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On discrete Sato-like theory with some specializations for finite fields

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Abstract

The construction presented in the article: leads to a discrete Sato-like theory in a simple form valid for any field, indicate a possibility of estimation the number of multisoliton solutions among all solutions of the dKP equation and gives (for special cases) an multisolitonic evolution" according to dKP equation starting from a given "initial" state - a potentially useful for application to natural phenomena.

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1 Introduction

Discrete integrable systems [4] attract attention of many researchers. Such a name is usually related to soliton equations with discrete independent variables (typically $\mathbb{Z}^n$ or its subsets) and continuous dependent variables (usually $\mathbb{C}$ or $\mathbb{R}$). An interesting subclass of such systems is those with discrete dependent variables, called ultradiscrete systems or integrable cellular automata. There are two systematic approaches to this subject: one based on so called limiting procedure [10, 6, 8] and second using algebro-geometric approach over finite fields [3, 1, 2].

Integrable systems may be described in many different ways and one of the most elegant is the Sato theory [7, 11]. While ultradiscrete systems obtained through limiting procedure fit this scheme very nicely (see [8]) the finite fields version needs some adjustment. In other words, desired formulation of the Sato theory should make no use of a notion of continuous sets and its limits (like Miwa transformation).

The idea of description of integrable cellular automata with values over a finite field was a motivation for work presented in this article. General way of such construction is outlined in

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Section 2. Then, following [9] and using linearity of the construction Section 3 shows how to obtain multisoliton solutions. Section 4 contain considerations related to special properties of a finite field valued solutions. We finish with some remarks in Section 5.

2 Elements of discrete Sato theory

For a function $w : \mathbb{Z}^3 \ni (x,y,z) \mapsto w(x,y,z) \in \mathbb{F}_q$ we require to satisfy a linear problem in a form of a nth order difference equation

$$(T_x^n + \alpha_{n-1}T_x^{n-1} + \ldots + \alpha_1 T_x + \beta)w = 0 \tag{1}$$

and a following dispersion relation

$$a(T_x - 1)w = b(T_y - 1)w = c(T_z - 1)w \tag{2}$$

with arbitrary fixed different nonzero constants $a, b, c \in \mathbb{F}_q$. Here, by $T_x w = w_x$ we denote a shift in the variable x: $T_x w(x,y,z) = w_x(x,y,z) = w(x+1,y,z)$. Similarly for y and z.

For n arbitrary independent solutions $w_1, w_2, \ldots, w_n$ to the linear problem (1) define the function τ by the following determinant

$$\tau := \begin{vmatrix} w_1 & w_{1,x} & \cdot & \cdot & w_{1,(n-1)x} \\ w_2 & w_{2,x} & \cdot & \cdot & w_{2,(n-1)x} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ w_n & w_{n,x} & \cdot & \cdot & w_{n,(n-1)x} \end{vmatrix},$$

where $w_{k,mx} := T_x^m w_k$. By the Cramers formulas for β we get

$$\beta = T_x\tau/\tau.$$

Applying T_y to the linear problem (1) we get

$$T_y(T_x^n + \alpha_{n-1}T_x^{n-1} + \ldots + \alpha_1 T_x + \beta)w = (T_x^n + \alpha_{n-1,y}T_x^{n-1} + \ldots + \alpha_{1,y}T_x + \beta_{,y})T_y w = 0,$$

where T_x and T_y commute and $\alpha_{k,y}$, $\beta_{,y}$ - shifted α_k, β. But $T_y w$ can be calculated from the dispersion relation (2), so

$$(T_x^n + \alpha_{n-1,y}T_x^{n-1} + \ldots + \alpha_{1,y}T_x + \beta_{,y})(T_x + \gamma^{(y)})w = 0 \text{ where } \gamma^{(y)} = \frac{b-a}{a}.$$

The last equation could be also factorized as

$$(T_x + C^{(y)})(T_x^n + \alpha_{n-1}T_x^{n-1} + \ldots + \alpha_1 T_x + \beta)w = 0$$

and we can compare coefficients in these two cases. In particular, for the lowest order we get

$$C^{(y)} = \gamma^{(y)}\frac{\beta_y}{\beta}.$$

Similarly, one can do the same with variable z and obtain completely analogous formulas with y replaced by z.

Now, considering a condition of commutation $T_yT_zf = T_zT_yf$ we get

$$T_yT_zf = (T_x + C_y^{(z)})(T_x + C^{(y)})f$$

and a similar formula for T_zT_y. Comparing them we arrive at the following **consistency condition**

$$C_y^{(z)} + C_x^{(y)} = C_z^{(y)} + C_x^{(z)}.$$

In terms of τ the consistency condition takes a form

$$\begin{aligned}(\tau_{yz}\tau_x)_x[(b-a)\tau_{xy}\tau_z - (c-a)\tau_{xz}\tau_y] &= \\ = (\tau_{yz}\tau_x)[(b-a)\tau_{xy}\tau_z - (c-a)\tau_{xz}\tau_y]_x\end{aligned}$$

This means that the following expresion is independent on x:

$$\delta(y,z) := \frac{(a-b)\tau_{xy}\tau_z + (c-a)\tau_{xz}\tau_y}{\tau_{yz}\tau_x}.$$

A change of variables defined by

$$\tau \longrightarrow -\left(\frac{\delta(y,z)}{(b-c)}\right)^{-yz}\tau$$

makes the consistency condition into the dKP equation:

$$(a-b)\tau_{xy}\tau_z + (b-c)\tau_{yz}\tau_x + (c-a)\tau_{xz}\tau_y = 0. \tag{3}$$

For $n = 2$, having two arbitrary independent solutions w_1 and w_2 to the linear problem (1) by the Cramers formulas we get

$$\alpha = \begin{vmatrix} w_1 & w_{1,xx} \\ w_2 & w_{2,xx} \end{vmatrix} / \tau \quad \text{and} \quad \beta = \begin{vmatrix} w_{1,x} & w_{1,xx} \\ w_{2,x} & w_{2,xx} \end{vmatrix} / \tau,$$

where the function τ stands for determinant (2) of 2×2 matrix. Obviously relation (2) holds. From comparison of coefficients, in addition to the relation (2) we get

$$\begin{aligned}\alpha_y + \gamma^{(y)} &= \alpha_x + C^{(y)}, \\ \beta_y + \gamma^{(y)}\alpha_y &= \beta_x + C^{(y)}\alpha,\end{aligned}$$

and analogous set for z instead of y. After substitution of $C^{(y)}$ and $C^{(z)}$, we can eliminate α_y and α_z from the above equations. Then we arrive at two linear equations for α and α_x containing βs. So we have an algebraic equation for α in terms of β, shifted βs and constants. It follows τ-function is regarded as a basic object in this construction.

Summarizing: solutions of the linear difference equation (1) satisfying the (linear) dispersion relation (2) give rise to the τ-function being a solution of the dKP equation (3). Let us point out, all considerations in this section remain valid for arbitrary field.

3 Multisoliton solutions over a finite field

Now, following [9], we make use of the linearity underlying this construction. For any λ, the following function $w(\lambda)$ solves the dispersion relation (2)

$$w(\lambda) = \left(1 + \frac{\lambda}{a}\right)^x \left(1 + \frac{\lambda}{b}\right)^y \left(1 + \frac{\lambda}{c}\right)^z$$

Linearity of the linear problem (1) and of the dispersion relation (2) provide that functions

$$w_i = \sum_j a_i^j w(\lambda_j) \tag{4}$$

give rise to the solution τ of the dKP equation. Solutions obtained in this way (however, so far in case of $\mathbb{C}$) are called **multisoliton solutions.**

Consider a number of summands in the formula (4). For known period (in each variable) of a solution in a fixed field $\mathbb{F}_q$ the number of possible parameters λ_i is bounded, since there exist a bound for a degree of extension field $\mathbb{E}$ containing these λ_i. (In fact periodicity is related to the field, a period $T = q^l - 1$.) Due to this property one can list of all possible λ_i and hence to fix a multisoliton solution means to fix the coefficients a_i^j.

There is ambiguity in this notation: many different w_i will give the same τ-function. (This leads to the notion of a Grassmaniann.) To remove it we will use the following form ("representation") for functions w_i building τ as o determinant of $N \times N$ matrix:

$$\begin{aligned} w_1 &:= c_0 w(\lambda_1) + \sum_{j>N} a_1^j w(\lambda_j), \\ w_i &:= w(\lambda_i) + \sum_{j>N} a_i^j w(\lambda_j), \quad i = 2, 3, \ldots, N. \end{aligned}$$

for a chosen order of the parameters λ_i. (Reordering if necessary.)

The natural question arises: how many free parameters do we need to fix a solution? Notice $\lambda_i \in \{-a, -b, -c\}$ does not contribute to the sum, since the corresponding term is zero. So the total number of λ is $\Lambda = |\mathbb{E}| - 3$. There are two possibilities $\mathbb{E} = \mathbb{F}_q$ and $\mathbb{E} \supsetneq \mathbb{F}_q$ but in these two cases the answer is the same

$$1 + ((q-3) - N)N \quad \text{for} \quad N = 1, 2, \ldots, N. \tag{5}$$

In the second case parameters are counted as over $\mathbb{F}_q$. For such a combination of parameters from $\mathbb{E}$ (being an extension of degree k of $\mathbb{F}_q$) we need to fix only one coefficient $a_i^j \in \mathbb{E}$ (the remaining are fixed by $\mathbb{F}_q$-rationality) and to fix that one coefficient we need k equations over $\mathbb{F}_q$ or equivalently k parameters from $\mathbb{F}_q$.

4 Evolution of multisoliton solutions over a finite field

If τ is a solution of the dKP equation (3), we may think $\tau(x, y, t)$ gives "an evolution" in the variable $t = z$ consistent with the dKP equation. Consider the following problem Assume: τ is periodic in all three variables with known common period T and values of $\tau(x, y, 0) \in \mathbb{F}_q$ are given for all x and y. Can we evolve these "initial values" with agreement to the dKP equation? Unfortunately these question is not properly stated. In fact a proper solution of initial value problem (see [5]) requires more then we can present here. However, we could solve it, if we restrict ourselves to the **multisoliton** solutions. The rest of this section will give partial answer to the above problem. Although it will not solve the problem completely, a proposed restricted solution is very simple and gives a starting point for potential applications.

Since

$$\tau := \begin{vmatrix} w_1 & w_{1,x} & \cdot & \cdot & w_{1,(n-1)x} \\ w_2 & w_{2,x} & \cdot & \cdot & w_{2,(n-1)x} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ w_n & w_{n,x} & \cdot & \cdot & w_{n,(n-1)x} \end{vmatrix},$$

and

$$T_x^k w_i = \sum_{j=1}^{\Lambda} a_i^j \left(1 + \frac{\lambda_j}{a}\right)^k w(\lambda_j),$$

then we have

$$\tau = \sum_{i_1 < i_2 < \ldots < i_N = 1}^{\Lambda} c_N(i_1, \ldots, i_N) w(\lambda_{i_1}) w(\lambda_{i_2}) \ldots w(\lambda_{1_N}),$$

where $c_N(i_1, i_2, \ldots, i_N)$ some coefficients built of a_i^j and constants; they are not algebraically independent .

To fix a τ from "initial data" (for any chosen N) we perform two steps:

- from array of $\tau(x, y, 0)$ to $c_N(i_1, i_2, \ldots, i_N)$
  - just solve linear equations obtained for different choices x and y; sometimes the number of initial data may be to small (that is why we are in not general case),
- from $c_N(i_1, i_2, \ldots, i_N)$ to a_i^j
  - we need to know a structure of $c_N(i_1, i_2, \ldots, i_N)$.

From the definition of τ as a determinant we get

$$\begin{aligned} \tau &= \sum_{\sigma \in Perm(N)} \sum_{j_1, j_2, \ldots, j_N = 1}^{\Lambda} sgn(\sigma) a_{\sigma(1)}^{j_1} a_{\sigma(2)}^{j_2} \cdots a_{\sigma(N)}^{j_N} \\ &\times \left(1 + \frac{\lambda_{j_2}}{a}\right) \left(1 + \frac{\lambda_{j_3}}{a}\right)^2 \cdots \left(1 + \frac{\lambda_{j_N}}{a}\right)^{N-1} \\ &\times w(\lambda_{j_1}) w(\lambda_{j_2}) \ldots w(\lambda_{j_N}) \end{aligned}$$

Notice: only all different $j_1, j_2, \ldots, j_N$ contribute to the sum, hence after some rearrangements, we arrive at

$$\begin{aligned} \tau &= \left(\sum_{l_1, l_2, \ldots, l_N = 1}^{\Lambda} \right)' a_1^{l_1} a_2^{l_2} \ldots a_N^{l_N} w(\lambda_{l_1}) w(\lambda_{l_2}) \ldots w(\lambda_{l_N}) \\ &\times \sum_{\sigma \in Perm(N)} sgn(\sigma) \left(1 + \frac{\lambda_{l_{\sigma(2)}}}{a}\right) \cdots \left(1 + \frac{\lambda_{l_{\sigma(N)}}}{a}\right)^{N-1} \end{aligned}$$

(the prime in the summation above means all $l_1, \ldots, l_N$ are different).

Now we would like to change

$$\left(\sum_{l_1,l_2,\ldots,l_N}\right)' = \sum_{\text{set}\{l_1,l_2,\ldots,l_N\}} \sum_{\text{Perm}\{l_1,l_2,\ldots,l_N\}}$$

Define an antisymmetric object $p_N(l_1, l_2, \ldots, l_N)$ by

$$p_N(l_1, l_2, \ldots, l_N) :=$$
$$\sum_{\sigma \in Perm(N)} sgn(\sigma) \left(1 + \frac{\lambda_{l_{\sigma(2)}}}{a}\right) \ldots \left(1 + \frac{\lambda_{l_{\sigma(N)}}}{a}\right)^{N-1}.$$

So we have

$$\begin{aligned} \tau &= \sum_{l_1<l_2<\ldots<l_N=1,}^{\Lambda} w(\lambda_{l_1}) w(\lambda_{l_2}) \ldots w(\lambda_{l_N}) \\ &\times \sum_{\mu \in \text{Perm}(N)} a_1^{l_{\mu_1}} a_2^{l_{\mu_2}} \ldots a_N^{l_{\mu_N}} p_N(l_1, l_2, \ldots, l_N). \end{aligned}$$

Since p_N is antisymmetric, it changes the sign under transposition of arguments:

$$\begin{aligned} \tau &= \sum_{l_1<l_2<\ldots<l_N=1,}^{\Lambda} w(\lambda_{l_1}) w(\lambda_{l_2}) \ldots w(\lambda_{l_N}) \\ &\times \; p_N(l_1, l_2, \ldots, l_N) \sum_{\mu \in \text{Perm}(N)} sgn(\mu) a_1^{l_{\mu_1}} a_2^{l_{\mu_2}} \ldots a_N^{l_{\mu_N}} \end{aligned}$$

Finally, coefficients $c_N(l_1, l_2, \ldots, l_N)$ are of the form

$$c_N(l_1, l_2, \ldots, l_N) =$$
$$= p_N(l_1, l_2, \ldots, l_N) \sum_{\mu \in \text{Perm}(N)} sgn(\mu) a_1^{l_{\mu_1}} a_2^{l_{\mu_2}} \ldots a_N^{l_{\mu_N}}$$

Summarizing: the coefficients a_i^j can be found from $c_N(l_1, l_2, \ldots, l_N)$ according to the following formulas

$$\begin{aligned} c_0 &= \frac{c_N(1, 2, \ldots, N)}{p_N(1, 2, \ldots, N)} \\ a_k^x &= (-1)^{N-k} \frac{c_N(1, \ldots, k-1, k+1, \ldots, N, x)}{p_N(1, \ldots, k-1, k+1, \ldots, N, x)} \end{aligned}$$

for $k = 1, 2, \ldots, N$ and $x = N+1, \ldots, \Lambda$.

5 Conclusions and prospects

We have presented a couple of ideas for developing a theory of integrable systems over finite fields. Although any such a field can be algebraically extended, for fixed field of rationality of solutions or for fixed period in presented construction there is a natural limit for the degree of extension, hence the number of free parameters is finite. An example of estimation according this kind of reasoning is the formula (5). Notice that (for any N) the number given by (5) is smaller of the number of free "initial data" $(q^l - 1)^2$, and it makes solving the problem of evolution considered in Section 4 possible.

A counting of number of free parameters to be fixed for the general solutions (which require solving initial value problem) and for multisoliton solutions will give the answer to the following new question. How big is the set of multisoliton solutions with comparison to the set of all solutions? An answer to the question require further research, which we believe bring us to the new topics in the theory of integrable systems.

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