\downarrow 0} \int_\Omega \nabla\beta(u) \cdot \nabla\zeta_\delta dx = 0. \tag{1.3}$$

If the function $\nabla\beta(u)$ makes sense on the boundary $\partial\Omega$, then the above equality means (1.2). Hence, we may interpret the Neumann boundary condition in the form of (1.3).

2. Abstract Cauchy Ploblems

We costruct nonlinear evolution operators associate with (IBVP) in terms of the nonlinear semigroup theory (ex. [6], [11]). To see this, we convert (IBVP) to a time-dependent abstract Cauchy probrem in $L^1(\Omega)$. For this purpose, we define the following differential operator $\mathcal{A}(t)$.

Definition 2.1. *Let $t \in [0,T]$. We say that $v \in \mathscr{D}(\mathcal{A}(t))$ and $w = \mathcal{A}(t)v$ iff $v, w \in L^\infty(\Omega) \cap BV(\Omega)$ and*

$$-\langle \nabla\beta(v), \operatorname{sgn}(v-k)\nabla\varphi\rangle + \langle A(\cdot,t,v) - A(\cdot,t,k), \operatorname{sgn}(v-k)\nabla\varphi\rangle$$
$$-\langle B(\cdot,t,v) + \nabla \cdot A(\cdot,t,k), \operatorname{sgn}(v-k)\varphi\rangle \geq \langle w, \operatorname{sgn}(v-k)\varphi\rangle$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)^+$, $k \in \mathbb{R}$.

Here, $\langle\cdot,\cdot\rangle$ means the duality pairing between $L^p(\Omega)$ and $L^q(\Omega)$ with $1/p + 1/q = 1$. $\mathcal{A}(t)$ are all single-valued for $t \in [0,T]$. Therefore, we consider the following abstract Cauchy problem in $L^1(\Omega)$:

$$\text{(ACP)} \quad \begin{cases} (d/dt)u(t) = \mathcal{A}(t)u(t) & \text{for } t \in (0,T), \\ u(0) = v \in L^\infty(\Omega) \cap BV(\Omega), \end{cases}$$

where $(d/dt)u(t)$ is understood to be the derivative of $u(\cdot)$ in a generalized sense. Also, we separate the class $L^\infty(\Omega)$ into a family of absolutely convex closed subsets defined by

$$X_r = \{v \in L^\infty(\Omega) \ ; \ ||v||_\infty \leq r\} \quad \text{for } r > 0. \tag{2.1}$$

In view of the BV theory, $\mathcal{A}(t)$ is quasi-dissipative with respect to the L^1-norm $||\cdot||_1$. To see this, we first show the following energy estimate for $\nabla\beta(v)$.

Lemma 2.1. *If $v \in \mathscr{D}(\mathcal{A}(t))$, then $||\nabla\beta(v)||_2 < \infty$.*

In fact, we substitute the regularized positive and negative part of diffusion function $[\beta(v)]_+^\eta$, $[\beta(v)]_-^\eta$ to the test function φ in the definition of BV-entropy solutions respectively. Using the Inflow-Outflow condition and certain known technical estimates, we obtain the above result. Using the above Lemma, we get the following result.

Theorem 2.1. *Let $r > 0$, $t \in [0,T]$ and $\lambda \in (0, \alpha'^{-1})$. Then for any pair $v, w \in \mathscr{D}(\mathcal{A}(t)) \cap X_r$, we have*

$$(1-\lambda\alpha')\|v-w\|_1 \le \|(I-\lambda\mathcal{A}(t))v-(I-\lambda\mathcal{A}(t))w\|_1.$$

The proof of Theorem 2.1 is obtained in a way similar to the proof of uniqueness of BV-entropy solutions (see Section 6). Hence, we write the resolvent operator of $\mathcal{A}(t)$ by $\mathcal{J}(\lambda;t) = (I-\lambda\mathcal{A}(t))^{-1}$ for $\lambda \in (0, \alpha'^{-1})$.

3. Difference approximation

3.1. **Difference operators.** We construct an evolution operator associated with $(IBVP)$ through difference approximations. We begin by defining three types of difference operators :

$$D_i^+(\ell)v = \ell^{-1}[v(\cdot+\ell\mathbf{e}_i)-v],\ D_i^-(\ell)v = \ell^{-1}[v-v(\cdot-\ell\mathbf{e}_i)],$$
$$D_i^0(\ell)v = (2\ell)^{-1}[v(\cdot+\ell\mathbf{e}_i)-v(\cdot-\ell\mathbf{e}_i)].$$

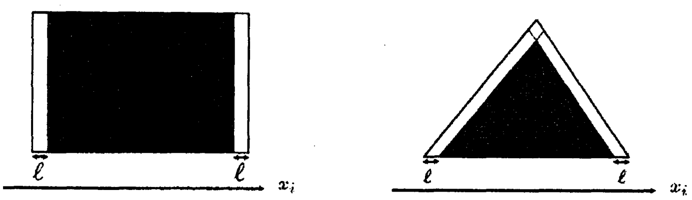
Here, $\ell > 0$ denotes a spatial mesh size. These operators are called by the forward, backward and central difference operators, respectively. Moreovre, we are the averaging operator defined by

$$D_i^a(\ell)v = (2N)^{-1}[v(\cdot+\ell\mathbf{e}_i)+v(\cdot-\ell\mathbf{e}_i)].$$

In this paper, we consider this equation on bounded domains. Therefore, it is nontrivial to see how the Neumann bondary condition can be formulated in terms of difference approximations. To this end, we define the domains of difference approximations by :

$$\Omega_\ell^i \equiv \{x \in \Omega \mid x \pm \theta\ell\mathbf{e}_i \in \Omega \text{ for all } 0 \le \theta \le 1\}.$$

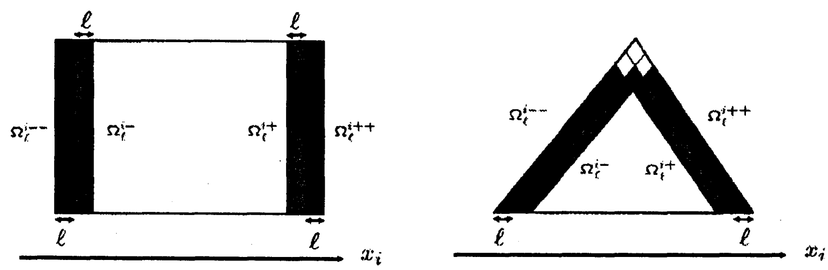
For instance, we think of the following simple figures in the case of Ω is a rectangle or a triangle, which we denote by Ω_ℓ^i and displayed as the shaded parts below:



We then consider the difference approximations in Ω_ℓ^i and try to formulate the Neumann boundary condition in $\Omega\backslash\Omega_\ell^i$. For this purpose we need more symbols:

$$\Omega_\ell^{i\pm\pm} \equiv \{x \pm \ell\mathbf{e}_i \in \Omega\backslash\Omega_\ell^i \mid x \in \Omega_\ell^i\},\quad \Omega_\ell^{i\pm} \equiv \{x \in \Omega_\ell^i \mid x \pm \ell\mathbf{e}_i \in \Omega\backslash\Omega_\ell^i\} \text{ (respectively)}.$$

These approximate domains may be depicted as below:



Remark 3.1. *We regard the domain $\Omega\backslash\Omega_\ell^i$ as the boundary layer of Ω. In paticuler, we make an attempt to impose the Neumann boundary condition in $\Omega_\ell^{i\pm}$ and $\Omega_\ell^{i\pm\pm}$.*

We first define the difference operators $C_{r,h}^i(t)\colon X_r \to L^\infty(\Omega)$ in the ith direction, $r>0$, $t\in[0,T]$, $h\in(0,h_r)$, by

$$(3.1)\qquad C_{r,h}^i(t)v = \begin{cases} \hat{C}_{r,h}^i(t)v & \text{in } \Omega_\ell^i\backslash[\Omega_\ell^{i+}\cup\Omega_\ell^{i-}] \\ \hat{C}_{r,h}^i(t)v + \hat{v}_\ell^i & \text{in } \Omega_\ell^{i+}\cup\Omega_\ell^{i-} \\ \hat{\hat{v}}_\ell^i & \text{in } \Omega\backslash\Omega_\ell^i, \end{cases}$$

for $v\in X_r$. Here $\hat{C}_{r,h}^i(t)$ are the core difference operators

$$\hat{C}_{r,h}^i(t)v = D_i^-(\ell)D_i^+(\ell)(\beta(v)+\varepsilon_r(h)v) - D_i^0(\ell)A^i(\cdot,t,v) - D_i^a(\ell)B(\cdot,t,v) \text{ in } \Omega_\ell^i.$$

where $\varepsilon_r(h)$ is defined later. Moreover, we choose $\hat{v}_\ell^i\in L^\infty(\Omega_\ell^{i+}\cup\Omega_\ell^{i-})$ and $\hat{\hat{v}}_\ell^i\in L^\infty(\Omega\backslash\Omega_\ell^i)$ satisfing appropriate conditions related to the Neumann boundary condition. In this paper we employ the elements

$$\hat{v}_\ell^i = \begin{cases} \dfrac{1}{\ell^2}[\beta(v)-\beta(v(\cdot-\ell\mathbf{e}_i))] & \text{in } \Omega_\ell^{i-}, \\ 0 & \text{in } \Omega_\ell^{i+}\backslash\Omega_\ell^{i-}. \end{cases}$$

$$\hat{\hat{v}}_\ell^i = \begin{cases} -\dfrac{1}{\ell^2}[\beta(v)-\beta(v(\cdot-\ell\mathbf{e}_i))] & \text{in } \Omega_\ell^{i++}, \\ 0 & \text{in } \Omega\backslash[\Omega_\ell^i\cup\Omega_\ell^{i++}]. \end{cases}$$

Then we obtain the following estimate :

$$\begin{aligned} &\int_\Omega \operatorname{sgn}(v-k)C_{r,h}^i(t)v\varphi dx \\ &= \int_{\Omega_\ell^i} \operatorname{sgn}(v-k)\hat{C}_{r,h}^i(t)v\varphi dx + \int_{\Omega_\ell^{i+}\cup\Omega_\ell^{i-}} \operatorname{sgn}(v-k)\hat{v}_\ell^i\varphi dx + \int_{\Omega\backslash\Omega_\ell^i} \operatorname{sgn}(v-k)\hat{\hat{v}}_\ell^i\varphi dx \\ &\le -\int_{\Omega_\ell^i} \operatorname{sgn}(v-k)[D_i^+(\ell)\beta(v)][D_i^+(\ell)\varphi]dx + \varepsilon_r(h)\int_{\Omega_\ell^i} D_i^-(\ell)D_i^+(\ell)|v-k|\varphi dx \\ &\quad + \int_{\Omega_\ell^i} \operatorname{sgn}(v-k)[A^i(x,t,v)-A^i(x,t,k)]D_i^0(\ell)\varphi dx \\ &\quad - \int_{\Omega_\ell^i} \operatorname{sgn}(v-k)[D_i^a(\ell)B(x,t,v)+D_i^0(\ell)A^i(x,t,k)]\varphi dx. \end{aligned}$$

for $\varphi\in C_0^\infty(\mathbb{R}^N)^+$ and $k\in\mathbb{R}$. On the right-hand side of the above inequality, the second term converges to zero as $\ell\to 0$. Hence, we clear integral near the boundary terms and may get that the operator $C_{r,h}^i(t)$ form to employ the entropy solutions.

Using the difference operator $C_{r,h}^i(t)$ in the ith direction, we define the aimed difference operator $C_{r,h}(t)\colon X_r\to L^\infty(\Omega)$, $r>0$, $t\in[0,T]$, $h\in(0,h_r)$, by

$$C_{r,h}(t)v = v + h\sum_{i=1}^N C_{r,h}^i(t)v.$$

The difference operators $\mathcal{C}_{r,h}(t)$ are of Lax-Friedrichs type in the following sense :

$$\mathcal{C}_{r,h}(t)v = v + h\sum_{i=1}^{N}[D_i^-(\ell)D_i^+(\ell)(\beta(v)+\varepsilon_r(h)v) - D_i^0(\ell)A^i(\cdot,t,v) - D_i^a(\ell)B(\cdot,t,v)]$$

in $\bigcap_{i=1}^N \Omega_\ell^i$. These difference operators are employed to construct entropy solutions.

3.2. **The CFL condition.** The difference operators $\mathcal{C}_{r,h}(t)$ are constructed in the sense of explicit Euler. Therefore, it is necessary to formulate the CFL condition for the stable computation. In order to discribe the CFL condition, we prepare several symbols. In view of condition (H.1), we write

$$\begin{cases} K_r = \text{ess.sup}\{\beta'(\xi)\ ;\ |\xi| \le r\}, \\ M_r = \max\{1, \text{suprema of } |A^i|,\ |\partial_i A^i|,\ |a^i|,\ |\partial_{ij}A^i|,\ |\partial_j a^i|, \\ \qquad\qquad |B|,\ |\partial_i B|,\ |b| \text{ over } Q_r \text{ and } i,j=1,\dots,N\}. \end{cases}$$

The first part of the CFL condition is imposed for the mesh ratio h/ℓ^2. We choose a monotone nonincreasing function $r \in (0,\infty) \mapsto \delta_r \in (0,1)$ such that

$$\delta_r \equiv h/\ell^2, \quad 0 < \delta_r < \min\left\{\frac{1}{2NK_r}, \frac{1}{NM_r}\right\}.$$

The second part of the CFL condition is imposed for the mesh size in time. We fix $\omega > 0$ which will be defined later and define

$$h_r = \min\left\{\frac{\omega}{3}, \delta_r\left(\frac{K_r}{M_r}\left(\frac{1}{2NK_r} - \delta_r\right)\right)^2\right\}.$$

The CFL condition is then stated as follows: If $v \in X_r$ in (DS), then we impose that $h \in (0, h_r)$. Moreover, we define the coefficient of artifitail viscosity term as

$$\varepsilon_r(h) = M_r(h/\delta_r)^{1/2} = M_r\ell. \tag{3.2}$$

3.3. **Properties of the difference operators $\mathcal{C}_{r,h}(t)$.** We have deal with several estimates of the difference operators $\mathcal{C}_{r,h}(t)$. Firstly, we have the following L^∞-estimates :

Proposition 3.1 (L^∞-estimate). *Let $r > 0$, $t \in [0,T]$ and $h \in (0,h_r)$. Then we have*

$$||\mathcal{C}_{r,h}(t)v||_\infty \le (1+\alpha h)||v||_\infty + h(||C||_\infty + 1). \tag{3.3}$$

for $v \in X_r$. Here, we write C for the function

$$C(x,t) = -\nabla\cdot A(x,t,0) - B(x,t,0) \quad \textit{for } (x,t) \in \overline{\Omega}\times[0,T]$$

and define $||C||_\infty \equiv ||C||_{L^\infty(\Omega\times[0,T])}$.

Secondly, we estimate the Lipschitz constants of $\mathcal{C}_{r,h}(t)$.

Proposition 3.2 (Lipschitz continuity in L^1). *Let $r > 0$, $t \in [0,T]$ and $h \in (0,h_r)$. Then we have*

$$||\mathcal{C}_{r,h}(t)v - \mathcal{C}_{r,h}(t)w||_1 \le (1+h\alpha''_{\ell,r})||v-w||_1, \tag{3.4}$$

for $v, w \in X_r$. Here, $\alpha''_{\ell,r} \equiv (1+\hat{\omega}_\ell)\alpha' + M_rC_\Omega$, $\hat{\omega}_\ell << \frac{1}{2}$ is constant satisfying $\hat{\omega}_\ell \to 0$ as $\ell \downarrow 0$ and C_Ω is a positive constant depending on the domain Ω.

We next treat the total variations of $\mathcal{C}_{r,h}(t)$. To this end, we introduce a discrete version of the functional $|D\cdot|(\Omega)$ which represents the total variations of elements of X_r in a discrete sense as follows :

$$|D^0(\nu)v|(\Omega_\nu) \equiv \sup\{\sum_{k=1}^{N}\int_{\Omega_\nu^k} vD_k^0(\nu)\varphi_k dx\ ;\ \varphi=(\varphi_1,\ldots,\varphi_N)\in[C_0^\infty(\Omega_\nu^k\backslash[\Omega_\nu^{k+}\cap\Omega_\nu^{k-}])]^N,$$
$$\max_{1\le k\le N}||\varphi_k||_\infty \le 1\}$$

for $v\in L^1(\Omega)$ and $\nu>0$. As easily seen, it holds that for $v\in L^\infty(\Omega)$

$$|D^0(\nu)v|(\Omega_\nu)=\sum_{k=1}^{N}\int_{\Omega_\nu^k}|D_k^0(\nu)v|dx,\quad \text{and that}\quad |Dv|(\Omega)<\infty \quad\text{if}\quad \sup_{\nu>0}|D^0(\nu)v|(\Omega_\nu)<\infty.$$

Using the above relations, we obtain BV estimates of $\mathcal{C}_{r,h}(t)$.

Proposition 3.3 (BV estimates). *Let $r>0$, $t\in[0,T]$, and $h\in(0,h_r)$. For $v\in X_r$ and $\nu\in(0,1)$, we have*

$$|D^0(\nu)\mathcal{C}_{r,h}(t)v|(\Omega_\nu)\le(1+h\alpha''_{\ell,r})|D^0(\nu)v|(\Omega_\nu)+\omega_r h(|D^0(\ell)v|(\Omega_\ell)+|\Omega|), \tag{3.5}$$

where $\omega_r = M_rN(N+1)$.

Finally, we give a result on the time dependence of the difference operators $C_{r,h}(t)$.

Proposition 3.4 (Time dependence). *Let $r>0$, $s,t\in[0,T]$, $h\in(0,h_r)$ and $v\in X_r$. Then we have*

$$||\mathcal{C}_{r,h}(s)v-\mathcal{C}_{r,h}(t)v||_1\le h(|D^0(\ell)v|(\Omega_\ell)+|\Omega|)\omega_r\rho(|s-t|;r).$$

The proof is obtained by rearranging $\mathcal{C}_{r,h}(t)v$ and applying Proposition 3.3.

4. Resolvents of the operators $\mathcal{A}(t)$

In this section, we discuss the resolvents of the operators $\mathcal{A}(t)$. First, we define the implicit difference operators $\mathcal{A}_{r,h}(t)$ as follows :

$$\mathscr{D}(\mathcal{A}_{r,h}(t))=X_r,\quad \mathcal{A}_{r,h}(t)=h^{-1}[\mathcal{C}_{r,h}(t)-I]=\sum_{i=1}^{N}C_{r,h}^i(t)v$$

for $r>0$, $t\in[0,T]$ and $h\in(0,h_r)$. We here make an attempt to define the difference operators $\mathcal{A}_{r,h}(t)$ which would approximate the differential operators $\mathcal{A}(t)$ formulated as above. In fact, it is seen that the differece operators $\mathcal{A}_{r,h}(t)$ converges the differential operators $\mathcal{A}(t)$ as $h\downarrow 0$ in the following sense:

$$(I-\lambda\mathcal{A}_{R,h}(t))^{-1}v\to(I-\lambda\mathcal{A}(t))^{-1}v\quad \text{in } L^1 \text{ as } h\downarrow 0$$

for $v\in X_r$ and $\lambda>0$ with $0<\lambda<\min\{\alpha^{-1},\hat{\alpha}_R(h_R)^{-1}\}$.

Here we introduce functions $\alpha(\cdot)$ and $\alpha_r(\cdot)$, $r>0$, on the interval $(0,1]$ difined by

$$\alpha_r(h) = h^{-1}(e^{\alpha''_{\ell,r}h}-1) \qquad \text{for } h\in(0,1], \tag{4.1}$$
$$\hat{\alpha}_r(h) = h^{-1}(e^{(\alpha''_{\ell,r}+(1+\hat{\omega}_\ell)\omega_r)h}-1) \qquad \text{for } h\in(0,1],\ \text{respectively}. \tag{4.2}$$

Remark 4.1. *For $r > 0$, $t \in [0,T]$ and $h \in (0, h_R)$, the operator $\mathcal{A}_{r,h}(t) - \alpha_r(h)I$ is dissipative on X_r with respect to the norm $||\cdot||_1$ by Proposition 3.2, namely,*

$$(1 - \lambda\alpha_r(h))||v - w||_1 \leq ||(I - \lambda\mathcal{A}_{r,h}(t))v - (I - \lambda\mathcal{A}_{r,h}(t))w||_1 \tag{4.3}$$

for $v, w \in X_r$ and $\lambda > 0$. This shows that the operator $I - \lambda\mathcal{A}_{r,h}(t)$ is injective on X_r for $\lambda \in (0, \alpha_r(h)^{-1})$ and has the inverse operator. In what follows, we write the resolvent operator as

$$\mathcal{J}_{R,h}(\lambda; t) = (I - \lambda\mathcal{A}_{R,h}(t))^{-1} \tag{4.4}$$

for $R > 0$, $t \in [0,T]$, $h \in (0, h_R)$ and $\lambda \in (0, \alpha_R(h)^{-1})$. By virtue of the definition of the difference operator $\mathcal{A}_{R,h}(t)$, $\mathcal{J}_{R,h}(\lambda, t)$ is an operator from $L^\infty(\Omega)$ to X_R.

4.1. **Properties of resolvent operator.** We first investigate the domains of the resolvents $\mathcal{J}_{R,h}(\lambda; t)$ and their growth with respect to λ. To this end, we employ the following functions:

$$g(\lambda; r) = (1 - \lambda\alpha)^{-1}\{r + \lambda(||C||_\infty + 1)\} \text{ for } r > 0 \text{ and } \lambda \in (0, \alpha^{-1}), \tag{4.5}$$

where $||C||_\infty$ is introduced in Proposition 3.1.

Proposition 4.1 (Growth of $\mathcal{J}_{R,h}$ in the spaces L^∞ and L^1 with respect to λ). *Let $R > 0$, $t \in [0,T]$, $h \in (0, h_R)$ and $0 < \lambda < \{\alpha^{-1}, \alpha_R(h)^{-1}\}$. Then we have :*

(a) *If $v \in L^\infty(\Omega)$ and $g(\lambda, ||v||_\infty) \leq R$, then*

$$v \in \mathscr{D}(\mathcal{J}_{R,h}(\lambda; t)), \quad ||\mathcal{J}_{R,h}(\lambda; t)v||_\infty \leq g(\lambda, ||v||_\infty).$$

(b) *If $v, w \in L^\infty(\Omega)$ and $g(\lambda, ||v||_\infty) \vee g(\lambda, ||w||_\infty) \leq R$, then*

$$||\mathcal{J}_{R,h}(\lambda; t)v - \mathcal{J}_{R,h}(\lambda; t)w||_1 \leq (1 - \lambda\alpha(h))^{-1}||v - w||_1.$$

Using Propositions 3.1 and 3.2, the above Proposition is varified. In fact, we consider the following map

$$\mathfrak{T}(v)z = h(h + \lambda)^{-1}v + \lambda(h + \lambda)^{-1}\mathcal{C}_{R,h}(t)z \quad \text{for } z \in X_R.$$

The operator $\mathfrak{T}(v)$ maps X_R into itself and is a strict contraction with respect to $||\cdot||_1$. Hence, using the fixed point theorem, we obtain the first part of (a). The last part of (a) and (b) are clear.

Secondly, we make an attempt to estimate the discrete version of the total variations of the set $\{\mathcal{J}_{R,h}(\lambda; t)v \; ; \; h \in (0, h_R)\}$.

Lemma 4.1 (Discrete version of total variations of $\mathcal{J}_{R,h}(\lambda; t)v$). *Let $R > 0$, $t \in [0,T]$, $h \in (0, h_R)$ and $0 < \lambda < \{\alpha^{-1}, \hat{\alpha}_R(h)^{-1}\}$. If $v \in L^\infty(\Omega)$ and $g(\lambda, ||v||_\infty) \leq R$, then*

$$\begin{aligned}(1 - \lambda\hat{\alpha}_R(h))|D^0(\nu)\mathcal{J}_{R,h}(\lambda; t)v|(\Omega) &\leq |D^0(\nu)v|(\Omega) \\ &+ \lambda\omega_R(1 - \lambda\hat{\alpha}_R(h))^{-1}|D^0(\ell)v|(\Omega) + \lambda\omega_R(1 + \lambda\omega_R(1 - \lambda\hat{\alpha}_R(h))^{-1})|\Omega|.\end{aligned} \tag{4.6}$$

The proof of this Lemma is essentially used in Proposition 3.3. By virtue of Lemma 4.1, we obtain the estimates of the total variations of $\mathcal{J}_{R,h}(\lambda; t)v$ in terms of the total variation of v. Notice that

$$|Dv|(\Omega) = \sup_{\nu \in (0,1)} |D^0(\nu)v|(\Omega_\nu)$$

hold for $v \in BV(\Omega)$.

Proposition 4.2 (Growth of $\mathcal{J}_{R,h}$ in the spaces $\mathcal{M}$). *Let $R > 0$, $t \in [0,T]$, $h \in (0, h_R)$ and $0 < \lambda < \{\alpha^{-1}, \hat{\alpha}_R(h)^{-1}\}$. If $v \in L^\infty(\Omega) \cap BV(\Omega)$ and $g(\lambda, ||v||_\infty) \leq R$, then*

$$|D\mathcal{J}_{R,h}(\lambda; t)v|(\Omega) \leq (1 - \lambda\hat{\alpha}_R(h))^{-1}(|Dv|(\Omega) + \lambda\omega_R|\Omega|).$$

Using the estimetes otained in the above two propositions and applying the compactness theorem in BV (see [4]), one may investigate the behavior in $L^1(\Omega)$ of $\mathcal{J}_{R,h}(\lambda; t)v$ as $h \downarrow 0$.

Proposition 4.3 (Convergence of the resolvents). *Let $R > 0$, $t \in [0,T]$ and $0 < \lambda < min\{\alpha^{-1}, \hat{\alpha}_R(h_R)^{-1}\}$. Let $v \in L^\infty(\Omega) \cap BV(\Omega)$ and $g(\lambda, ||v||_\infty) \leq R$. Then*

$$v \in \mathcal{R}(I - \lambda\mathcal{A}(t)) \quad and \quad \mathcal{J}_{R,h}(\lambda.t)v \to \mathcal{J}(\lambda; t)v \quad in\ L^1\ as\ h \downarrow 0.$$

We now combine the first part of Proposition 4.1 (a) and Proposition 4.3 to obtain the following result.

Proposition 4.4 (Range condition). *Let $R > 0$, $t \in [0,T]$, and $0 < \lambda < \{\alpha^{-1}, \hat{\alpha}_R(h_R)^{-1}\}$. Then*

$$\{v \in L^\infty \cap BV(\Omega)\ ;\ g(\lambda, ||v||_\infty) \leq R\} \subset \mathcal{R}(I - \lambda\mathcal{A}(t)).$$

Moreover, combining the second part of Proposition 4.1 (a), Proposition 4.2 and Proposition 4.3, the following assertions are obtained:

Proposition 4.5. *Let $R > 0$, $t \in [0,T]$, and $0 < \lambda < \{\alpha^{-1}, \hat{\alpha}_R(h_R)^{-1}\}$. If $v, w \in L^\infty(\Omega) \cap BV(\Omega)$ and $g(\lambda, ||v||_\infty) \vee g(\lambda, ||w||_\infty) \leq R$, then*

$$||\mathcal{J}(\lambda; t)v||_\infty \leq g(\lambda, ||v||_\infty),$$
$$||\mathcal{J}(\lambda; t)v - \mathcal{J}(\lambda; t)w||_1 \leq (1 - \lambda\alpha_R'')^{-1}||v - w||_1,$$
$$|D\mathcal{J}(\lambda; t)v|(\Omega) \leq (1 - \lambda(\alpha_R'' + \omega_R))^{-1}(|Dv|(\Omega) + \lambda\omega_R|\Omega|),$$

where $\alpha_r'' \equiv \alpha' + M_r C(\Omega)$.

Furthermore, the following result shows the denseness of $\mathcal{D}(\mathcal{A}(t))$.

Proposition 4.6 (Denseness). *Let $R > 0$ and $t \in [0,T]$. Then $X_R^\circ \cap BV(\Omega)$ and $\mathcal{D}(\mathcal{A}(t)) \cap X_R$ are both dense in X_R.*

Since the time-dependent evolution problems are treated in this paper, it is significant to establish the best possible result for the time-dependence of the resolvents of $\mathcal{A}(t)$'s.

Proposition 4.7 (Time Dependence). *Let $R > 0$, $s, t \in [0,T]$, $h \in (0, h_R)$ and let $0 < \lambda < \{\alpha^{-1}, \hat{\alpha}_R(h)^{-1}\}$. If $v \in L^\infty(\Omega)$ and $g(\lambda, ||v||_\infty) \leq R$. Then*

$$(4.7) \quad \begin{aligned} &(1 - \lambda\alpha_R(h))||\mathcal{J}_{R,h}(\lambda; s)v - \mathcal{J}_{R,h}(\lambda; t)v||_1 \\ &\qquad \leq \lambda(1 - \lambda\hat{\alpha}_R(h))^{-1}\{|D^0(\ell)v|(\Omega_\ell) + |\Omega|\}\omega_R\rho(|s-t|; R). \end{aligned}$$

For $R > 0$ and $\gamma > 0$, we choose a number λ_R satisfying $0 < \lambda_R < \min\{(2\alpha)^{-1}, \hat{\alpha}_R(h_R)^{-1}\}$ and define the function of time-dependence

$$(4.8) \qquad \theta_R^\gamma(\tau) = (1 - \lambda_R\hat{\alpha}_R(h_R))^{-1}(\gamma + |\Omega|)\omega_R\rho(\tau; R) \quad \text{for } \tau \geq 0.$$

Then, for $v \in L^\infty(\Omega)$, $g(\lambda; ||v||_\infty) \leq R$ and $|Dv|(\Omega) \leq \gamma$, we obtain

$$(4.9) \qquad (1 - \lambda\alpha_R(h))||\mathcal{J}_{R,h}(\lambda; s)v - \mathcal{J}_{R,h}(\lambda; t)v||_1 \leq \lambda\theta_R^\gamma(|s-t|).$$

5. Generation and approximation of the evolution operator

5.1. **Generation of the evolution operator.** We prove the generation theorem of the nonliner evolution operators through the results due to K.Kobayashi-S.Oharu [6] and K.Okamoto-S.Oharu [11]. We first introduce a family of positive functions $\{G_s(\cdot)\ ;\ s \in [0,T]\}$ defined by

$$G_s(r) = e^{2\alpha(T-s)}\{r + (T-s)(||C||_\infty + 1)\} \quad \text{for } r > 0. \tag{5.1}$$

These functions are modified forms of the growth functions $g(\lambda; r)$ defined in (4.5). The function $g(\lambda; r)$ represents the modulus of growth of the single resolvent $\mathcal{J}_{R,h}(\lambda; t)$. In what follows, we demonstrate that products of resolvents generate the evolution operater. Therefore, we involve the functions $G_s(r)$. Also, for $R > 0$ and $s \in [0,T]$, the symbol $X_{R,s}$ denotes an absolutely convex subset of $L^\infty(\Omega)$ defined by

$$X_{R,s} = \{v \in L^\infty(\Omega) \mid G_s(||v||_\infty) \leq R\}.$$

We see that $X_{R,s} \subset X_R^\circ$ for $s \in [0,T)$, and that $X_{R,s}$ is monotone increasing with respect to $s \in [0,T)$. The evolution operator which is constructed by means of the products of resolvents is also constructed by means of the products of finite-difference operators. This is important since so-constructed evolution operator provides physically right solutions to (IBVP).

We first prove that the family of difference operators $\{\mathcal{A}_{R,h}(t) \mid h \in (0,h_R),\ t \in [0,T]\}$ generates a family of solution operators $\{\mathcal{U}_{R,h}(t,s) \mid h \in (0,h_R)\}$ from $X_{R,s}$ into $X_{R,t}$ to the semidiscrete problem in $L^1(\Omega)$:

$$(\mathrm{ACP})_{R,h} \qquad \begin{cases} L^1(\Omega)\text{-}(d/dt)u(t) = \mathcal{A}_{R,h}(t)u(t) & \text{for } t \in (s,T), \\ u(s) = v \in X_{R,s}. \end{cases}$$

In what follows, we follow the discussions of the generation of evolution operators along Kobayashi-Oharu [6]. First, we obtain the following technical but crucial lemma for the induction argument which is basic to the generation theorem.

Lemma 5.1. *Let $R > 0$, $s,t \in [0,T]$, $h \in (0,h_R)$, $\lambda,\mu \in (0,\lambda_R)$ and $\gamma > 0$. Let $v,w \in L^\infty(\Omega) \cap BV(\Omega)$ and suppose that*

$$\begin{aligned} &g(\lambda; ||v||_\infty) \leq R, \quad (1-\lambda\hat{\alpha}_R(h))^{-1}(|Dv|(\Omega) + \lambda\omega_R|\Omega|) \leq \gamma, \\ &g(\mu; ||w||_\infty) \leq R, \quad (1-\mu\hat{\alpha}_R(h))^{-1}(|Dw|(\Omega) + \mu\omega_R|\Omega|) \leq \gamma. \end{aligned} \tag{5.2}$$

Then we have

$$\begin{aligned} &[1-\lambda\mu(\lambda+\mu)^{-1}\alpha_R(h)] \cdot ||\mathcal{J}_{R,h}(\lambda;s)v - \mathcal{J}_{R,h}(\mu;t)w||_1 \\ &\qquad \leq \lambda(\lambda+\mu)^{-1}||\mathcal{J}_{R,h}(\lambda;s)v - w||_1 + \mu(\lambda+\mu)^{-1}||v - \mathcal{J}_{R,h}(\mu;t)w||_1 \\ &\qquad\quad + \lambda\mu(\lambda+\mu)^{-1}e^{(\lambda+\mu)\alpha_R(h)}\theta_R^\gamma(|s-t|). \end{aligned}$$

Let $R > 0$, $s,\hat{s} \in [0,T)$, $t \in [s,T]$, $\hat{t} \in [\hat{s},T]$ and $\gamma > 0$. Let $v \in X_{R,s}$, $w \in X_{R,\hat{s}}$ and suppose that

$$\exp\left[(1-\lambda_R\hat{\alpha}_R(h_R))^{-1}T\hat{\alpha}_R(h_R)\right](|Dv|(\Omega) + T\omega_R|\Omega|) \leq \gamma,$$

$$\exp\left[(1-\lambda_R\hat{\alpha}_R(h_R))^{-1}T\hat{\alpha}_R(h_R)\right](|Dw|(\Omega) + T\omega_R|\Omega|) \leq \gamma.$$

Fix any $h \in (0, h_R)$ and any $\lambda, \mu \in (0, \lambda_R)$. We define a double indexed sequence $(a_{k.\ell})$ of nonnegative numbers by

$$a_{k,\ell} = \left\| \prod_{i=1}^{k} \mathcal{J}_{R,h}(\lambda; t_i)v - \prod_{j=1}^{\ell} \mathcal{J}_{R,h}(\mu; \hat{t_j})w \right\|_1 \tag{5.3}$$

for $t_i = s + i\lambda$, $\hat{t_j} = \hat{s} + j\mu$, $0 \le k \le [(t-s)/\lambda]$, $0 \le \ell \le [(\hat{t} - \hat{s})/\mu]$. From Proposition 4.1(a), it follows that $v \in \mathscr{D}(\prod_{i=1}^{k} \mathcal{J}_{R,h}(\lambda; t_i))$ so far as v satisfies (5.2). Hence the elements $\prod_{i=1}^{k} \mathcal{J}_{R,h}(\lambda; t_i)$, $0 \le k \le [(t-s)/\lambda]$, make sense. Moreover, for $0 \le k \le [(t-s)/\lambda]$,

$$\begin{aligned} &\|\textstyle\prod_{i=1}^{k} \mathcal{J}_{R,h}(\lambda; t_i)v\|_\infty \le (1-\lambda\alpha)^{-k}(\|v\|_\infty + k\lambda(\|C\|_\infty + 1)) \le R, \\ &|D\textstyle\prod_{i=1}^{k} \mathcal{J}_{R,h}(\lambda; t_i)v|(\Omega) \le (1 - \lambda\hat{\alpha}_R(h))^{-k}(|Dv|(\Omega) + k\lambda\omega_R|\Omega|) \le \gamma. \end{aligned} \tag{5.4}$$

Likewise, the corresponding estimates for $\prod_{j=1}^{\ell} \mathcal{J}_{R,h}(\mu; \hat{t_j})w$ are obtained for w and $0 \le \ell \le [(\hat{t} - \hat{s})/\mu]$.

Now the application of Lemma 5.1 implies the relations

$$\begin{aligned} [1 - \lambda\mu(\lambda+\mu)^{-1}\alpha_R(h)] \cdot a_{k,\ell} &\le \lambda(\lambda+\mu)^{-1}a_{k,\ell-1} + \mu(\lambda+\mu)^{-1}a_{k-1,\ell} \\ &\quad + 3\lambda\mu(\lambda+\mu)^{-1}e^{(\lambda+\mu)\alpha_R(h)}\theta_R^\gamma(|t_k - \hat{t}_\ell|). \end{aligned} \tag{5.5}$$

In view of (5.4) and (5.5) for the sequence $(a_{k,\ell})$, we obtain the following lemma.

Lemma 5.2 (Comparison lemma). *Let $R, s, \hat{s}, t, \hat{t}$ and γ be as above. Let $0 < \varepsilon \le T$, $|\eta| < \varepsilon$, $0 < \kappa < \varepsilon - |\eta|$, $\tau \in [0, T]$ and $\nu \in (0, \lambda_R)$. Let $z \in L^\infty(\Omega) \cap BV(\Omega)$ and suppose that*

$$g(\lambda; \|z\|_\infty) \le R \quad \text{and} \quad (1 - \nu\hat{\alpha}_R(h))^{-1}(|Dz|(\Omega) + \nu\omega_R|\Omega|) \le \gamma.$$

Let $0 \le k \le [(t-s)/\lambda]$, $0 \le \ell \le [(\hat{t} - \hat{s})/\mu]$ and put

$$c_{k,\ell}(\eta) = \left((t_k - \hat{t}_\ell - \eta)^2 + k\lambda^2 + \ell\mu^2\right)^{1/2}.$$

Then for the sequence $a_{k.\ell}$ defined by (5.3) we obtain the following comparison result :

$$\begin{aligned} (1 - \lambda\alpha_R(h))^k(1 - \mu\alpha_R(h))^\ell \cdot a_{k,\ell} &\le \|v - \mathcal{J}_{R,h}(\nu; \tau)z\|_1 + \|\mathcal{J}_{R,h}(\nu; \tau)z - w\|_1 \\ &+ c_{k,\ell}(s - \hat{s})\left(\nu^{-1}\|\mathcal{J}_{r,h}(\nu; \tau)z - z\|_1 + 3e^{(\lambda+\mu+\nu)\alpha_R(h)}\theta_R^\gamma(T)\right) \\ &+ 3e^{(\lambda+\mu+\nu)\alpha_R(h)}\ell\mu\left(\kappa^{-1}\theta_R^\gamma(T)c_{k,\ell}(\eta) + \theta_R^\gamma(\varepsilon)\right). \end{aligned}$$

Using this lemma, we first construct an evolution operator $\mathcal{U}_{R,h}(t,s)$ associated with the Cauchy problem $(ACP)_{R,h}$ for the family of operators $\{\mathcal{A}_{R,h}(t)\}$.

Proposition 5.1. *Let $R > 0$, $s \in [0, T)$ and $h \in (0, h_R)$. For each $v \in X_{R,s}$, the product formula*

$$\mathcal{U}_{R,h}(t,s)v \equiv L^1(\Omega)\text{-}\lim_{\lambda \downarrow 0} \prod_{i=1}^{[(t-s)/\lambda]} \mathcal{J}_{R,h}(\lambda; s + i\lambda)v \tag{5.6}$$

hold for each $t \in [s, T]$ and the convergence is uniform on $[s, T]$ with respect to t. Furthermore, the family $\{\mathcal{U}_{R,h}(t,s)\ ;\ 0 \le s \le t \le T\}$ has the properties (a) through (e) below:

(a) *For* $s,t \in [0,T]$ *with* $s \le t$, $\mathcal{U}_{R,h}$ *maps* $X_{R,s}$ *into* $X_{R,t}$ *and*

$$\mathcal{U}_{R,h}(r,r) = I, \quad \mathcal{U}_{R,h}(t,s)\mathcal{U}_{R,h}(s,r) = \mathcal{U}_{R,h}(t,r) \quad \text{on } X_{R,r}$$

for $0 \le r \le s \le t \le T$.

(b) *For* $s \in [0,T)$ *and* $v \in X_{R,s}$, $\mathcal{U}_{R,h}(\cdot,s)v \in C^1([s,T];L^1(\Omega))$ *and*

$$L^1(\Omega)\text{-}(\partial/\partial t)\mathcal{U}_{R,h}(t,s)v = \mathcal{A}_{R,h}(t)\mathcal{U}_{R,h}(t,s)v \quad \text{for } t \in [s,T].$$

(c) *For* $v \in X_{R,s}$,

$$||\mathcal{U}_{R,h}(t,s)v||_\infty \le e^{\alpha(t-s)}(||v||_\infty + (t-s)||C||_\infty).$$

(d) *For* $v,w \in X_{R,s}$,

$$||\mathcal{U}_{R,h}(t,s)v - \mathcal{U}_{R,h}(t,s)w||_1 \le e^{\alpha_R(h)(t-s)}||v-w||_1.$$

(e) *For* $v \in X_{R,s} \cap BV(\Omega)$,

$$|D\mathcal{U}_{R,h}(t,s)v|(\Omega) \le e^{\hat{\alpha}_R(h)(t-s)}(|Dv|(\Omega) + (t-s)\omega_R|\Omega|).$$

Remark 5.1. *We have proved the continuity of* $\mathcal{U}_{R,h}(t,s)$ *with respect to* t *in Proposition 5.1 (b). Actually, for each* $v \in X_{R,0}$, $L^1(\Omega)$*-valued function* $\mathcal{U}_{R,h}(t,s)v$ *is continuous over* $0 \le s \le t \le T$.

Also, we use the convergence of products of resolvents (Proposition 4.3) and in tern construct the evolution operator $\{\mathcal{U}(t,s) \; ; \; 0 \le s \le t \le T\}$.

Theorem 5.1. *There exists an evolution operator* $\{\mathcal{U}(t,s) \; ; \; 0 \le s \le t \le T\}$ *on the subspace* $L^\infty(\Omega)$ *of* $L^1(\Omega)$ *such that*

$$\mathcal{U}(t,s)v \equiv L^1(\Omega)\text{-}\lim_{\lambda \downarrow 0} \prod_{i=1}^{[(t-s)/\lambda]} \mathcal{J}(\lambda; s+i\lambda)v \tag{5.7}$$

holds for $v \in L^\infty(\Omega) \cap BV(\Omega)$, $0 \le s \le t \le T$ *and the convergence is uniform for* $0 \le s \le t \le T$.

5.2. **Approximation theorem and product formula.** We are now in a position to obtain the desired nonlinear evolution operator $\mathcal{U}(t,s)$. We can also prove the two types of convergence theorems for the evolution operators. The next theorem is the Tortter type convergence theorem of the evolution operator :

Theorem 5.2. *Let* $R > 0$, $s \in [0,T)$ *and* $h \in (0,h_R)$. *For* $v \in X_{R,s}$, *we have*

$$||\mathcal{U}_{R,h}(t,s)v - \mathcal{U}(t,s)v||_1 \to 0 \quad \text{as } h \downarrow 0 \tag{5.8}$$

and the convergence is uniform for $t \in [s,T]$.

Applying Proposition 5.1 and Theorem 5.2, we prove the several properties of the evolution operator. This is the first main theorem together with Theorem 5.1.

Theorem 5.3. *Let* $\{\mathcal{U}(t,s) \; ; \; 0 \le s \le t \le T\}$ *be the two parameter family of operators defined by* (5.7). *Then it is an evolution operator on the subspace* $L^\infty(\Omega)$ *of* $L^1(\Omega)$ *with the following properties:*

(*a*) *For* $r,s,t \in [0,T]$ *with* $r \le s \le t$, $\mathcal{U}(t,s)$ *maps* $L^\infty(\Omega)$ *into itself,*

$$\mathcal{U}(s,s) = I \quad \text{and} \quad \mathcal{U}(t,s)\mathcal{U}(s,r) = \mathcal{U}(t,r) \quad \text{on } L^\infty(\Omega).$$

(*b*) *For* $v \in L^\infty(\Omega)$, *the* $L^1(\Omega)$*-valued function* $\mathcal{U}(t,s)v$ *is continuous over the triangle* $0 \le s \le t \le T$.

(*c*) *For* $v \in L^\infty(\Omega)$,

$$||\mathcal{U}(t,s)v||_\infty \le e^{\alpha(t-s)}(||v||_\infty + (t-s)||C||_\infty).$$

(*d*) *For* $v, w \in X_{R,s}$,

$$||\mathcal{U}(t,s)v - \mathcal{U}(t,s)w||_1 \le e^{\alpha''_R(t-s)}||v-w||_1.$$

(*e*) *For* $v \in X_{R,s} \cap BV(\Omega)$,

$$|D\mathcal{U}(t,s)v|(\Omega) \le e^{(\alpha''_R+\omega_R)(t-s)}(|Dv|(\Omega) + (t-s)\omega_R|\Omega|).$$

We next demonstrate that the product formula in terms of the difference operators $\mathcal{C}_{r,h}(t)$.

Proposition 5.2. *Let* $R > 0$, $s \in [0,T)$ *and* $h \in (0, h_R)$. *For* $v \in X_{R,s}$,

$$\left\| \prod_{i=0}^{[(t-s)/h]-1} \mathcal{C}_{R,h}(s+ih)v - \mathcal{U}_{R,h}(t,s)v \right\|_1 \to 0 \quad \text{as } h \downarrow 0 \tag{5.9}$$

and the convergence is uniform for $t \in [s,T]$.

Namely, we get the Chernoff type convergence theorem for the evolution operator.

Theorem 5.4. *Let* $R > 0$, $s \in (0,T]$ *and* $v \in X_{R,s}$. *Then*

$$\mathcal{U}(t,s)v = \lim_{h\downarrow 0} \prod_{i=0}^{[(t-s)/h]} \mathcal{C}_{R,h}(s+ih)v \quad \text{in } L^1(\Omega) \tag{5.10}$$

and the convergence is uniform for $t \in [s,T]$.

Applying Proposition 5.1 and Theorem 5.2, one can show that the evolution operator $\{\mathcal{U}(t,s)\ ;\ 0 \le s \le t \le T\}$ provides entropy solutions to (IBVP).

Proposition 5.3. *For* $v \in L^\infty(\Omega)$, *the* $L^1(\Omega)$*-valued function* $u(\cdot) \equiv \mathcal{U}(\cdot,0)v$ *gives a entropy solution to the problem (IBVP).*

Finally, we give the proof of the following Proposition. For each $R > 0$ and $t \in [0,T]$ we define

$$\widehat{D}_R(t) = \{v \in X_R^\circ \cap BV(\Omega)\ ;\ \liminf_{\lambda\downarrow 0} \lambda^{-1}||\mathcal{J}(\lambda;t)v - v||_1 < \infty\}.$$

This set is called the generalized domain. It is easily seen that $\mathscr{D}(\mathcal{A}(t)) \cap X_R^\circ \subset \widehat{D}_R(t)$ and $\widehat{D}_R(t)$ is independent of t. We write $\widehat{D}_R$ for the common set.

Proposition 5.4. *Suppose that* $\limsup_{\tau\downarrow 0} \tau^{-1}\rho(\tau;r) < \infty$ *for all* $r > 0$. *Let* $v \in L^\infty(\Omega) \cap BV(\Omega)$. *Assume that there is a positive number* $R = R(v)$ *such that* $v \in X_{R,0}$ *and* $v \in \widehat{D}_R$. *Then the function* $u(x,t) = [\mathcal{U}(t,0)v](x)$ *gives a BV-entropy solution to (IBVP).*

6. Uniqueness of BV-entropy solutions

In this section, we prove the uniqueness of BV-entropy solutions.

Theorem 6.1. *Let u, v be a pair of BV-entropy solutions with initial values u_0 and v_0, respectively. Then under the growth condition $(H.2)$, we have*

$$||u(\cdot,t)-v(\cdot,t)||_1 \le e^{\alpha' t}||u_0(\cdot)-v_0(\cdot)||_1$$

for some positive constant α'. In particular, a BV-entropy solution is unique.

We set $Q \equiv \Omega \times [0,T]$ and $\partial Q \equiv \partial\Omega \times [0,T]$. Furthermore, we set the degenerate domain of β as follows :

$$E = \{k \in \mathbb{R} \mid \beta'(k) = 0\}, \qquad \mathscr{E}_u \equiv \{x \in \Omega \mid \nabla_x \beta(u(x)) = 0\} \tag{6.1}$$

and $Q\backslash\mathscr{E}_u \equiv (\Omega\backslash\mathscr{E}_u)\times[0,T]$. Since β is monotone nondecreasing, $\mathcal{L}^N(E) \ge 0$ hold. Hence, there may exist $x, y \in E$ such that

$$\mathrm{sgn}(x-y) \ne \mathrm{sgn}(\beta(x)-\beta(y)).$$

This causes difficulties for treating our degenerate equations in the case of β is monotone nondecreasing.

We then introduce a family of functions $\{\Phi_j\ ;\ j \in \mathbb{N}\} \subset C^2(\mathbb{R})$ defined by

$$\Phi_j(s) = \begin{cases} -(j\pi/2)^{-1}\cos[(j\pi/2)s] + j^{-1} & (|s| \le j^{-1}), \\ |s| & (|s| \ge j^{-1}). \end{cases}$$

These functions approximate the function $\Phi(s) = |s|$ and have the following properties:

$$\lim_{j\to\infty} \Phi_j(s) = |s|, \quad \lim_{j\to\infty} \Phi_j'(s) = \mathrm{sgn}(s), \quad \lim_{j\to\infty} s\Phi_j''(s) = 0, \tag{6.2}$$

$$\Phi_j''(s) \ge 0, \quad \mathrm{supp}\ \Phi_j'' = [-j^{-1}, j^{-1}], \quad |s\Phi_j''(s)| \le \pi/2. \tag{6.3}$$

Let $\xi \otimes \theta \in \mathscr{D}^+(\mathbb{R}^N) \otimes \mathscr{D}^+(0,T)$. In addition, we introduce symmetric functions $\rho^{(1)} \in \mathscr{D}^+(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \rho^{(1)}dx = 1$ and $\mathrm{supp}[\rho^{(1)}] \subset [-1,1]$. Similarly, we use spherically symmetric functions $\rho^{(N)} \in \mathscr{D}^+(\mathbb{R}^N)$ satsfying $\int_{\mathbb{R}^N} \rho^{(N)}dx = 1$ and $\mathrm{supp}[\rho^{(N)}(x)] \subset \{|x| \le 1\}$. Let $\delta > 0$ be a small number such that $\rho_\delta^{(1)}(x) = (1/\delta)\rho(x/\delta)$ and $\rho_\delta^{(N)}(x) = (1/\delta^N)\rho^{(N)}(x/\delta)$ is continuous on $\mathbb{R}$ and $\mathbb{R}^N$ with the properties that $\int \rho_\delta^{(1)}(t)\theta(t)dt \to \theta(0)$ and $\int \rho_\delta^{(N)}(x)\xi(x)dx \to \xi(0)$ as $\delta \downarrow 0$. We then define

$$\xi_\delta(x,y) = \xi\left(\frac{x+y}{2}\right)\rho_\delta^{(N)}\left(\frac{x-y}{2}\right) \tag{6.4}$$

in $\mathscr{D}^+(\mathbb{R}^{2N})$ and

$$\theta_\delta(t,s) = \theta\left(\frac{t+s}{2}\right)\rho_\delta^{(1)}\left(\frac{t-s}{2}\right) \tag{6.5}$$

belonging to $\mathscr{D}^+((0,T)^2)$. We now employ the test functions φ_δ defined by

$$\varphi_\delta(x,y,t,s) = \xi_\delta(x,y)\theta_\delta(t,s). \tag{6.6}$$

To show Theorem 6.2, we employ the following Lemma which is first shown by J. Carrillo [3] and is a key estimate for proving the uniqueness of entropy solutions associated with the degenerate parabolic equation.

Lemma 6.1. *Let u be a BV-entropy solution. Then, the following equality hold :*

$$\int_\Omega \operatorname{sgn}(u-k)[-(u-k)\varphi_t + \nabla\beta(u)\cdot\nabla\varphi - [A(x,t,u)-A(x,t,k)]\cdot\nabla\varphi$$
$$+[B(x,t,u)+\nabla\cdot A(x,t,k)]\varphi]dx = -\int_\Omega \Phi_j''(\beta(u)-\beta(k))|\nabla\beta(u)|^2\varphi dx,$$

for $\varphi \in \mathscr{D}^+(\mathbb{R}^N \times (0,T))$ and $k \notin E$.

Using Lemma 6.1, we obtain the following Proposition. To prove the uniqueness of entropy solutions, this result is crucial.

Theorem 6.2. *For u, v are BV-entropy solutions to (P), the next inequality hold :*

$$\int_0^T\int_\Omega |u-v|\varphi_t dxdt \geq \int_0^T\int_\Omega \operatorname{sgn}(u-v)([\nabla\beta(u)-\nabla\beta(v)]\cdot\nabla\varphi \tag{6.7}$$
$$-[A(x,t,u)-A(x,t,v)]\cdot\nabla\varphi + [B(x,t,u)-B(x,t,v)]\varphi)dxdt,$$

for $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))^+$.

Outline of the proof of Uniqueness Theorem.

Let u, v be a pair of BV-entropy solutions. We put $k = v(y,s)$ and $\varphi = \varphi_\delta(x,y,t,s)$ in the definition of BV-entropy solution u. Then

$$I_1 \equiv \int_{Q\times Q} \operatorname{sgn}(u(x,t)-v(y,s))[-(u-v)(\varphi_\delta)_t + \nabla_x\beta(u)\cdot\nabla_x\varphi_\delta$$
$$-[A(x,t,u)-A(x,t,v)]\cdot\nabla_x\varphi_\delta + [B(x,t,u)+\nabla_x\cdot A(x,t,v)]\varphi_\delta]dxdydtds.$$

Similarlly, we also have following :

$$I_2 \equiv \int_{Q\times Q} \operatorname{sgn}(v(y,s)-u(x,t))[-(v-u)(\varphi_\delta)_s + \nabla_y\beta(v)\cdot\nabla_y\varphi_\delta$$
$$-[A(y,s,v)-A(y,s,u)]\cdot\nabla_y\varphi_\delta + [B(y,s,v)+\nabla_y\cdot A(y,s,u)]\varphi_\delta]dxdydtds.$$

We then set $I \equiv I_1 + I_2$. Using the estimates for I, we get the desired result. By the definition of BV-entropy solution u, the following inequality hold :

$$\int_{Q\times\mathscr{E}_v\times[0,T]} \operatorname{sgn}(u(x,t)-v(y,s))[-(u-v)(\varphi_\delta)_t + \nabla_x\beta(u)\cdot\nabla_x\varphi_\delta$$
$$-[A(x,t,u)-A(x,t,v)]\cdot\nabla_x\varphi_\delta + [B(x,t,u)+\nabla_x\cdot A(x,t,v)]\varphi_\delta]dxdydtds \leq 0.$$

Hence, in view of the domain of integral and the above estimate, we have

$$I_1 = \int_{Q\times Q\backslash\mathscr{E}_v} + \int_{Q\times\mathscr{E}_v\times[0,T]} \leq \int_{Q\times Q\backslash\mathscr{E}_v}.$$

Application of Lemma 6.1 on the right-hand side, then implies

$$I_1 \leq -\int_{Q\times Q\backslash\mathscr{E}_v} \Phi_j''(\beta(u)-\beta(v))|\nabla\beta(u)|^2\varphi_\delta dxdydtds.$$

Moreover, since $\nabla\beta(u)$ includes the integrand on the right-hand side, the integral on the right-hand side is equal to

$$-\int_{Q\backslash\mathscr{E}_u\times Q\backslash\mathscr{E}_v} \Phi_j''(\beta(u)-\beta(v))|\nabla\beta(u)|^2\varphi_\delta dxdydtds.$$

Similarly, we get the next estimate.

$$I_2 \leq -\int_{Q\backslash\mathscr{E}_u\times Q\backslash\mathscr{E}_v} \Phi_j''(\beta(v)-\beta(u))|\nabla\beta(v)|^2\varphi_\delta dxdydtds.$$

Consequently, we deduce the estimate

$$I \leq -\int_{Q\backslash\mathscr{E}_u\times Q\backslash\mathscr{E}_v} \Phi_j''(\beta(u)-\beta(v))[|\nabla\beta(u)|^2+|\nabla\beta(v)|^2]\varphi_\delta dxdydtds. \tag{6.8}$$

Next, we consider the diffusion, convection and remaining integral terms respectively. In this paper, we only consider the diffusion term as follows :

$$\begin{aligned}
&\int_{Q\times Q} \mathrm{sgn}(u(x,t)-v(y,s))[\nabla_x\beta(u)\cdot\nabla_x\varphi_\delta - \nabla_y\beta(v)\cdot\nabla_y\varphi_\delta]dxdydtds\\
&= \int_{Q\times Q} \mathrm{sgn}(u(x,t)-v(y,s))[\nabla_x\beta(u)-\nabla_y\beta(v)]\cdot[\nabla_x\varphi_\delta+\nabla_y\varphi_\delta]dxdydtds\\
&\quad -\int_{Q\times Q} \mathrm{sgn}(u(x,t)-v(y,s))[\nabla_x\beta(u)\cdot\nabla_y\varphi_\delta - \nabla_y\beta(v)\cdot\nabla_x\varphi_\delta]dxdydtds
\end{aligned}$$

We then estimate the second term on the right-hand side of the above equality; in paticuler, we wish to estimate the first term of the integrand $\nabla_x\beta(u)\cdot\nabla_y\varphi_\delta$. Considering that $\nabla_x\beta(u)$ is included the integrand and using the relation $\mathrm{sgn}(u-k)=\mathrm{sgn}(\beta(u)-\beta(k))$ for $k\notin E$, we have

$$\begin{aligned}
&-\int_{Q\times Q} \mathrm{sgn}(u-v)\nabla_x\beta(u)\cdot\nabla_y\varphi_\delta dxdydtds\\
&= -\int_{Q\backslash\mathscr{E}_u\times Q} \mathrm{sgn}(\beta(u)-\beta(v))\nabla_x\beta(u)\cdot\nabla_y\varphi_\delta dxdydtds
\end{aligned}$$

Moreover, using the Gauss divergence theorem, the above integral is equal to

$$\begin{aligned}
&-\lim_j[\int_{Q\backslash\mathscr{E}_u}\int_{\partial Q} \Phi_j'(\beta(u)-\beta(v))\nabla_x\beta(u)\cdot\mathbf{n}(y)\varphi_\delta d\mathcal{H}^{N-1}dydtds\\
&\qquad +\int_{Q\backslash\mathscr{E}_u\times Q} \Phi_j''(\beta(u)-\beta(v))\nabla_x\beta(u)\cdot\nabla_y\beta(v)\varphi_\delta dxdydtds] \equiv I_{d,1}.
\end{aligned}$$

Using the boundary-layer sequence $\{\zeta_{\delta'}\}$, the first term on the right-hand side is equal to

$$\lim_j\lim_{\delta'\downarrow 0}\left[\int_{Q\backslash\mathscr{E}_u}[\int_Q \nabla_x\beta(u)\cdot\nabla\zeta_{\delta'}(y)\Phi_j'(\beta(u)-\beta(v))\xi_\delta\theta_\delta dxdt]dyds\right].$$

We here notice that the convergence as $\delta\downarrow 0$ of the above integral is uniform with respect to $\delta'>0$ and so that by the Lemma of Moor we have

$$\lim_{\delta\downarrow 0}\lim_j\lim_{\delta'\downarrow 0}[\text{the above integral}] = \lim_{\delta'\downarrow 0}\lim_j\lim_{\delta\downarrow 0}[\text{the above integral}].$$

By the Lebesgue convergence theorem and the Neumann boundary condition (1.3), the right-hand side goes to 0. Similarly, we get the next identity

$$\begin{aligned}
&\int_{Q\times Q} \operatorname{sgn}(u-v)\nabla_y\beta(v)\cdot\nabla_x\varphi_\delta dxdydtds \\
&= \lim_j[\int_{Q\setminus\mathscr{E}_v}\int_{\partial Q}\Phi_j'(\beta(u)-\beta(v))\nabla_y\beta(v)\cdot\mathbf{n}(x)\varphi_\delta d\mathcal{H}^{N-1}dxdtds \\
&\qquad -\int_{Q\times Q\setminus\mathscr{E}_v}\Phi_j''(\beta(u)-\beta(v))\nabla_y\beta(v)\cdot\nabla_x\beta(u)\varphi_\delta dxdydtds] \equiv I_{d,2}.
\end{aligned}$$

In the same way as above, the first term on the right-hand side of the above equality converges to 0 as $\delta\downarrow 0$.

Finally, we derive the left hand side on the aimed inequality (6.7). We begin by transforming the left-hand side of the very first identity of the present proof.

$$\int_{Q\times Q}(u(\varphi_\delta)_t - v(\varphi_\delta)_s)dxdydtds = -\int_{Q\times Q}\varphi_\delta D_t u dxdyds + \int_{Q\times Q}\varphi_\delta D_s v dxdydt.$$

In order to treat the right-hand side, we need the next lemma :

Lemma 6.2. *For a BV solution u of $(IBVP)$, the following equality hold :*

$$\int_\Omega[\int_0^T\varphi_\delta D_t u]dx = -\int_0^T\int_\Omega[\int_k^u\Phi_j'(\beta(r)-\beta(k))dr]\xi_\delta(x,y)\frac{\partial\theta_\delta}{\partial t}(t,s)dxdt, \tag{6.9}$$

for all $k\in\mathbb{R}$.

Consequently, we sum up the above results.

$$\begin{aligned}
I = &\int_{Q\times Q}\operatorname{sgn}(u-v)(-[u(\varphi_\delta)_t - v(\varphi_\delta)_s] + [\nabla_x\beta(u)-\nabla_y\beta(v)]\cdot[\nabla_x\varphi_\delta+\nabla_y\varphi_\delta] \\
&-[A(x,t,u)-A(y,t,v)]\cdot[\nabla_x\varphi_\delta+\nabla_y\varphi_\delta] + [B(x,t,u)-B(y,t,v)]\varphi_\delta)dxdydtds \\
&+I_{d,1}+I_{d,2}.
\end{aligned}$$

Since the inequality (6.8), we get the following estimate :

$$\begin{aligned}
&\lim_{\delta\downarrow 0}[\int_{Q\times Q}\operatorname{sgn}(u-v)(-[u(\varphi_\delta)_t - v(\varphi_\delta)_s] + [\nabla_x\beta(u)-\nabla_y\beta(v)]\cdot[\nabla_x\varphi_\delta+\nabla_y\varphi_\delta] \\
&\quad -[A(x,t,u)-A(y,t,v)]\cdot[\nabla_x\varphi_\delta+\nabla_y\varphi_\delta] + [B(x,t,u)-B(y,t,v)]\varphi_\delta)dxdydtds] \\
&\le -\lim_{\delta\downarrow 0}\lim_j\int_{Q\setminus\mathscr{E}_u\times Q\setminus\mathscr{E}_v}\Phi_j''(\beta(u)-\beta(v))[\nabla_x\beta(u)+\nabla_y\beta(v)]^2\varphi_\delta dxdydtds.
\end{aligned}$$

Hence, we obtain the next result :

$$\begin{aligned}
\int_Q|u-v|\xi\theta_t dxdt \ge &\int_Q\operatorname{sgn}(u-v)([\nabla\beta(u)-\nabla\beta(v)]\cdot\nabla\xi\theta \\
&-[A(x,t,u)-A(x,t,v)]\cdot\nabla\xi\theta + [B(x,t,u)-B(x,t,v)]\xi\theta)dxdt.
\end{aligned}$$

Finally, we set $\varphi(x,t)=\xi(x)\theta(t)$ to get the desired result. □

From the inequality (6.7), we may show the uniequness of BV-entropy solutions of (IBVP). In fact, we put $\xi \in \chi_{\overline{\Omega}}(x) \otimes \mathscr{D}^+(0,T)$. Then we obtain the following estimate :

$$(6.10) \qquad (d/dt)\int_0^t \int_\Omega |u-v|dx \leq \alpha' \int_0^t \int_\Omega |u-v|dxdt.$$

Notice that we have used the growth condition (H.2). Hence, Gronwall's inequality can be applied to show the continuous dependence of BV-entropy solutions on initial values. Consequently, the proof of Theorem 6.1 is now complete.

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