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THE STRUCTURE OF GOOD PROGRAMS IN THE RSS MODEL

ALEXANDER J. ZASLAVSKI

ABSTRACT. In this paper we obtain results on the good choice of techniques in the long-run in the model proposed by Robinson, Solow and Srinivasan. We study the structure of good programs and obtain its full description.

1. INTRODUCTION

In the late sixties and early seventies Robinson [9, 10], Okishio [7] and Stiglitz [14-16] studied the problem of optimal economic growth in a model of an economy originally formulated by Robinson [9], Solow [11] and Srinivasan [13] (henceforth, the RSS model). In his recent revisit of Srinivasan [13], Solow [2000] asks for a solution to the "Ramsey problem for this model." In their recent paper [4] Khan and Mitra addressed this general question in the setting of the modern theory of optimal intertemporal allocation initiated by Ramsey [8] and von Neumann [17] and developed by Brock [2], Gale [3], McKenzie [6] and von Weizsacker [18]. Khan and Mitra [4] solved the Ramsey problem for the RSS model. They established the existence of a golden-rule stock, with support prices, and showed that the golden-rule stock is unique. Using the methods of Brock [2] and McKenzie [6] they showed the existence of good programs and maximal programs, and furthermore, established their convergence to a subset of the transition set, the so-called von Neumann facet, consisting of all plans which have "zero-value-loss" at the golden-rule support prices. Khan and Mitra [4] obtained a description of the von Neumann facet and showed that it is a subset of a line in the Euclidean space of dimension $2n$. In Zaslavski [19] some of their results were extended. In this paper we study the structure of good programs which are approximate solutions of the corresponding infinite horizon optimal control problems. This knowledge is important since in general optimal programs may fail to exist while good programs do exist even for nonconcave models [20]. On the other hand in real situations it is not difficult to obtain good programs (especially when it is known their asymptotic behavior) while it is almost impossible to construct optimal programs. We believe that our results can be useful for construction of good programs for RSS models.

We begin with some preliminary notation. Let R (R_+) be the set of real (non-negative) numbers. We shall work in a finite-dimensional Euclidean space R^n with non-negative orthant $R^n_+ = \{x \in R^n : x_i \geq 0, i = 1, \dots, n\}$. For any $x, y \in R^n$, let the inner product $xy = \sum_{i=1}^n x_i y_i$, and $x >> y$, $x > y$, $x \geq y$ have their usual meaning. Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element

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of R^n_+ all of whose coordinates are unity. For any $x \in R^n$, let $||x||$ denote the Euclidean norm of x.

Let $a = (a_1, \dots, a_n) >> 0$, $b = (b_1, \dots, b_n) >> 0$, $b_1 \geq b_2, \dots, \geq b_n$, $d \in (0, 1)$, $c_i = b_i/(1 + da_i)$, $i = 1, \dots, n$.

We consider an economy capable of producing a finite number n of alternative types of machines. For every $i = 1, \dots, n$, one unit of machine of type i requires $a_i > 0$ units of labor to construct it, and together with one unit of labor, each unit of it can produce $b_i > 0$ units of a single consumption good. Thus, the production possibilities of the economy are represented by an (labor) input-coefficients vector, $a = (a_1, \dots, a_n) >> 0$ and an output-coefficients vector, $b = (b_1, \dots, b_n) >> 0$. Without loss of generality we assume that the types of machines are numbered such that $b_1 \geq b_2 \cdots \geq b_n$.

We shall assume that all machines depreciate at a rate $d \in (0, 1)$. Thus the effective labor cost of producing a unit of output on a machine of type i is given by $(1 + da_i)/b_i$: the direct labor cost of producing unit output, and the indirect cost of replacing the depreciation of the machine in this production. We shall work with the reciprocal of the effective labor cost, the effective output that takes the depreciation into account, and denote it by c_i for the machine of type i. Throughout this paper, we shall assume that there is a uniqie machine type σ at which effective labor cost $(1 + da_i)/b_i$ is minimal, or at which the effective output per man $b_i/(1 + da_i)$ is maximal. Thus we shall assume:

There exists $\sigma \in \{1, \dots, n\}$ such that for all

$$i \in \{1, \dots, n\} \setminus \{\sigma\},\ c_\sigma > c_i. \tag{1.1}$$

For each nonnegative integer t let $x(t) = (x_1(t), \dots, x_n(t)) \geq 0$ denote the amounts of the n types of machines that are available in time-period t, and let $z(t+1) = (z_1(t+1), \dots, z_n(t+1)) \geq 0$ be the gross investments in the n types of machines during period $(t+1)$. Hence, $z(t+1) = (x(t+1) - x(t)) + dx(t)$, the sum of net investment and of depreciation. Let $y(t) = (y_1(t), \dots, y_n(t))$ be the amounts of the n types of machines used for production of the consumption good, $by(t)$, during period $(t+1)$. Let the total labor force of the economy be stationary and positive. We shall normalize it to be unity. Clearly, gross investment, $z(t+1)$ representing the production of new machines of the various types, will require $az(t+1)$ units of labor in period t. Also $y(t)$ representing the use of available machines for manufacture of the consumption good, will require $ey(t)$ units of labor in period t. Thus, the availability of labor constrains employment in the consumption and investment sectors by $az(t+1) + ey(t) \leq 1$. Note that the flow of consumption and of investment (new machines) are in gestation during the period and available at the end of it. We now give a formal description of this technological structure.

Definition 1.1. A sequence $\{x(t), y(t)\}_{t=0}^\infty$ is called a program if for each integer $t \geq 0$

$$(x(t), y(t)) \in R^n_+ \times R^n_+,\ x(t+1) \geq (1-d)x(t), \tag{1.2}$$

$$0 \leq y(t) \leq x(t),\ a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1.$$

Definition 1.2. Let T_1, T_2 be integers such that $0 \le T_1 < T_2$. A pair of sequences

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R^n_+$ and for each integer t satisfying $T_1 \le t < T_2$ relations (1.2) hold.

Let $w : [0, \infty) \to R$ be a continuous strictly increasing concave and differentiable function which represents the preferences of the planner.

Set

$$\Omega = \{(x, x') \in R^n_+ \times R^n_+ : \ x' - (1-d)x \ge 0 \\ \text{and } a(x' - (1-d)x) \le 1\}.$$

We have a correspondence $\Lambda : \Omega \to R^n_+$ given by

$$\Lambda(x, x') = \{y \in R^n_+ : \ 0 \le y \le x \text{ and } ey \le 1 - a(x' - (1-d)x)\}.$$

For any $(x, x') \in \Omega$ define

$$u(x, x') = \max\{w(by\} : \ y \in \Lambda(x, x')\}.$$

Definition 1.3. A golden-rule stock is $\widehat{x} \in R^n_+$ such that $(\widehat{x}, \widehat{x})$ is a solution to the problem:

maximaze $u(x, x')$ subject to

(i) $x' \ge x$; (ii)) $(x, x') \in \Omega$.

In Khan and Mitra [4] it was established the following result.

Theorem 1.1. *There exists a unique golden-rule stock* $\widehat{x} = (1/(1 + da_\sigma))e(\sigma)$.

It is not difficult to see that $\widehat{x}$ is a solution to the problem

$$w(by) \to \max, \ y \in \Lambda(\widehat{x}, \widehat{x}).$$

Set

$$\widehat{y} = \widehat{x}.$$

For $i = 1, \dots, n$ set

$$\widehat{q}_i = a_i b_i/(1 + da_i), \ \widehat{p}_i = w'(b\widehat{x})\widehat{q}_i. \tag{1.3}$$

In Khan and Mitra [4, Lemma 1] it was established the following important auxiliary result.

Lemma 1.1. $w(b\widehat{x}) \ge w(by) + \widehat{p}x' - \widehat{p}x$ *for any* $(x, x') \in \Omega$ *and for any* $y \in \Lambda(x, x')$.

For any $(x, x') \in \Omega$ and any $y \in \Lambda(x, x')$ set

$$\delta(x, y, x') = \widehat{p}(x - x') - (w(by) - w(b\widehat{y})). \tag{1.4}$$

By Lemma 1.1, $\delta(x, y, x') \ge 0$ for any $(x, x') \in \Omega$ and any $y \in \Lambda(x, x')$.

We use the following notion of good programs introduced by Gale [3].

Definition 1.4. A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called good if there exists $M \in R$ such that

$$\sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) \ge M \text{ for all } T \ge 0.$$

A program is called bad if

$$\lim_{T \to \infty} \sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) = -\infty.$$

The following proposition was established in Khan and Mitra [4, Proposition 4].

Proposition 1.1. *Any program that is not good is bad.*

Also it was shown in Khan and Mitra [4, Proposition 2] that for any initial stock $x_0 \in R^n_+$ there exists a good program $\{x(t), y(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$.

The following result was obtained in Khan and Mitra [4, Proposition 7].

Proposition 1.2. *A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good if and only if*

$$\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) < \infty.$$

In Zaslavski [19] it was established the following two results.

Theorem 1.2. *Let the function w be strictly concave. Then for each good program $\{x(t), y(t)\}_{t=0}^{\infty}$,*

$$\lim_{t\to\infty} (x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

Let

$$\xi_\sigma = 1 - d - (1/a_\sigma). \tag{1.5}$$

Theorem 1.3. *Let $\xi_\sigma \neq -1$. Then for each good program $\{x(t), y(t)\}_{t=0}^{\infty}$,*

$$\lim_{t\to\infty} (x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

In the sequel we need the following auxiliary result. (For its proof see Khan and Mitra [4, Lemma 1].)

Lemma 1.2. *Let $(x, x') \in \Omega$, $y \in \Lambda(x, x')$, $z = x' - (1-d)x$. Then*

$$b\widehat{y} - by - \widehat{q}(x' - x) = c_\sigma(1 - ey - az) + \sum_{i=1}^{n} (c_\sigma - c_i) y_i$$

$$+ \sum_{i=1}^{n} (c_\sigma - c_i) a_i z_i + d\widehat{q}(x - y).$$

Remark 1.1. Since the function w is strictly increasing it is easy to see that $w'(b\widehat{x}) \neq 0$.

2. Main results

In this section we state the main results of the paper which will be proved in Section 3. These results provide a full description of the structure of good programs.

We begin with the following result.

Theorem 2.1. *Let a program $\{x(t), y(t)\}_{t=0}^{\infty}$ be good. Then for each $i \in \{1, \dots, n\} \setminus \{\sigma\}$,*

$$\sum_{t=0}^{\infty} x_i(t) < \infty,$$

$$\sum_{t=0}^{\infty} (x_\sigma(t) - y_\sigma(t)) < \infty$$

and the sequence $\{\sum_{t=0}^{T-1} x_\sigma(t) - T(1 + da_\sigma)^{-1}\}_{T=1}^{\infty}$ is bounded.

Theorem 2.2. *Let the function w be linear. Then a program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good if and only if for each $i \in \{1, \dots, n\} \setminus \{\sigma\}$,*

$$\sum_{t=0}^{\infty} x_i(t) < \infty,$$

$$\sum_{t=0}^{\infty} (x_\sigma(t) - y_\sigma(t)) < \infty$$

and the sequence $\{\sum_{t=0}^{T-1} x_\sigma(t) - T(1 + da_\sigma)^{-1}\}_{T=1}^{\infty}$ is bounded.

Finally we will establish the following result.

Theorem 2.3. *Let $w \in C^2$, $w''(b\widehat{x}) \neq 0$ and for any good program $\{u(t), v(t)\}_{t=0}^{\infty}$,*

$$\lim_{t \to \infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Then a program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good if and only if for each $i \in \{1, \dots, n\} \setminus \{\sigma\}$,

$$\sum_{t=0}^{\infty} x_i(t) < \infty,$$

$$\sum_{t=0}^{\infty} (x_\sigma(t) - y_\sigma(t)) < \infty,$$

$$\sum_{t=0}^{\infty} (y_\sigma(t) - \widehat{x}_\sigma(t))^2 < \infty,$$

and the sequence $\{\sum_{t=0}^{T-1} x_\sigma(t) - T(1 + da_\sigma)^{-1}\}_{T=1}^{\infty}$ is bounded.

It should be mentioned that if a program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good, then according to the results of [4], for each $i \in \{1, \dots, n\} \setminus \{\sigma\}$ we have $\lim_{t \to \infty} x_i(t) = 0$ while our results implies that $\sum_{t=0}^{\infty} x_i(t) < \infty$.

3. Proofs of Theorems 2.1-2.3

Since Lemma 1.2 is an important ingredient in the proofs of the main results we present here its proof following [4].

Proof of Lemma 1.2.

$$\begin{aligned}
b\widehat{y} - by - \widehat{q}(x' - x) &= c_\sigma - by - \widehat{q}(x' - x) \\
&= c_\sigma(1 - ey - az) + c_\sigma ey + c_\sigma az - by - \widehat{q}(x' - x) \\
&= c_\sigma(1 - ey - az) + \sum_{i=1}^{n} (c_\sigma - c_i) y_i + \sum_{i=1}^{n} c_i y_i + \sum_{i=1}^{n} (c_\sigma - c_i) a_i z_i \\
&\quad + \sum_{i=1}^{n} c_i a_i z_i - \widehat{q}(x' - x) - by
\end{aligned}$$

$$= c_\sigma(1 - ey - az) + \sum_{i=1}^{n}(c_\sigma - c_i)y_i + \sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i$$

$$+ \sum_{i=1}^{n} c_i y_i + \sum_{i=1}^{n} c_i a_i z_i + \widehat{q}(x - ((1-d)x + z)) - by$$

$$= c_\sigma(1 - ey - az) + \sum_{i=1}^{n}(c_\sigma - c_i)y_i + \sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i + d\widehat{q}(x - y).$$

Lemma 1.2 is proved.

Proposition 3.1. *Let $m_0 > 0$. Then there exists $m_1 > 0$ such that for each natural number T and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies $x_0 \le m_0 e$ the inequality $x(t) \le m_1 e$ holds for all integers $t \in [0, T]$.*

For the proof see Khan and Mitra [4, Proposition 1].

Proof of Theorem 2.1. Since the program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good it follows from Proposition 1.2 that

$$\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) < \infty. \tag{3.1}$$

For $t = 0, 1, \dots$ set

$$z(t) = x(t+1) - (1-d)x(t). \tag{3.2}$$

Since w is concave for each $z \in [0, \infty)$

$$w(z) - w(b\widehat{x}) \le w'(b\widehat{x})(z - b\widehat{x}). \tag{3.3}$$

Since $w'(b\widehat{x}) \ne 0$ and w is an increasing we have

$$w'(b\widehat{x}) > 0. \tag{3.4}$$

By (1.4), (3.3), (1.3), Lemma 1.2 and (3.2) for each integer $t \ge 0$

$$\delta(x(t), y(t), x(t+1)) = \widehat{p}(x(t) - x(t+1)) - (w(by(t)) - w(b\widehat{y}))$$

$$\ge \widehat{p}(x(t) - x(t+1)) + w'(b\widehat{x})(b\widehat{x} - by(t))$$

$$= w'(b\widehat{x})[b\widehat{x} - by(t) + \widehat{q}(x(t) - x(t+1))]$$

$$= w'(b\widehat{x})[c_\sigma(1 - ey(t) - az(t)) + \sum_{i=1}^{n}(c_\sigma - c_i)y_i(t)$$

$$+ \sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) + d\widehat{q}(x(t) - y(t))]. \tag{3.5}$$

Combined with (3.1), (3.4), (1.1) and (1.2) the relation (3.5) implies that

$$\infty > \sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) \geq w'(b\widehat{x}) \sum_{t=0}^{\infty} [c_\sigma(1 - ey(t) - az(t))$$

$$+\sum_{i=1}^{n}(c_\sigma - c_i)y_i(t) + \sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) + d\widehat{q}(x(t) - y(t))]. \tag{3.6}$$

Relations (3.6), (3.4), (1.1) and (1.2) imply that for all $i \in \{1, \dots, n\} \setminus \{\sigma\}$

$$\sum_{t=0}^{\infty} y_i(t) < \infty, \tag{3.7}$$

$$\sum_{t=0}^{\infty}(x_i(t) - y_i(t)) < \infty, \; i = 1, \dots, n. \tag{3.8}$$

In view of (3.7) and (3.8)

$$\sum_{t=0}^{\infty} x_i(t) < \infty \text{ for all } i \in \{1, \dots, n\} \setminus \{\sigma\}. \tag{3.9}$$

Relations (3.6), (3.4), (1.1) and (1.2) imply that

$$\sum_{t=0}^{\infty}(1 - ey(t) - az(t)) < \infty.$$

This inequality, (1.2) and (3.2) imply that

$$\sum_{t=0}^{\infty}(1 - y_\sigma(t) - a_\sigma(x_\sigma(t+1) - (1-d)x_\sigma(t))) < \infty.$$

Therefore there is $M_1 > 0$ such that for each natural number T

$$M_1 > \sum_{t=0}^{T-1}(1 - y_\sigma(t) - a_\sigma(x_\sigma(t+1) - x_\sigma(t) + dx_\sigma(t)))$$

$$= T - \sum_{t=0}^{T-1} y_\sigma(t) - a_\sigma(x_\sigma(T) - x_\sigma(0)) - da_\sigma \sum_{t=0}^{T-1} x_\sigma(t) \geq 0.$$

It follows from this inequality, (3.8), (3.9) and Proposition 3.1 that there is $M_2 > 0$ such that for each natural number T

$$|T - (\sum_{t=0}^{T-1} x_\sigma(t))(1 + da_\sigma)| < M_2.$$

Theorem 2.1 is proved.

Proof of Theorem 2.2. Assume that a program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good. Then by Theorem 2.1,

$$\sum_{t=0}^{\infty} x_i(t) < \infty \text{ for all } i \in \{1, \dots, n\} \setminus \{\sigma\}, \tag{3.10}$$

$$\sum_{t=0}^{\infty} (x_\sigma(t) - y_\sigma(t)) < \infty, \tag{3.11}$$

and the sequence $\{T(1 + da_\sigma)^{-1} - \sum_{t=0}^{T-1} x_\sigma(t)\}_{T=1}^{\infty}$ is bounded.

Now assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program, (3.10) and (3.11) hold and that the sequence $\{T(1 + da_\sigma)^{-1} - \sum_{t=0}^{T-1} x_\sigma(t)\}_{T=1}^{\infty}$ is bounded. We prove that the program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good. For $t = 0, 1, \dots$ set

$$z(t) = x(t+1) - (1-d)x(t). \tag{3.12}$$

Since the sequence $\{T(1 + da_\sigma)^{-1} - \sum_{t=0}^{T-1} x_\sigma(t)\}_{T=1}^{\infty}$ is bounded it follows from Proposition 3.1 and (3.11) that the sequence

$$\{T - \sum_{t=0}^{T-1} y_\sigma(t) - da_\sigma \sum_{t=0}^{T-1} x_\sigma(t) - a(x_\sigma(T) - x_\sigma(0))\}_{T=1}^{\infty}$$

$$= \{\sum_{t=0}^{T-1} (1 - y_\sigma(t) - a_\sigma(x_\sigma(t+1) - (1-d)x_\sigma(t))\}_{T=1}^{\infty}$$

is also bounded. Together with (3.12) and (1.2) this implies that

$$\sum_{t=0}^{\infty} (1 - y_\sigma(t) - a_\sigma z_\sigma(t)) < \infty.$$

Combined with (1.2) this inequality implies that

$$\sum_{t=0}^{\infty} (1 - ey(t) - az(t)) < \infty. \tag{3.13}$$

It follows from (1.4), the linearlity of w, (1.3), (3.12) and Lemma 1.2 that for each integer $t \geq 0$

$$\begin{aligned}
\delta(x(t), y(t), x(t+1)) &= \widehat{p}(x(t) - x(t+1)) - (w(by(t)) - w(b\widehat{y})) \\
&= \widehat{p}(x(t) - x(t+1)) - w'(b\widehat{x})(by(t) - b\widehat{x}) \\
&= w'(b\widehat{x})[-by(t) + b\widehat{x} + \widehat{q}(x(t) - x(t+1))] \\
&= w'(b\widehat{x})[c_\sigma(1 - ey(t) - az(t)) + \sum_{i=1}^{n} (c_\sigma - c_i) y_i(t)
\end{aligned}$$

$$+\sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) + d\widehat{q}(x(t) - y(t))]. \tag{3.14}$$

Clearly $w'(b\widehat{x}) > 0$. By (3.14), (3.13), (1.1), (1.2), (3.12), (3.10), (3.11)

$$\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) < \infty.$$

Together with Proposition 1.2 this inequality implies that $\{x(t), y(t)\}_{t=1}^{\infty}$ is a good program. Theorem 2.2 is proved.

Proof of Theorem 2.3. Assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program. Let $t \geq 0$ be an integer. Set

$$z(t) = x(t+1) - (1-d)x(t). \tag{3.15}$$

In view of (1.4)

$$\delta(x(t), y(t), x(t+1)) = \widehat{p}(x(t) - x(t+1)) - (w(by(t)) - w(b\widehat{y})). \tag{3.16}$$

Since $w \in C^2$ it follows from the Taylor's theorem that there exists

$$\gamma_t \in [\min\{by(t), b\widehat{y}\},\ \max\{by(t), b\widehat{y}\}] \tag{3.17}$$

such that

$$w(by(t)) - w(b\widehat{y}) = w'(b\widehat{y})(by(t) - b\widehat{y}) + 2^{-1}w''(\gamma_t)(by(t) - b\widehat{y})^2. \tag{3.18}$$

Relations (3.16), (3.18), (1.3), (3.15) and Lemma 1.2 imply that

$$\delta(x(t), y(t), x(t+1)) = w'(b\widehat{x})(b\widehat{y} - by(t)) - 2^{-1}w''(\gamma_t)(by(t) - b\widehat{y})^2$$

$$+w'(b\widehat{y})\widehat{q}(x(t) - x(t+1))$$

$$= -2^{-1}w''(\gamma_t)(by(t) - b\widehat{y})^2 + w'(b\widehat{y})[b\widehat{y} - by(t) + \widehat{q}(x(t) - x(t+1))]$$

$$= -2^{-1}w''(\gamma_t)(by(t) - b\widehat{y})^2 + w'(b\widehat{y})[c_\sigma(1 - ey(t) - az(t)) + \sum_{i=1}^{n}(c_\sigma - c_i)y_i(t)$$

$$+\sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) + d\widehat{q}(x(t) - y(t))]. \tag{3.19}$$

Since w is concave, increasing and $w'(b\widehat{x}) \neq 0$ we have

$$w'(b\widehat{x}) = w'(b\widehat{y}) > 0,\ w''(\gamma_t) \leq 0. \tag{3.20}$$

Assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is good. Then

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x}). \tag{3.21}$$

By Proposition 1.2

$$\sum_{t=0}^{\infty} \delta(x(t), y(t), x(t+1)) < \infty. \tag{3.22}$$

Since $w''(b\widehat{y}) \neq 0$ and w is concave we conclude that

$$w''(b\widehat{y}) < 0. \tag{3.23}$$

Since $w \in C^2$ it follows from (3.21), (3.23), (3.17) that there exists a natural number t_0 such that for each integer $t \geq t_0$

$$w''(\gamma_t) \leq 2^{-1} w''(b\widehat{y}). \tag{3.24}$$

It follows from (3.19), the choice of t_0 and (3.24) that for each integer $t \geq t_0$

$$\delta(x(t), y(t), x(t+1)) \geq -(4)^{-1} w''(b\widehat{y})(by(t) - b\widehat{y})^2$$

$$+w'(b\widehat{y})[c_\sigma(1 - ey(t) - az(t)) + \sum_{i=1}^{n}(c_\sigma - c_i)y_i(t)$$

$$+\sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) + d\widehat{q}(x(t) - y(t))]. \tag{3.25}$$

It follows from (3.25), (3.22), (3.23), (3.20), (1.2), (3.15) and (1.1) that

$$\sum_{t=0}^{\infty}(x_i(t) - y_i(t)) < \infty, \ i = 1, \dots, n, \tag{3.26}$$

$$\sum_{t=0}^{\infty} y_i(t) < \infty, \ i \in \{1, \dots, n\} \setminus \{\sigma\}, \tag{3.27}$$

$$\sum_{t=0}^{\infty}(by(t) - b\widehat{y})^2 < \infty. \tag{3.28}$$

Relations (3.26) and (3.27) imply that

$$\sum_{t=0}^{\infty} x_i(t) < \infty, \ i \in \{1, \dots, n\} \setminus \{\sigma\}. \tag{3.29}$$

By Theorem 2.1 the sequence $\{T(1 + da_\sigma)^{-1} - \sum_{t=0}^{T-1} x_\sigma(t)\}_{T=1}^{\infty}$ is bounded. By (3.28) and (3.27)

$$\sum_{t=0}^{\infty}(y_\sigma(t) - \widehat{y}_\sigma)^2 < \infty. \tag{3.30}$$

Assume that (3.26), (3.29) and (3.30) hold and that the the sequence

$$\{T(1+da_\sigma)^{-1} - \sum_{t=0}^{T-1} x_\sigma(t)\}_{T=1}^{\infty}$$

is bounded. We show that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a good program. By (3.26), (3.29) and (3.30)

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x}). \tag{3.31}$$

Since $w'(b\widehat{y}) \neq 0$ and w is concave we have

$$w''(b\widehat{y}) < 0. \tag{3.32}$$

By (3.32), (3.31) and (3.17) there exists a natural number t_0 such that for each integer $t \geq t_0$

$$w''(\gamma_t) \geq 2w''(b\widehat{y}). \tag{3.33}$$

It follows from (3.19), (3.33) and the choice of t_0 that for each integer $t \geq t_0$

$$\delta(x(t), y(t), x(t+1)) \leq -w''(b\widehat{y})(by(t) - b\widehat{y})^2$$

$$+w'(b\widehat{y})[c_\sigma(1 - ey(t) - az(t)) + \sum_{i=1}^{n}(c_\sigma - c_i)y_i(t)$$

$$+\sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) + d\widehat{q}(x(t) - y(t))]. \tag{3.34}$$

By (3.20), (1.1), (3.15), (1.2), (3.29) and (3.26)

$$w'(b\widehat{y})\sum_{t=0}^{\infty}\sum_{i=1}^{n}(c_\sigma - c_i)y_i(t) < \infty,\ w'(b\widehat{y})\sum_{t=0}^{\infty}\sum_{i=1}^{n}(c_\sigma - c_i)a_i z_i(t) < \infty, \tag{3.35}$$

$$w'(b\widehat{y})\sum_{t=0}^{\infty} d\widehat{q}(x(t) - y(t)) < \infty.$$

By (3.30) and (3.29),

$$\sum_{t=0}^{\infty}(by(t) - b\widehat{y})^2 < \infty. \tag{3.36}$$

We show that

$$\sum_{t=0}^{\infty} c_\sigma(1 - ey(t) - az(t)) < \infty. \tag{3.37}$$

Clearly, it is sufficient to show that the sequence

$$\{\sum_{t=0}^{T-1}(1-y_\sigma(t)-a_\sigma(x_\sigma(t+1)-(1-d)x_\sigma(t))\}_{T=1}^\infty$$

is bounded. Since the sequence $\{T(1+da_\sigma)^{-1}-\sum_{t=0}^{T-1}x_\sigma(t)\}_{T=1}^\infty$ is bounded it follows from Proposition 3.1 and (3.26) that the sequence

$$\{\sum_{t=0}^{T-1}(1-y_\sigma(t)-a_\sigma(x_\sigma(t+1)-x_\sigma(t))+a_\sigma dx_\sigma(t))\}_{T=1}^\infty$$

$$=\{\sum_{t=0}^{T-1}(1-y_\sigma(t)-a_\sigma dx_\sigma(t))-a_\sigma(x(T)-x_\sigma(0))\}_{T=1}^\infty$$

is bounded. Thus (3.37) holds.

Relations (3.34), (3.32), (3.36), (3.20), (3.27) and (3.35) imply that

$$\sum_{t=0}^{\infty}\delta(x(t),y(t),x(t+1))<\infty.$$

Together with Proposition 1.2 this inequality implies that $\{x(t),y(t)\}_{t=0}^\infty$ is a good program. Theorem 2.3 is proved.

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DEPARTMENT OF MATHEMATICS, THE TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, 32000 HAIFA, ISRAEL

E-mail address: `ajzasl@tx.technion.ac.il`