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Computational Analysis of Orientations of Matroids

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Abstract

Matroids and oriented matroids are fundamental structures in discrete geometry. Analysis of orientations of matroids is important to investigate the relationship between matroids to oriented matroids. We introduce related results of our paper on the next two problems: testing orientability and enumerating orientations of matroids. First, we test orientability of matroids by formulating it as SAT problems. We determine orientability of matroids of rank 3 on $n \leq 12$ elements and of rank 4 on $n \leq 9$ elements. Next, we enumerate orientations of matroids again using a formulation as SAT. We show by computational experiments that rather degenerate matroids have much less orientations compared to uniform matroids.

1 Introduction

Matroids and oriented matroids are fundamental structures in discrete geometry as abstract generalizations of the combinatorial properties of geometric configurations such as vector configurations, point configurations and hyperplane arrangements. While matroids only encode incidence relations such as collinearity or coplanarity, oriented matroids in addition describe relative positions of objects. Briefly speaking, oriented matroids are extension of matroids with additional information. Thus, every oriented matroid has a corresponding matroid called an *underlying matroid*. Conversely, we call the oriented matroid as an *orientation* of the underlying matroid. While every oriented matroid has a unique underlying matroid, not every matroid has an orientation. Matroids with at least one orientation are called *orientable matroids*.

Table 1: Number of non-isomorphic simple matroids [9]

$r \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1		1											
2			1	1	1	1	1	1	1	1	1	1	1
3				1	2	4	9	23	68	383	5249	232928	28872972
4					1	3	11	49	617	185981	4884573865	*	*

r: rank n: size of the ground set $*$: non-determined value

Table 2: Number of non-isomorphic simple oriented matroids [8] [7]

$r \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2			1	1	1	1	1	1	1	1	1
3				1	2	4	17	143	4890	461053	95052532
4					1	3	12	206	181472	*	*

r: rank n: size of the ground set $*$: non-determined value

For computational analysis of discrete structures such as matroids and oriented matroids, enumeration of them is important as the source. In this paper, we investigate oriented matroids from the side of matroids. The motivation for this approach comes from the extensiveness of enumeration of matroids.

Enumeration of matroids has been studied for more than thirty years since Blackburn, Crapo and Higgs [4] determined all simple matroids on $n \leq 8$ elements in 1973 and several subsequent studies as [2] [11] have extended the coverage. The most recent result by the authors [9] extended the coverage by determining all more than 4×10^9 matroids of rank 4 on 10 elements. Table 1 summarizes the number of simple matroids of rank $r \leq 4$ on $n \leq 12$ elements determined by these studies.

Now we compare enumeration of matroids with that of oriented matroids. Oriented matroids of rank 3 on $n \leq 10$ elements and of rank 4 on $n \leq 8$ elements are determined by Finschi and Fukuda [8] [7] as in summarized in Table 2. We remark that in a case of uniform, ones of rank 3 on $n \leq 11$ elements are determined by Aichholzer and Krasser [1].

By comparing Table 1 with Table 2, we can easily see that the coverage of enumeration of matroids is more extensive than that of oriented matroids. We may think this difference occurs from the fact that the number of matroids is much less than that of oriented matroids of the same rank and on the ground set of the same size. Because of this extensiveness of enumeration of matroids, we can expect analysis of oriented matroids based on matroids can overwhelm the direct result from enumeration of oriented matroids.

In this paper, we are concerned with the next two problems as a bridge from matroids

Table 3: Results on orientability of matroids [10]

rank r	3					
size of the ground set n	7	8	9	10	11	12
# simple matroids	23	68	383	5249	232928	28872972
# orientable	22	65	365	5048	223515	26873051
# non-orientable	1	3	18	201	9413	1999921
ratio of non-orientable	0.043	0.044	0.046	0.038	0.040	0.069
rank r	4					
size of the ground set n	7	8	9	10 *		
# simple matroids	49	617	185981	*1000000*		
# orientable	48	583	173697	*731745*		
# non-orientable	1	34	12284	*268255*		
ratio of non-orientable	0.020	0.055	0.066	*0.268*		

* For a case with $r = 4$ and $n = 10$, we test randomly sampled 1000000 matroids.

to oriented matroids: testing orientability and enumerating orientations of matroids. For these problems, we introduce the results from the paper [10].

First, we discuss testing orientability. Orientability of matroids is important to understand the gap between matroids and oriented matroids. Although testing orientability of matroids is proved to be NP-complete [15], we propose a method which works satisfactory in practice. We formulate orientability of a matroid as a Boolean Satisfiability problem (SAT) and solve it with a practically well designed SAT solver. The method is based on the oriented matroid generation method by Bremner, Bokowski and Gévay [5] and Schewe [16]. The point in this method is the fact that although SAT is also NP-complete, practical heuristic methods are well studied.

We determine orientability of matroids of rank 3 on $n \leq 12$ elements and of rank 4 on $n \leq 9$ elements. The results are summarized in Table 3. Note that this result is more extensive than the result directly obtained from the existing enumeration of oriented matroids.

We obtain new insights on orientability of matroids. Immediate one is on the conjecture that asymptotically most matroids are non-orientable [3, p279]. If this conjecture is true, it is expected that the ratio of non-orientable matroids is higher among matroids on the ground set of the larger size. We can observe such a behavior especially clear in cases of rank 4.

Next, we discuss enumeration of orientations. Enumerating orientations serves as a local oriented matroid generation method in case only oriented matroids with the specific underlying matroids are concerned. The method is expected to generate oriented matroids on the rather large-sized ground set which are not contained in the existing oriented matroid enumeration. We can use the previously noted SAT formulation to find all orientations of an orientable matroid. It works well if a matroid does not have too many orientations. We present by a computational experiment that while uniform matroids have many orientations, most other matroids, especially highly degenerate ones, have much less orientations. The result suggests generating orientations goes well for many matroids in average except a few cases such as uniform ones.

2 Testing Orientability

In this section, we describe the method of testing orientability of matroids and the computational result.

A matroid M of rank r on E_n is orientable if and only if there is a chirotope χ satisfying following conditions: (i) the set $\{\{x_1, \ldots, x_r\} \mid \chi(x_1, x_2, \ldots, x_r) \neq 0\}$ is equal to the set of bases of M, and (ii) χ satisfies the 3-term Grassman-Plücker relations. The condition (i) determines whether the value of χ on each r-subset of E_n is zero or non-zero. Thus, the rest problem is whether there is an assignment of -1 and 1 to the values of χ on r-subset corresponding to bases of M.

Out method for testing orientability is to translate this problem into an instance of SAT problem and solve it with a general SAT solver. We use MiniSAT [6] in our experiment. The variables correspond to the values of χ on ordered bases of M and the Boolean values *true* and *false* correspond to 1 and -1, respectively. The constraints are 3-term Grassman-Plücker relations. The idea of using SAT is based on the methods for generation of oriented matroids by Bremner, Bokowski and Gévay [5] and Schewe [16]. However, our case is special in a point that we need only one variable for each r-subset because whether the value is zero or non-zero is fixed in a problem of testing orientability.

We add one constraint to narrow the search space. Because if χ is a chirotope then $-\chi$ is too, we may fix the value of χ on one freely selected ordered base. With this constraint, each orientation of a matroid corresponds to one satisfiable truth assignment of the SAT.

The results from [10] are summarized in Table 3. These results are developed as extension of [9]. We only investigate the cases with rank $r \geq 3$ because all matroids of rank at most 2 are orientable [3, p247-248]. Additionally, we only show the cases with $n \geq 7$ because all matroids on $n \leq 6$ elements are orientable. Because it is hard to determine orientability of all rank 4 matroids on 10 elements, we investigate randomly sampled 1,000,000 matroids to estimate the ratio of non-orientable ones, which is correct up to the range 0.001 for 95 %.

3 Enumerating Orientations

For enumeration of orientations of a matroid, we can use the SAT instance discussed in Section 2 again. Because each satisfiable truth assignment of the SAT instance corresponds to an orientation, enumeration of satisfiable truth assignments achieves the end. Note that orientations generated by this method include isomorphic ones. For distinctness, we call orientations including isomorphic ones as *labeled orientations.*

The computational cost for generation of orientations is at least proportional to the number of solutions in principle. Conversely, our generation method will still work well enough if the number of labeled orientations is small enough. Thus, we study the behavior of the number of labeled orientations of matroids by counting them.

While orientations generated by our method include isomorphic ones, there are many applications in which isomorph-free orientations are of interest and existing database of oriented matroids [8] [1] are designed to store them. Our method has redundancy for generation of isomorph-free ones. However, if the number of labeled orientations is small enough, our generation method still works well. On the other hand, if the number of isomorphism classes is too large, they are hard to enumerate intrinsically. Thus, we also investigate the number of isomorphism classes of orientations of matroids.

Figure 1 shows the number of orientations in two senses of simple matroids of rank 3 on 9 elements. First, we discuss the number of labeled orientations. We can observe easily that the number is positively correlated with the number of bases. While the uniform matroid (the rightmost in each graph) has as many as 4.0×10^{11} labeled orientations, the median is only 45056. It suggests our generation works well enough in average and especially well for highly-degenerate matroids.

Next, we compare the number of labeled orientations to that of isomorphism classes. Our computational result shows that while the ratio of the number of labeled orientations to that of isomorphism classes is about 9.0×10^7 on the uniform matroid, the median is only 583. Thus we can conclude that our generation method will achieve satisfactory performance enough even for generation of isomorphism classes on most matroids except a few matroids including uniform ones.

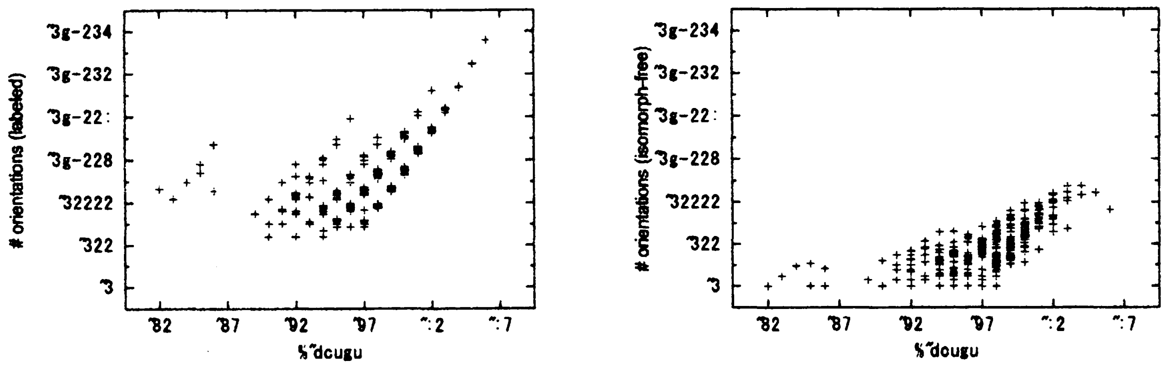

Figure 1: Number of orientations and bases of simple matroids on 9 elements

4 Concluding Remarks

In this paper, we present computational analysis of orientations of matroids using the database of matroids. We investigate the two problems: testing orientability and enumerating orientations. Solving these problems fills the gap between matroids and oriented matroids.

For oriented matroids, several studies have been made on practical methods for the realizability problem which work satisfactory for small instances including non-uniform ones [14] [13] [12]. Combining these methods with our method for enumerating orientations, we can determine realizability of matroids. This approach will be useful in investigation of properties of point configurations and hyperplane arrangement, which are fundamental interests in discrete geometry.

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