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The Counting Board Algebra and its Applications

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1 Japanese Mathematics in the *Edo* Period

- Two Mathematicians
  - 関孝和 SEKI, Takakazu ca.1640 – 1708
  - 建部賢弘 TAKEBE, Katahiro 1664 – 1739
- Two main sources from Chinese mathematics
  - 朱世傑：算学啓蒙, Zhu Shijie: *Suanxue Qimeng*, 1299,
    訓点 Japanese translations 1658, 1673,
    建部賢弘：算学啓蒙諺解大成 Annotation by TAKEBE Katahiro, 1690
  - 程大位：算法統宗, Cheng Dawei: *Suanfa Tongzong*, 1592,
    訓点 Japanese translation 1676

2 Influence of the *Suanfa Tongzong*

- Chapter names of the *Nine Chapters* are cited in Introduction of the *Suanfa Tongzong*. Japanese mathematicians of the Edo period did not have access to the *Nine Chapters*.
- The *Suanfa Tongzong* was very popular in Japan because it explained in detail how to manipulate the abacus.
- The book had great influence on the Japanese popular mathematics.

3 Influence of the *Suanxue Qimeng*

3.1 Decimal number system and Counting rods

- Red rods : positive integer
- Black rods : negative integer
- Number of rods : Absolute number of the integer

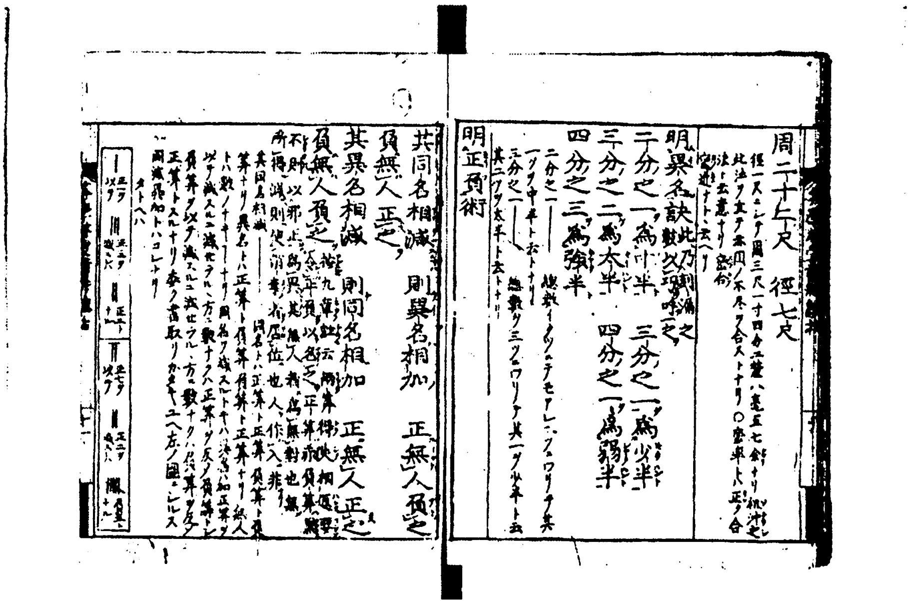


Figure 1: Positive and Negative integers in the *Suanxue Qimeng* (annotated by Takebe, 1690)

- "Subtract for the same, Add for the different" 同減異加： subtraction of integers
- "Subtract for the different, Add for the same" 異減同加： addition of integers

3.2 Extraction of a root in the *Suanxue Qimeng*

The last paragraph in the Preliminaries of the *Suanxue Qimeng* reads as follows: (See Figure 3 and 4.)

> **The Clarification of the method of extraction of a root** Place the number to be extracted in the Reality row. Using the lower rows, extract it by the ADDITION of integers. (同加異減) (Takebe's annotation: "This is a method known from antiquity!" Put the Quotient. Multiply it by the Corner row and ADDing till the Square row and subtract at the Reality row.)

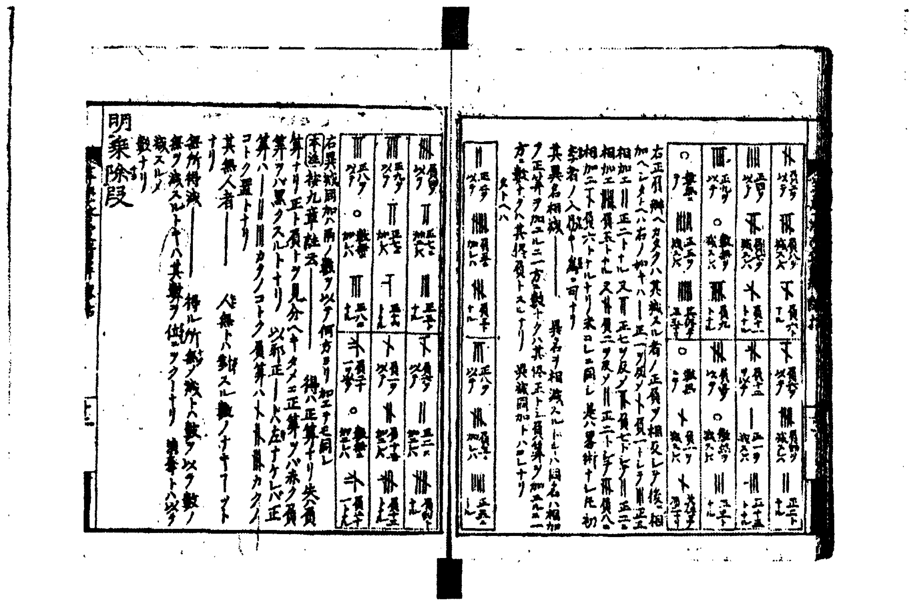
As seen at the last paragraph in the Preliminaries, the extraction of a square (cubic) root was assumed well established in the *Suanxue Qimeng*. In fact, there were — problems on the extraction of a square root and — problems on the extraction of a cubic root in Chapter — (少広) of the *Nine Chapters.*

3.3 Constructive Division (組み立て除法) (Horner's method)

Let $P(x) = a + bx + cx^2$ be a polynomial with numerical coefficients and q a number.

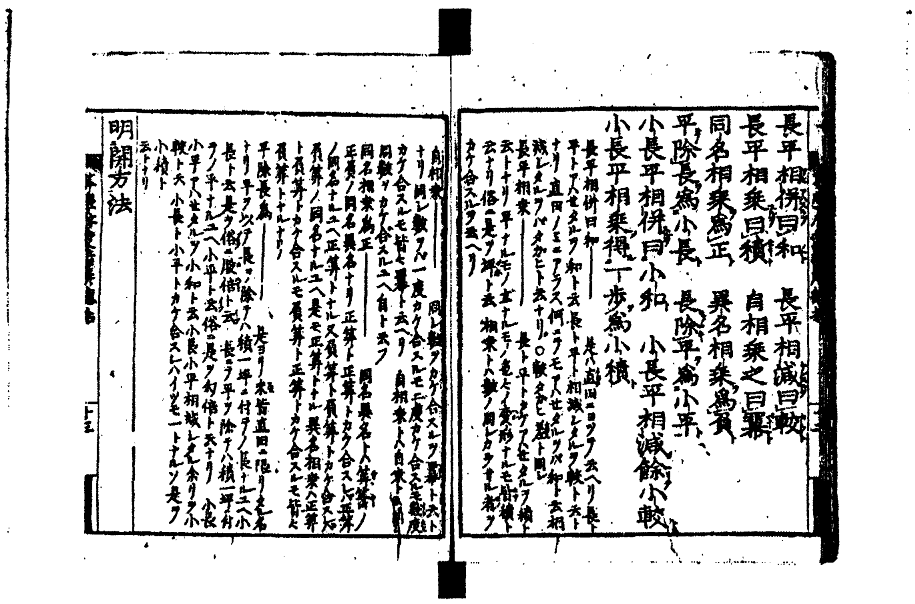
Seki and Takebe learned from the *Suanxue Qimeng* how to calculate the coefficients a', b', c' of

$$P(x) = a' + b'(x-q) + c'(x-q)^2$$



明乘除段

Figure 2: Long annotation by Takebe Katahiro on Subtraction and Addition of integers



長平相併曰和 長平相減曰較
長平相乘曰積 自相乘之曰冪
同名相乘為正 異名相乘為負
平除長為小長 長除平為小平
小長平相併曰小和 小長平相減餘小較
小長平相乘得一步為小積

明開方法

Figure 3: The Last paragraph in the Preliminaries of the *Suanxue Qimeng*.

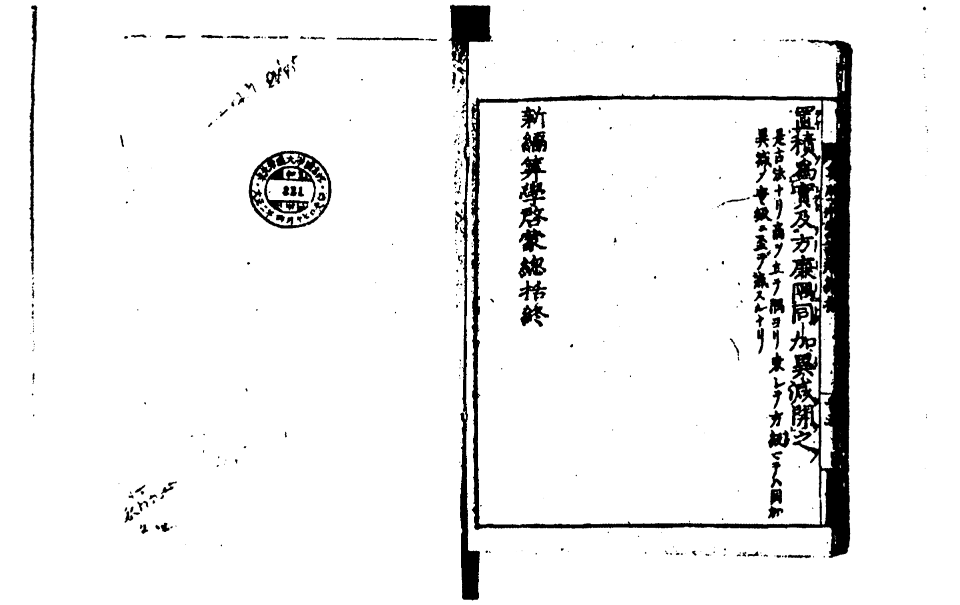


Figure 4: End of the Preliminaries of the *Suanxue Qimeng*.

from the given q, a, b, c. This algorithm is the constructive division.

3.4 Counting board (算盤)

億	千	百	十	万	千	百	十	一	分	厘	毛
								商			
								実			
								方			
								廉			
								隅			
								三			
								四			

A polynomial $a + bx + cx^2$ with numerical coefficients was represented by a column vector on a counting board.

	Quotient row 商
a	Reality row 実級
b	Square row 方級
c	Side row 廉級
	Corner row 隅級

$a' + b'(x - q) + c'(x - q)^2$ was represented as follows:

q	Quotient row 商
a'	Reality row 実級
b'	Square row 方級
c'	Side row 廉級
	Corner row 隅級

For the convenience we transpose the column vector to a row vector.

The calculation of a', b', c' from the q, a, b, c is a series of operations from the bottom to the top.

	a	b	c	
$q)$	a	b	c	
	$+)\ (b+cq)q$	$+)\ cq$		
	$a+(b+cq)q$	$b+cq$	c	
		$+)\ cq$		
q	$a+(b+cq)q$	$b+2cq$	c	

This algorithm can be written by a computer language: A row on a computing board is a memory in a computer. The counting board is a computer with a few memories:

```
Q=0:A=a:B=b:C=c:D=0
R=q:Q=Q+R
C=C+DR:B=B+CR:A=A+BR
C=C+DR:B=B+CR
C=C+DR
print Q,A,B,C,D
```

Let $P(x) = a+bx+cx^2 = a'+b'(x-q)+c'(x-q)^2 = a''+b''(x-q-q')+c''(x-q-q')^2$. To calculate a'', b'', c'' we apply the same algorithm two times. In a computer language, we can state the algorithm easily:

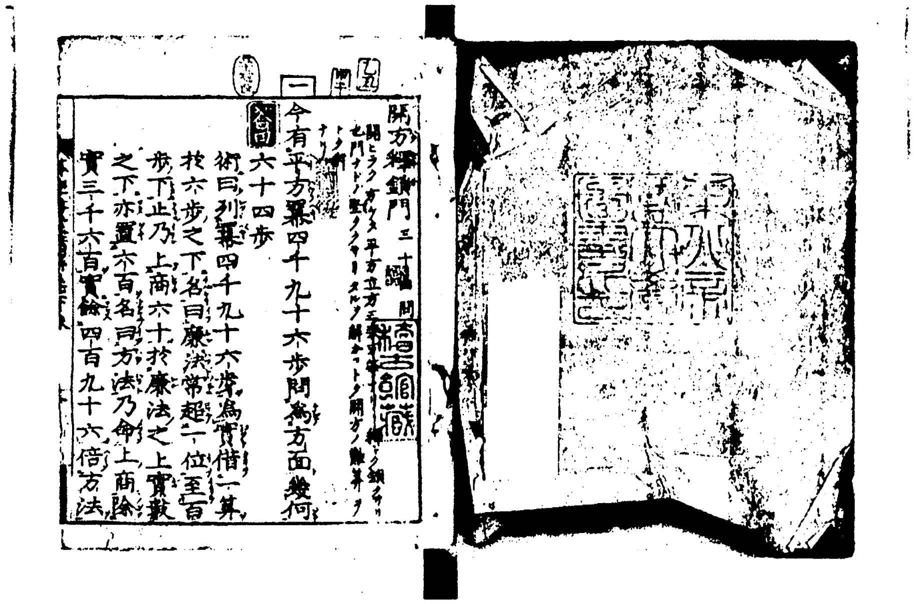
```
Q=0:A=a:B=b:C=c:D=0
R=q:Q=Q+R
C=C+DR:B=B+CR:A=A+BR
C=C+DR:B=B+CR
C=C+DR
R=q':Q=Q+R
C=C+DR:B=B+CR:A=A+BR
C=C+DR:B=B+CR
C=C+DR
print Q,A,B,C,D
```

After several operations, if the value of the Reality row A becomes 0, then the value of the Quotient row Q is a root of the algebraic equation $P(x) = 0$.

If $P(x) = a - bx$, this is the ordinary division. (Hence, this algorithm is called the constructive division. This algorithm is sometimes called Horner's method.)

If $P(x) = 2 - x^2$, this is the extraction of the square root. If $P(x) = 2 - x^3$, this is the extraction of the cubic root. These special cases were known since the *Nine Chapters*.

The Japanese mathematicians learned this constructive division from the *Suanxue Qimeng*.



開方釋鎖門 三十四問
今有平方積四千九十六步問爲方面幾何
答曰 六十四步
術曰列積四千九十六步爲實借一算
於六步之下名曰廉法常超一位至百
步下止乃上商六十於廉法之上實數
之下亦置六百名曰方法乃命上商除
實三千六百實餘四百九十六倍方法

Figure 5: The last chapter of the *Suanxue Qimeng*

A side remark Seki and Takebe DID NOT know the Cartesian Plane, NOR the graph of a function, NOR a tangent line. But Takebe KNEW that the Square row vanishes $b' = 0$ when a cubic polynomial $P(x)$ takes a maximal or minimal value. (The *Tetsujutsu Sankei* 『綴術算経』 Takebe, 1722.)

N.B. In advanced calculus, we know $a' = P(q)$, $b' = P'(q)$ and $c' = \dfrac{P''(q)}{2}$.

3.5 Detailed explanation of the extraction of a root

The method of the extraction of a root was mentioned briefly in the Preliminaries, but in Problem 1 of the last chapter it was described in details.

Problem 1 Calculate $\sqrt{4096}$. Answer: 64. (See Figure 5.)

Then follows Takebe's long annotation on the constructive division. The first step reads as follows: (see Figure 6.)

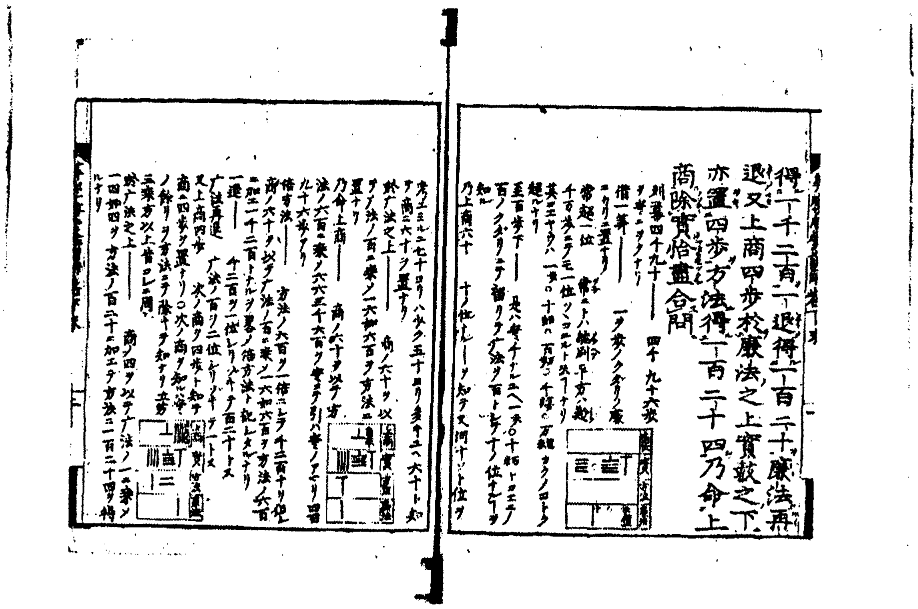
$$4096 - x^2 = 0 \quad \longrightarrow \quad 496 - 120(x-60) - (x-60)^2 = 0.$$

The second step reads as follows: (See Figure 7.)

$$496 - 120(x-60) - (x-60)^2 = 0 \quad \longrightarrow \quad 0 - 128(x-64) - (x-64)^2 = 0.$$

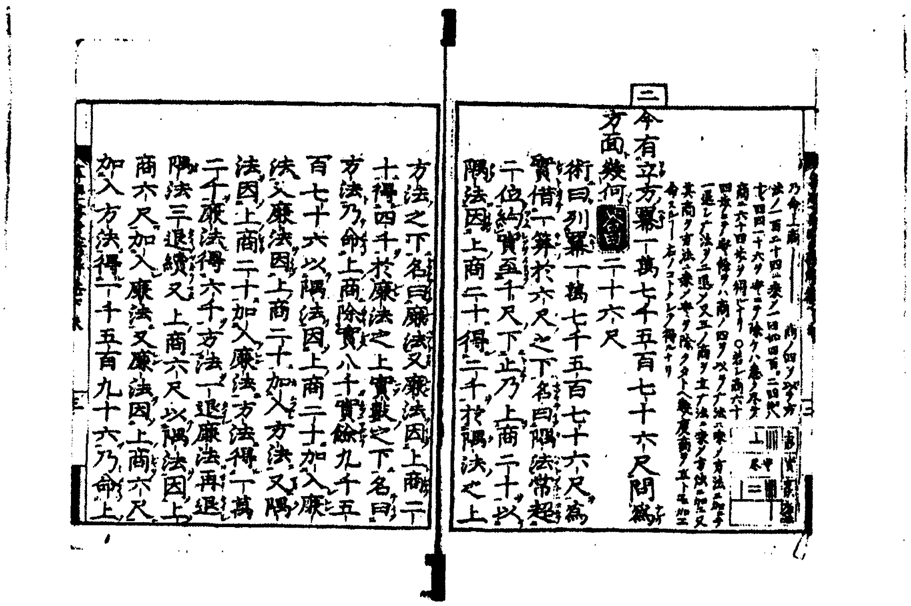
The Reality row becomes empty, the number 64 in the Quotient row is the solution.

Problem 2 Calculate $\sqrt[3]{17576}$. Answer 26. (See Figure 7.)



得一千二百一退得一百二十廉法再退又上商四步於廉法之上實數之下亦置四步方法得一百二十四乃命上商除實恰盡合問

Figure 6: The last chapter of the *Suanxue Qimeng* (cont'd)



二

今有立方積一萬七千五百七十六尺問爲方面幾何 答曰二十六尺

術曰列積一萬七千五百七十六尺爲實借一算於六尺之下名曰隅法常超二位約實至千尺下止乃上商二十以隅法因上商二十得二千於隅法之上方法之下名曰廉法又廉法因上商二十得四千於廉法之上實數之下名曰方法乃命上商除實八千實餘九千五百七十六以隅法因上商二十加入廉法又廉法因上商二十加入方法又隅法因上商二十加入廉法方法得一萬二千廉法得六千方法一退廉法再退隅法三退續又上商六尺以隅法因上商六尺加入廉法又廉法因上商六尺加入方法得一千五百九十六乃命上

Figure 7: The last chapter of the *Suanxue Qimeng* (cont'd)

3.6 Technique of the Celestial Element (天元術) and Counting Board Algebra (算盤代数)

In the constructive division, a configuration on the counting board (i.e., a column vector) represents an algebraic equation. For example,

$$\boxed{\begin{matrix}0\\-3\\1\\2\end{matrix}} \to \text{(by constructive division)} \to \boxed{\begin{matrix}1\\0\\5\\2\end{matrix}}$$

means the equation $-3+x+2x^2=0$ is transformed to the equivalent equation $0+5(x-1)+2(x-1)^2=0$, which has the root 1.

From the technique of the celestial element, Seki and Takebe learned a configuration on the computing board (i.e., a column vector) can represent a polynomial.

Around 1683-1685, Seki wrote the *Trilogy* (三部抄：解見題之法、解隠題之法、解伏題之法) to describe the rule of operations on column vectors:

Addition and scalar multiplication are vector operations:

$$\boxed{\begin{matrix}a\\b\\c\\d\end{matrix}} + \boxed{\begin{matrix}a'\\b'\\c'\\d'\end{matrix}} = \boxed{\begin{matrix}a+a'\\b+b'\\c+c'\\d+d'\end{matrix}}, \qquad r\times\boxed{\begin{matrix}a\\b\\c\\d\end{matrix}} = \boxed{\begin{matrix}ra\\rb\\rc\\rd\end{matrix}}$$

These corresponds to the addition and the scalar multiplication of polynomials:

$$(a+bx+cx^2+dx^3)+(a'+b'x+c'x^2+d'x^3) = ((a+a')+(b+b')x+(c+c')x^2+(d+d')x^3,$$

$$r\times(a+bx+cx^2+d^3) = ra+rbx+rcx^2+rdx^3$$

The multiplication of two configurations

$$\boxed{\begin{matrix}a\\b\\c\\d\end{matrix}} \times \boxed{\begin{matrix}a'\\b'\\c'\\d'\end{matrix}}$$

was assumed to be bilinear and commutative and to satisfy

$$\boxed{\begin{matrix}a\\b\\c\\d\end{matrix}} \times \boxed{\begin{matrix}1\\0\\0\\0\end{matrix}} = \boxed{\begin{matrix}a\\b\\c\\d\end{matrix}} \quad \text{(identity)}, \qquad \boxed{\begin{matrix}a\\b\\c\\d\end{matrix}} \times \boxed{\begin{matrix}0\\1\\0\\0\end{matrix}} = \boxed{\begin{matrix}0\\a\\b\\c\\d\end{matrix}} \quad \text{(shift)}$$

These correspond to

$$(a+bx+cx^2+dx^3)\times(1) = a+bx+c^2+dx^3,$$
$$(a+bx+cx^2+dx^3)\times(x) = 0+ax+bx^2+c^3+dx^4.$$

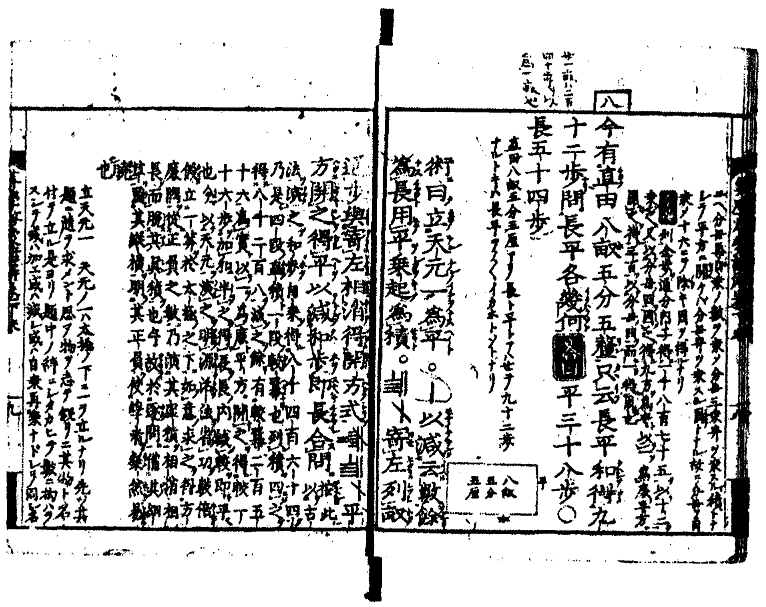

今有直田八畝五分五釐只云長平和得九十二步問長平各幾何
答曰 平三十八步
長五十四步
術曰立天元一為平以減云數餘為長用平乘起為積

Figure 8: Problem 8 開方釈鎖門、the *Suanxue Qimeng*

For example, $x \times x = 1x^2$ is described as follows:

$$\boxed{\begin{matrix}0\\1\end{matrix}} \times \boxed{\begin{matrix}0\\1\end{matrix}} = \boxed{\begin{matrix}0\\0\\1\end{matrix}}$$

In the annotation to the *Suanxue Qimeng* Takebe reproduced the theory of counting board algebra in Seki's *Trilogy.*

Problem 8 Suppose $x + y = 92$, $xy = 2052$. Find x and y.
Answer: $x = 38$, $y = 54$.
Procedure: $y = 92 - x$. $-2052 + x(92 - x) = -2052 + 92x - x^2 = 0$. Solve it. (See Figure 8.)

Takebe's annotation goes like this: let $y = 92 - x$. Solve $-2052 + x(92 - x) = -2052 + 92x - x^2 = 0$. Although the extraction of a root is an established method, he gave a step-by-step explanation. (See Figure 9.)

Then Takebe inserted the explication on the self multiplication $(a+bx)^2 = a^2 + 2abx + b^2x^2$. (See Figure 10.)

Takebe continued the explication of the multiplication of configurations on the counting board (i.e., polynomials).

$$(a + bx + cx^2)^2 = a^2 + 2abx + (b^2 + 2ac)x^2 + 2bcx^3 + c^2x^2$$

Multiplication: $(a + bx) \times (a' + b'x)$, $(a + bx + cx^2) \times (a' + b'x + c'x^2)$, $(a + bx)^3 = (a + bx)^2 \times (a + bx)$, and $(a + bx)^4 = ((a + bx)^2)^2$. (See Figures 10 and 11.)

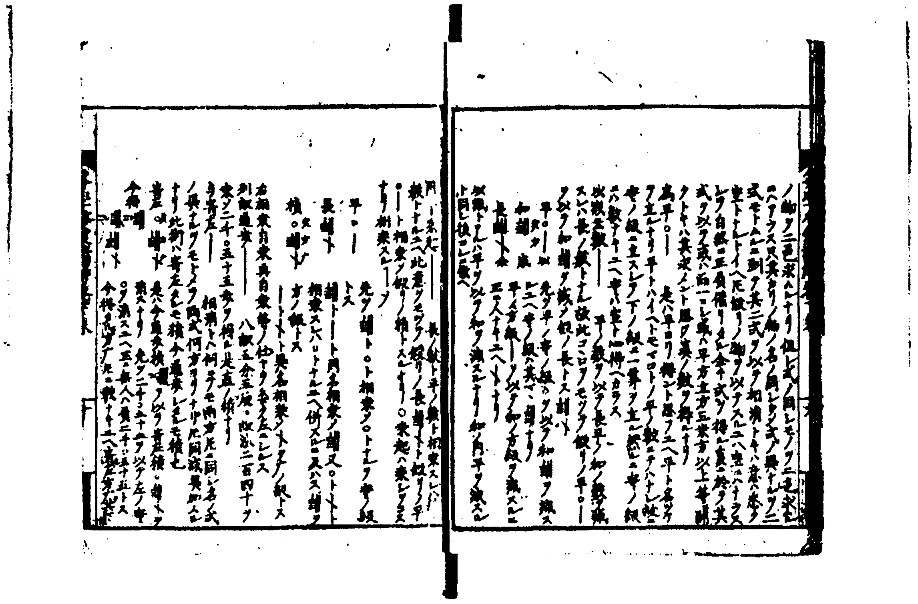


Figure 9: Problem 8 continued 開方积鎖門、the *Suanxue Qimeng*

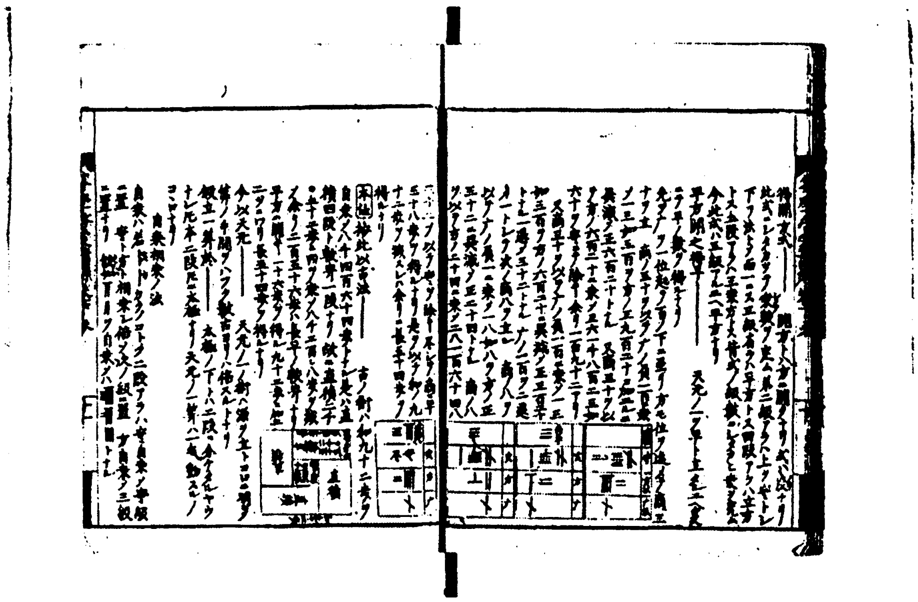


Figure 10: Self multiplication 開方积鎖門、the *Suanxue Qimeng*

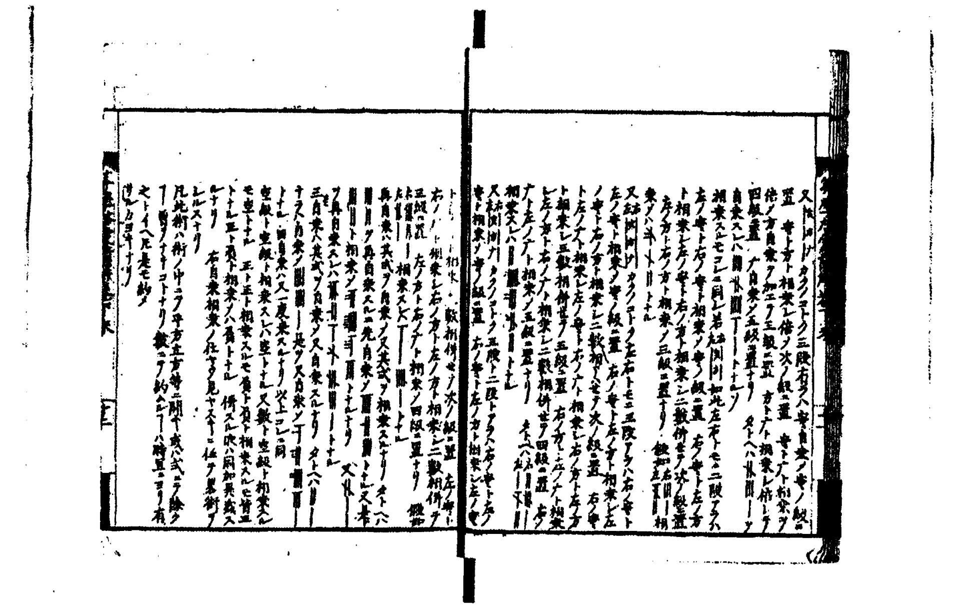


Figure 11: Multiplications 開方釈鎖門、the *Suanxue Qimeng*

4 Takebe's expansion formula for an inverse trigonometric function

Takebe found the Taylor's expansion formula for $(\arcsin\sqrt{t})^2$ in the *Tetsujutsu Sankei*. The formula is written as follows in notation of advanced calculus. (Takebe's original notation is given in Figure 12.

$$(\arcsin\sqrt{t})^2 = t + \frac{t^2}{3} + \frac{8t^3}{45} + \frac{4t^4}{35} + \frac{128t^5}{1575} + \frac{128t^6}{2079} + \frac{1024t^7}{21021} + \cdots$$

$$= t(1 + \frac{t}{3}(1 + \frac{8t}{15}(1 + \frac{9t}{14}(1 + \frac{32t}{45}(1 + \frac{25t}{33}(1 + \frac{72t}{91}))))))$$

Takebe discovered this without knowing the advanced calculus but by a numerically analytic method.

$$\begin{array}{cccccc} \frac{1}{3} & \frac{8}{15} & \frac{9}{14} & \frac{32}{45} & \frac{25}{33} & \frac{72}{91} \\[2ex] \frac{1}{1\cdot 3} & \frac{2\cdot 2^2}{3\cdot 5} & \frac{3^2}{2\cdot 7} & \frac{2\cdot 4^2}{5\cdot 9} & \frac{5^2}{3\cdot 11} & \frac{2\cdot 6^2}{7\cdot 13} \\[2ex] \frac{2\cdot 1^2}{2\cdot 3} & \frac{2\cdot 2^2}{3\cdot 5} & \frac{2\cdot 3^2}{4\cdot 7} & \frac{2\cdot 4^2}{5\cdot 9} & \frac{2\cdot 5^2}{6\cdot 11} & \frac{2\cdot 6^2}{7\cdot 13} \end{array}$$

- The second formula by Takebe: The *Tetsujutsu Sankei* (See Figure 13.)
- The 3rd formula by Takebe: The *Enri Kohai-jutsu*(『円理弧背術』, unknown date)

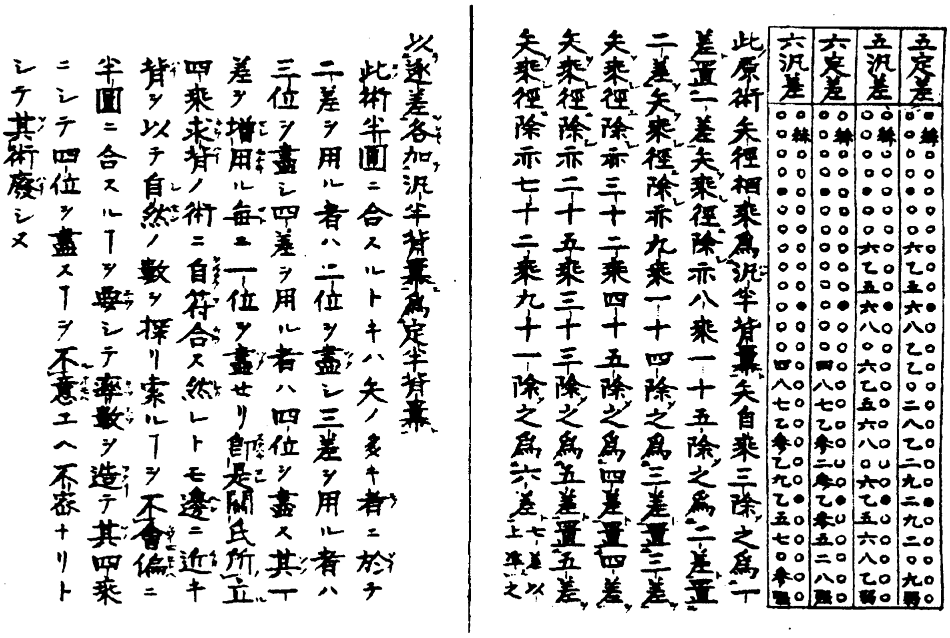


Figure 12: Taylor's expansion formula for $(\arcsin t)^2$ in the *Tetsujutsu Sankei.*

一差	二差	三差	四差	五差	六差	七差	八差	九差	十差
段奇	段偶	段奇	段偶	段奇	段偶	段奇	段偶	段奇	段偶
乘法一	乘法八	乘法九	乘法三十二	乘法二十五	乘法七十二	乘法四十九	乘法一百二十八	乘法八十一	乘法二百
除法三	除法一十五	除法一十四	除法四十五	除法三十三	除法九十一	除法六十	除法一百五十三	除法九十五	除法二百三十一

Figure 13: The rule governing the coefficients of the expansion. The *Tetsujutsu Sankei*

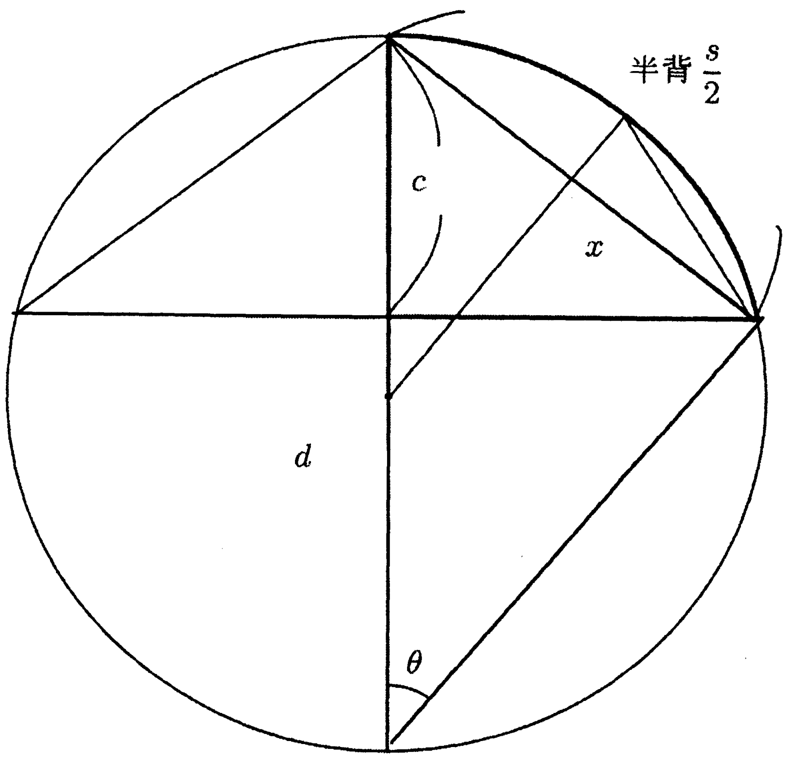
4.1 Sagitta (矢) and Half back arc (半背)

Takebe could calculate the length of the half back arc

$$\left(\frac{s}{2}\right)^2 = \left(d \arcsin\sqrt{c/d}\right)^2$$

with as accurate as he wants once the diameter d and the sagitta c are given. (Repeated use of acceleration)

In the study of circle (円理) the most important problem was to find a formula to represent $(s/2)^2$ in terms of d and c.



$$x : c = d : x, \quad x^2 = cd$$

$$\sin\theta = \frac{x}{d} = \sqrt{c/d}$$

$$\frac{d}{2}\theta = \frac{s}{4}$$

$$\frac{s}{2} = d\theta = d \arcsin\sqrt{c/d}$$

Half back arc (半背)

4.2 Discovery of an infinite expansion in the *Tetsujutsu Sankei*

Let $d = 10$ (1 尺) and $c = 10^{-5}$ (1 忽),

尺	寸	分	厘	毛	糸	忽
10	1	0.1	0.01	0.001	0.0001	0.00001

Takebe calculated the numerical value of the square of the half back arc $\left(\frac{s}{2}\right)^2$ by repeated application of acceleration method, and found the infinite series expansion relying on this numerical value.

5 Application of the constructive division in the search of the expansion formula

5.1 An algebraic method to find the infinite expansion in the *Enri Kohai-jutsu*

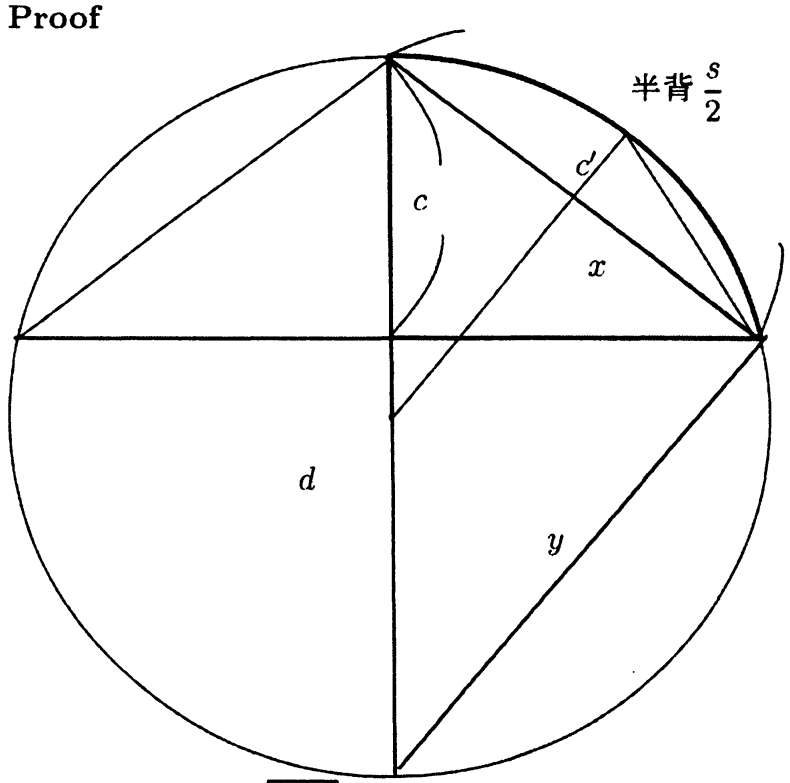
Let

$$g_1(t) = \frac{1-\sqrt{1-t}}{2}, \quad g_n(t) = g_1(g_{n-1}(t)), \quad n = 1, 2, 3, \cdots$$

The we have

$$\lim_{n\to\infty} 4^n g_n(t) = (\arcsin\sqrt{t})^2 \quad (0 \le t \le 1)$$

Proof



$$x : c = d : x, \quad x^2 = cd$$

$$x^2 + y^2 = d^2$$

$$y^2 = d^2 - cd$$

$$d/2 - c' = y/2$$

$$c' = (d - \sqrt{d^2 - cd})/2$$

Put $g(t) = \dfrac{1-\sqrt{1-t}}{2}$. Then the first sagitta c' is given by $c'/d = g(c/d)$. The 0-th approximation of $(s/2)^2$ is cd using the original sagitta c. Because the 0-th approximation of $(s/4)^2$ is $c'd$, the 1st approximation of $(s/2)^2$ is given by $4c'd = 4d^2g(c/d)$ by means of the first sagitta c'.

Because the first approximation of $(s/4)^2$ is $4c''d = 4d^2g(c'/d) = 4d^2g(g(c/d))$, the 2nd approximation of $(s/2)^2$ is given by $4^2c''d = 4^2d^2g(c'/d) = 4^2d^2g(g(c/d))$ by means of the second sagitta c''.

Consequently,

$$\lim_{n\to\infty} 4^n d^2 g_n(c/d) = \left(\frac{s}{2}\right)^2 = d^2(\arcsin\sqrt{c/d})^2$$

End of Proof

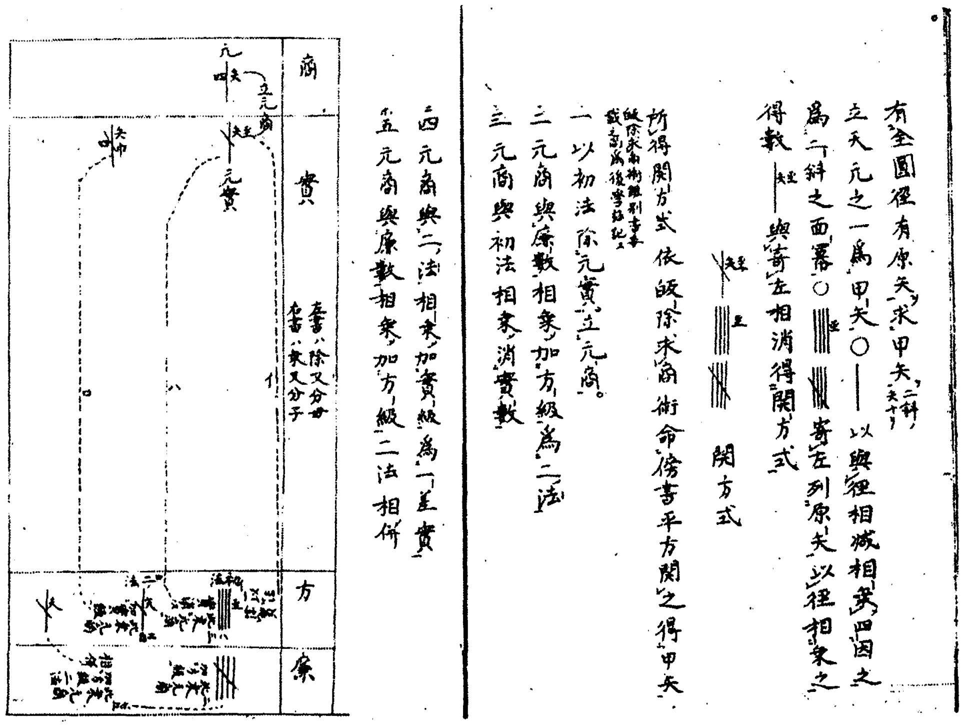


Figure 14: The *Enri Kohai jutsu*, p. 3-4, 東北大学, 狩野文庫 19929

5.2 Side writing method (傍書法) in the Japanese mathematics

Let $g(t) = \dfrac{1-\sqrt{1-t}}{2}$. Let c be the original sagitta, d the diameter and $t = c/d$. The first sagitta $x = g(c/d)/d$ satisfies

$$P(x) = -cd + 4dx - 4x^2 = 0.$$

This quadratic equation （開方式） was written like

卜矢径
||||径
卌

Note that letters were written on rows instead of integers. This is called the side writing method (傍書法). See the right hand side of Figure 14.

Note that $x = c/4$ is close to the root if c is small. Expand the quadratic equation in $x_1 = x - \dfrac{c}{4}$.

$$P_1(x_1) = P(x_1 + \frac{c}{4}) = -c^2/4 + (4d - 2c)x_1 - 4x_1^2 = 0$$

This procedure was explained as manipulation on the counting board. Note that letters were written on the rows. This enabled the Japanese mathematicians to manipulate one-variable polynomial with several symbols, thus polynomials of several variables. See the left hand side of Figure 14.

四丨矢	商
四卜 矢冪 (卜矢径)	実
卜矢 卜矢 ||||径	方
卌	廉

Note that $x_1 = c^2/(16d)$ is close to the root if c is small. We expand the quadratic equation in $x_2 = x_1 - c^2/(16d) = x - c/4 - c^2/(16d)$.

$$P_2(x_2) = P_1(x_2 + c^2/(16d)) = -\frac{c^3}{8d} - \frac{c^4}{64d^2} + (4d - 2c - \frac{c^2}{2d})x_2 - 4x_2^2 = 0$$

This procedure was explained as manipulation on the counting board. Although cumbersome, Takebe manimulated one variable polynomial with coefficients of polynomials in the sagitta (矢) c and the diameter (径) d. See the left hand side of Figure 15.

一十六 径丨 矢冪 四丨矢	商
六十四 径冪 卜 矢三冪 八径卜 矢再冪 (四卜 矢冪) 〇	実
四径卜 矢冪 四径卜 矢冪 卜卜矢 ||||径	方
卌	廉

Note that $x_2 = c^3/(32d^2)$ is close to the root if c is small. Expand the quadratic equation in $x_3 = x_2 - \frac{c^3}{32d^2}$.

$$\begin{aligned} P_3(x_3) &= P_2(x_3 + \frac{c^3}{32d^2}) \\ &= -\frac{5c^4}{64d^2} - \frac{c^5}{64d^3} - \frac{c^6}{256d^4} + (4 - 2c - \frac{c^2}{2d} - \frac{c^3}{4d^2})x_3 - 4x_3^2 = 0 \end{aligned}$$

Expand the quadratic equation in $x_4 = x_3 - \frac{5c^4}{256d^3}$.

$$\begin{aligned} P_4(x_4) &= P_3(x_4 + \frac{5c^4}{256d^3}) \\ &= -\frac{7c^5}{128d^3} - \frac{7c^6}{512d^4} - \frac{5c^7}{1024d^5} - \frac{25c^8}{16384d^6} \\ &+ (4 - 2c - \frac{c^2}{2d} - \frac{c^3}{4d^2} - \frac{5c^4}{32d^3})x_4 - 4x_4^2 = 0 \end{aligned}$$

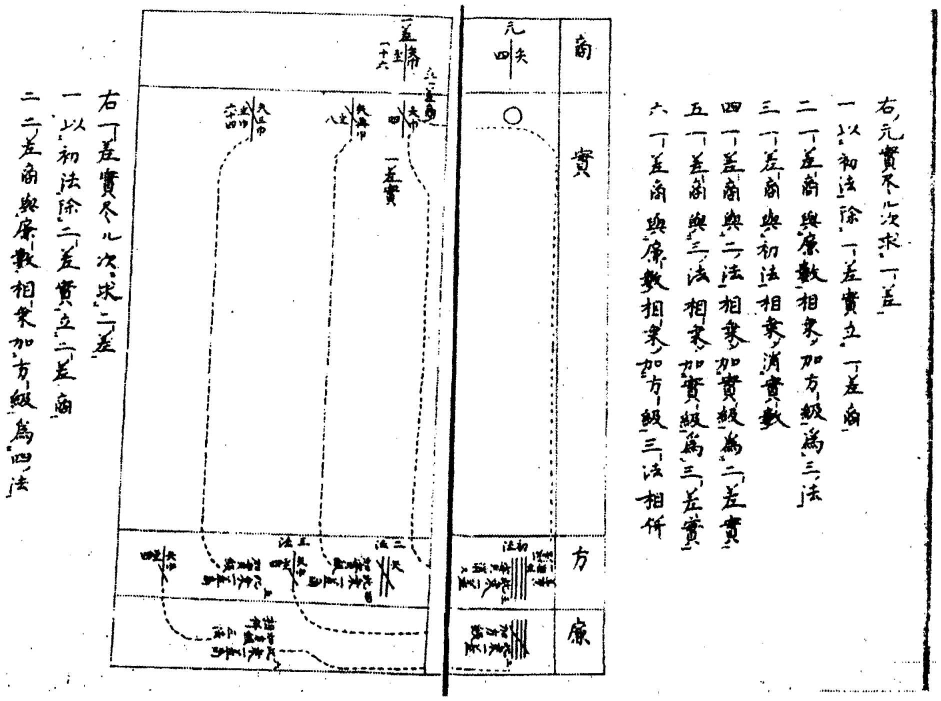


Figure 15: The *Enri Kohai jutsu*, p. 4-5, 東北大学, 狩野文庫 19929

Expand the quadratic equation in $x_5 = x_4 - \frac{7c^5}{512d^4}$.

$$\begin{aligned} P_5(x_5) &= P_4(x_5 + \frac{7c^5}{512d^4}) \\ &= -\frac{21c^6}{512d^4} - \frac{3c^7}{256d^5} - \frac{81c^8}{16384d^6} - \frac{35c^9}{16384d^6} - \frac{49c^{10}}{65536d^8} \\ &+ (4 - 2c - \frac{c^2}{2d} - \frac{c^3}{4d^2} - \frac{5c^4}{32d^3} - \frac{7c^5}{64d^4})x_5 - 4x_5^2 = 0 \end{aligned}$$

In this way, Takebe could obtain the Taylor expansion of the 1st sagitta $g(t)$ as long as possible: (Take $c = t$ and $d = 1$.)

$$\begin{aligned} g(t) &= \frac{t}{4} + \frac{t^2}{16} + \frac{t^3}{32} + \frac{5t^4}{256} + \frac{7t^5}{512} + \cdots \\ &= \frac{t}{4}(1 + \frac{t}{4}(1 + \frac{3t}{6}(1 + \frac{5t}{8}(1 + \frac{7t}{10}(1 + \cdots))))) \end{aligned}$$

Therefore the 1st approximation of the square of the half back arc is given by

$$4g(t) = t + \frac{t^2}{4} + \frac{t^3}{8} + \frac{5t^4}{64} + \frac{7t^5}{128} + \cdots$$

In the *Enri Kohai-jutsu* Takebe calculated the terms up to t^{11} and remarked the rule governing the coefficients (binomial coefficients).

5.3 The expansion of the second sagitta c''

In the *Enri Kohai-jutsu* the constructive division was applied repeatedly to $Q(y) = -g(t) + 4y - 4y^2 = -\frac{t}{4} - \frac{t^2}{16} - \frac{t^3}{32} - \frac{5t^4}{256} - \frac{7t^5}{512} - \cdots + 4y - 4y^2 = 0$ because $g(g(t))$ is a solution of the quadratic equation $Q(y) = 0$. First, expand it in $y_1 = y - \frac{t}{4 \cdot 4}$.

$$\begin{aligned} Q_1(y_1) &= Q(y_1 + \frac{t}{16}) \\ &= -\frac{5t^2}{64} - \frac{t^3}{32} - \frac{5t^4}{256} - \frac{7t^5}{512} - \frac{21t^6}{2048} + (4 - \frac{t}{2})y_1 - 4y_1^2 = 0 \end{aligned}$$

Second, expand it in $y_2 = y_1 - \frac{5t^2}{64 \cdot 4}$.

$$\begin{aligned} Q_2(y_2) &= Q_1(y_2 + \frac{5t^2}{256}) \\ &= -\frac{21t^3}{512} - \frac{345t^4}{16384} - \frac{7t^5}{512} - \frac{21t^6}{2048} + (4 - \frac{t}{2} - \frac{5t^2}{32})y_2 - 4y_2^2 = 0 \end{aligned}$$

Third, expand it in $y_3 = y_2 - \frac{21t^3}{512 \cdot 4}$.

$$\begin{aligned} Q_3(y_3) &= Q_2(y_2 + \frac{21t^3}{2048}) \\ &= -\frac{429t^4}{16384} - \frac{1001t^5}{65536} - \frac{11193t^6}{1048576} + (4 - \frac{t}{2} - \frac{5t^2}{32} - \frac{21t^3}{256})y_3 - 4y_3^2 = 0 \end{aligned}$$

Fourth, expand it in $y_4 = y_3 - \frac{429t^4}{16384 \cdot 4}$.

$$\begin{aligned} Q_4(y_4) &= Q_3(y_4 + \frac{429t^4}{65536}) \\ &= -\frac{2431t^5}{131072} - \frac{24531t^6}{2097152} - \frac{9009t^7}{16777216} - \frac{184041t^8}{1073741824} \\ &\quad + (4 - \frac{t}{2} - \frac{5t^2}{32} - \frac{21t^3}{256} - \frac{429t^4}{8192})y_4 - 4y_4^2 = 0 \end{aligned}$$

Fifth, expand it in $y_5 = y_4 - \frac{2431t^5}{131072 \cdot 4}$.

$$\begin{aligned} Q_5(y_5) &= Q_4(y_5 + \frac{2431t^5}{524148}) \\ &= -\frac{29393t^6}{2097152} - \frac{5291t^7}{4194304} - \frac{592449t^8}{1073741824} - \frac{1042899t^9}{4294967296} - \frac{5909761t^{1}0}{68719476736} \\ &\quad + (4 - \frac{t}{2} - \frac{5t^2}{32} - \frac{21t^3}{256} - \frac{429t^4}{8192} - \frac{2431t^5}{65536})y_5 - 4y_5^2 = 0 \end{aligned}$$

Repeating in this way, Takebe found the second sagitta c'' to be

$$g(g(t)) = \frac{t}{16} + \frac{5t^2}{256} + \frac{21t^3}{2048} + \frac{429t^4}{65536} + \cdots .$$

The 2nd approximation of the square of the half back arc $(s/2)^2$ is given by

$$16g(g(t)) = t + \frac{5t^2}{16} + \frac{21t^3}{128} + \frac{429t^4}{4096} + \cdots$$

Takebe calculated up to t^{11} term in the *Enri Kohai-jutsu.*

5.4 Further approximations

1st sagitta $g_1(t) = g(t) = a_0t + a_1t^2 + a_2t^3 \cdots$
2nd sagitta $g_2(t) = g(g(t)) = b_0t + b_1t^2 + b_2t^3 + \cdots$
3rd sagitta $g_3(t) = g(g(g(t))) = c_0 + c_1t^2 + c_2t^3 + \cdots$
4th sagitta $g_4(t) = g(g(g(g(t))))$
5th sagitta $g_5(t) = g(g(g(g(g(t)))))$
6th sagitta $g_6(t) = g(g(g(g(g(g(t))))))$
7th sagitta $g_7(t) = g(g(g(g(g(g(g(t)))))))$
8th sagitta $g_8(t) = g(g(g(g(g(g(g(g(t))))))))$
9th sagitta $g_9(t) = g(g(g(g(g(g(g(g(g(t)))))))))$
10th sagitta $g_{10}(t) = g(g(g(g(g(g(g(g(g(g(t))))))))))$

To find the coefficients of $g(g(g(t)))$ Takebe remarked the following rules:

$$b_0 = a_0/4, \quad b_1 = (a_0^2/4 + a_1)/4, \quad b_2 = (a_0^3/8 + a_0a_1/2 + a_2)/4, \quad \cdots$$

Using the same rules Takebe calculated the coefficients c_k's from b_k's.

Repeating this procedures, finally Takebe calculated the coefficients of $4^{10}g_{10}(t)$ and taking the limit, rediscovered the coefficients of $(\arcsin t)^2$ in the *Enri Kohai-jutsu.*

6 Summary in Japanese

- 中算
  - 『算法統宗』、『算学啓蒙』、『楊揮算法』、(『九章算術』)、(『四元玉鑑』)
  - 開方術、天元術
- 関孝和 (SEKI, Takakazu)
  - 『発微算法』、『三部抄』
  - 算盤代数、係数：数値（天元術）→ 文字（傍書法）
  - 終結式 (resultant)、行列式 (determinant)
- 建部賢弘 (TAKEBE, Katahiro)
  - 『発微算法演段諺解』、『算学啓蒙諺解大成』、『綴術算経』、『円理弧背術』
  - 円周率、半背冪、$\arcsin t$、2項展開

References

[1] Mitsuo Morimoto: *Differentiation and Integration in Takebe Katahiro's Mathematics*, to appear in the *Surikaisekiken Kokyuroku.* See the references in this article.