数理解析研究所講究録
第 1664 巻 2009 年 117-125

境界付きルジャンドル曲面の特異性幾何と射影双対
Singularity geometry of Legendre surfaces with boundaries and projective duality

石川 剛郎 (いしかわ・ごうお)
北海道大学大学院理学研究院数学部門
Goo ISHIKAWA
Department of Mathematics, Hokkaido University
[E-mail: ishikawa(at)math.sci.hokudai.ac.jp or ishikawa-goo(at)sci.hokudai.ac.jp]

Abstract

We are interested in the interaction between singularity and geometry. In this monograph we recall several results on generic Legendre surfaces with boundaries and their projective duality. Moreover, as an application, we study the flat extension problem of a surface with boundary in Euclidean 3-space and clarify its relation to the envelope generated by the boundary and the singularities of tangent developables. In our treatment, a local geometry of surface-curve causes a global (or non-local) effect to the singularities of the envelope via projective (or Legendre) duality. Thus we give examples of results on the interaction between singularity and geometry and between local and global.

This monograph is the announcement of results obtained in [10]. Refer [10] for their detailed proofs.

1 Introduction.

The projective duality between the projective 3-space $\mathbf{R}P^3 = P(\mathbf{R}^4)$ and the dual projective 3-space $\mathbf{R}P^{3*} = P(\mathbf{R}^{4*})$ is given by the incidence manifold

$$I = \{([x], [y]) \in \mathbf{R}P^3 \times \mathbf{R}P^{3*} \mid x \cdot y = 0\},$$

and projections $\pi_1 : I \to \mathbf{R}P^3$ and $\pi_2 : I \to \mathbf{R}P^{3*}$. The space I is identified with the space $PT^*\mathbf{R}P^3$ of contact elements of $\mathbf{R}P^3$ and with $PT^*\mathbf{R}P^{3*}$ as well. It is endowed with the natural contact structure

$$D = \{x \cdot dy = 0\} = \{dx \cdot y = 0\} \subset TI \cong T(PT^*\mathbf{R}P^3).$$

A C^∞ surface S in $\mathbf{R}P^3$ lifts uniquely to a Legendre surface L in I which is an integral submanifold to D:

$$L = \{([x], [y]) \in I \mid [x] \in S, [y] \text{ determines } T_{[x]}S \text{ as a projective plane}\}.$$

Then L projects to $\mathbf{R}P^{3*}$ by π_2. The "front" $S^\vee = \pi_2(L)$, as a parametrized surface with singularities, is called the *projective dual* or *Legendre transform* of S.

If we start with a surface S with boundary γ in $\mathbf{R}P^3$, then the Legendre lift L also has the boundary Γ:

$$\Gamma = \{([x],[y]) \in L \mid [x] \in \gamma\} = \partial L.$$

Then L is a Legendre surface and Γ is an integral curve to the contact distribution D:

$$T\Gamma \subset TL \subset D \subset TPT^*\mathbf{R}P^3.$$

Now we have a Legendre surface with boundary in I and two Legendre fibrations π_1, π_2:

$$\begin{array}{ccccccc}
(L,\Gamma) & \subset & PT^*\mathbf{R}P^3 & \cong & I^5 & \cong & PT^*\mathbf{R}P^{3*} \\
\downarrow & & \downarrow \quad {\scriptstyle\pi_1} \swarrow & & & \searrow {\scriptstyle\pi_2} & \downarrow \\
(S,\gamma) & \subset & \mathbf{R}P^3 & & & & \mathbf{R}P^{3*}
\end{array}$$

Then the basic result follows:

Theorem 1.1 *For a generic Legendre surface with boundary (L,Γ) in the incident manifold $I^5 \cong PT^*\mathbf{R}P^3 \cong PT^*\mathbf{R}P^{3*}$ with respect to C^∞ topology, we have*
(1) The singularities of $\pi_1|_L$ and $\pi_2|_L$ are just cuspidal edges and swallowtails.
(2) The diffeomorphism types of the pair $(\pi_1|_L, \pi_1|_{L_2})$ (resp. $(\pi_2|_L, \pi_2|_{L_1})$) of germs at points on Γ are given by B_2, B_3 and C_3.
(3) Both $\pi_1|_\Gamma$ and $\pi_2|_\Gamma$ are generically immersed space curves in the sense of Scherbak ("Scherbak-generic") [19], in $\mathbf{R}P^3$ and $\mathbf{R}P^{3*}$ respectively. Singularities of $\pi_1|_{L_2}$ and $\pi_2|_{L_1}$ are only cuspidal edges and swallowtails. Swallowtails are not on $\pi_1(L)$ (resp. $\pi_2(L)$).

Remark 1.2 We can show that moreover the singular loci of $\pi_1|_L$ and $\pi_2|_L$, and Γ are in general position in L. Moreover the swallowtail points of $\pi_1|_L$ and $\pi_2|_L$ are not on the intersections of the above three curves.

We write $\gamma = \pi_1(\Gamma)$ and $\widehat{\gamma} = \pi_2(\Gamma)$, and call $\widehat{\gamma}$ the *dual-boundary* to γ. We use the notions of the dual curve c^* and the dual surface $c^\vee$ to a space curve c in $\mathbf{R}P^3$ or in $\mathbf{R}P^{3*}$. Note that $\widehat{\gamma}$ is different from the dual curve γ^* to γ and it is defined only when γ is regarded as a surface-curve.

Now again let (S,γ) be a surface with boundary in $\mathbf{R}P^3$. We consider the one-parameter family of tangent planes along the boundary γ to the surface S and consider the envelope of the family. Then we have

Theorem 1.3 *If (S,γ) is generic, then the envelope of the one-parameter family of tangent planes to S along γ is the dual surface $(\widehat{\gamma})^\vee$ of the dual-boundary $\widehat{\gamma}$. The envelope is the tangent developable to the dual curve $(\gamma^\vee)^*$ to the dual-boundary $\gamma^\vee$. Moreover there are only cuspidal edge singularities and swallowtail singularities on the envelope.*

The above basic theorems (Theorems 1.1 and 1.3) provide the strong motivation as well as the clear framework for the applications stated below. Therefore we give the key idea for the proofs of Theorems 1.1 and 1.3 in the next section of this paper to assure ourselves.

Now, motivated by the above results, we find "landmarks" on the boundary in a generic surface: Besides with parabolic points, we observe osculating-tangent points and swallowtail-tangent points. Here a *parabolic point* is just the intersection of the parabolic locus and the boundary.

A point on the boundary of a surface is called an *osculating-tangent point* if the boundary, regarded as a space curve, has the osculating plane and it coincides with the tangent plane to the surface.

A point on the boundary of a surface is called a *swallowtail-tangent point* if the tangent plane at the point to the surface contacts with the envelope at the swallowtail point of the envelope. It turns out to be that a point $t = t_1$ of the parametric boundary γ is a swallowtail-tangent point if and only if, at $t = t_1$, the dual curve $(\widehat{\gamma})^*$ to the dual-boundary $\widehat{\gamma}$ is defined and $(\widehat{\gamma})^*$ has a singularity of type $(2, 3, 4)$. (See the next section.)

We apply the above basic projective-contact results to a problem of Euclidean geometry of surfaces with boundary in $\mathbf{R}^3$; the flat extension problem:

Problem: Let (S, γ) be a C^∞ surface with boundary γ in $\mathbf{R}^3$. Find a C^1 *extension* $\widetilde{S}$ of S such that $\widetilde{S} \setminus \mathrm{Int}S$ is a C^∞ surface with boundary γ with the Gauss curvature

$$K|_{\widetilde{S}\setminus \mathrm{Int}S} \equiv 0.$$

Remark 1.4 In general, for a hypersurface $y = f(x_1, \ldots, x_n)$ in $\mathbf{R}^{n+1}$, the Gauss-Kronecker curvature is given by

$$K = \frac{(-1)^n \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)}{\left[1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2\right]^{\frac{n+2}{n}}}.$$

Therefore, for a C^2-extension $\widetilde{S}$, K must be continuous on $\widetilde{S}$. Thus, if S is not flat in itself, then we have to impose just C^1-condition to the flat extensions $\widetilde{S}$.

The geometric method to find an extension of (S, γ) along the boundary γ is to take tangent planes to S along γ and to take the envelope of the one-parameter family of tangent planes.

A surface with boundary (S, γ) has a local flat extension across non-osculating-tangent points. Moreover a global obstruction occurs by singularities of the envelope, in particular, by self-intersection loci. Thus a swallowtail point of the envelope provides "a global obstruction with local origin" for the flat extension problem.

With this motivation, we characterise the osculating tangent points and the swallowtail tangent points in terms of Euclidean invariant of the surface-boundary γ of S.

We will recall three fundamental invariants κ_1, κ_2 and κ_3 of the boundary γ. Actually κ_1 is the *geodesic curvature*, κ_2 is the *normal curvature* and κ_3 is the *geodesic torsion* of γ, up to sign.

Then our characterisation is given by

Theorem 1.5 *Let (S, γ) be a generic C^∞ surface with boundary in Euclidean three space $\mathbf{R}^3$. Then the osculating-tangent point on γ is characterised by the condition $\kappa_2 = 0$. Moreover there exists a characterization the swallowtail-tangent points in terms of $\kappa_1, \kappa_2, \kappa_3$ and their derivatives of order ≤ 3.*

In fact we have

Theorem 1.6 (Euclidean generic characterisation of swallowtail-tangent) *Let (S, γ) be a generic C^∞ surface with boundary in Euclidean three space $\mathbf{R}^3$. A swallowtail-tangent point of γ is characterised by the condition*

(I) $\kappa_2 \neq 0$,

(II) $\kappa_1^2\kappa_3(\kappa_2^2+\kappa_3^2)+\kappa_2(\kappa_2^2+\kappa_3^2)\kappa_1'-3\kappa_1\kappa_3^2\kappa_2'+3\kappa_1\kappa_2\kappa_3\kappa_3'+2\kappa_3(\kappa_2')^2-2\kappa_2\kappa_2'\kappa_3'-\kappa_2\kappa_3\kappa_2''+\kappa_2^2\kappa_3'' = 0$,

(III) $2\kappa_1\kappa_2^3(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) + 2\kappa_1\kappa_3(2\kappa_2^2 + \kappa_3^2)\kappa_1' + (3\kappa_2^2 - 2\kappa_3^2)\kappa_1'\kappa_2' + 5\kappa_2\kappa_3\kappa_1'\kappa_3' + 3\kappa_1\kappa_2(\kappa_3')^2 + \kappa_2(3\kappa_1\kappa_2 + \kappa_2^2 + \kappa_3^2)\kappa_1'' + 3\{\kappa_1(-\kappa_2^2 - \kappa_3^2 + \kappa_2\kappa_3) + 3(\kappa_3\kappa_2' - \kappa_2\kappa_3')\}\kappa_3'' + \kappa_2(\kappa_2 - 2\kappa_3)\kappa_3''' \neq 0$.

Remark 1.7 The surface is necessarily hyperbolic at a boundary point with $\kappa_2 = 0, \kappa_3 \neq 0$.

The fundamental construction to observe such characterisaions is as follows:
The unit tangent bundle

$$T_1\mathbf{R}^3 = \{(x, v) \mid x \in \mathbf{R}^3, v \in T_x\mathbf{R}^3, \|v\| = 1\} \cong \mathbf{R}^3 \times S^2,$$

to the Euclidean three space $\mathbf{R}^3$ has the contact structure $\{vdx = 0\} \subset T(T_1\mathbf{R}^3)$. We have analogous double Legendre fibrations as in above projective framework:

$$\begin{array}{ccccccc}
 & & PT^*\mathbf{R}P^3 & \hookleftarrow & T_1\mathbf{R}^3 & & \\
 & & & \pi_1 \swarrow & & \searrow \pi_2 & \\
\mathbf{R}P^3 & \supset & \mathbf{R}^3 & & & & \mathbf{R} \times S^2 \hookrightarrow \mathbf{R}P^3,
\end{array}$$

where π_1 is the bundle projection and π_2 is defined by $\pi_2(x, v) = (-x \cdot v, v)$, $\mathbf{R} \times S^2$ being identified with the space of co-orineted affine planes in $\mathbf{R}^3$. Note that $T_1\mathbf{R}^3$ is mapped to $PT^*(\mathbf{R}P^3)$ by $\Phi : (x, v) \mapsto ([1, x], [-x \cdot v,\ v])$ as a double covering on the image, that the mapping $\Phi : T_1\mathbf{R}^3 \to PT^*(\mathbf{R}P^3)$ is a local contactomorphism, and that $\mathbf{R} \times S^2$ is mapped to $\mathbf{R}P^3$ by $(r, v) \mapsto [r, v]$ as a double covering on the image which is $\mathbf{R}P^3 \setminus \{[1, 0, 0, 0]\}$.

Any co-oriented surface with boundary (S, γ) in $\mathbf{R}^3$ lifts to a Legendre surface with boundary (L, Γ) in $T_1\mathbf{R}^3$ uniquely. A generic surface in $\mathbf{R}^3$ induces a generic Legendre surface. The lifted Legendre surface (L, Γ) projects to a front with boundary (boundary-front) in $\mathbf{R} \times S^2$ by π_2. Actually the "local contact nature" of the double Legendre fibrations is the same, as is noted above, in projective and in Euclidean framework.

Remark 1.8 There exists no invariant metrics on $T_1\mathbf{R}^3$ and on $\mathbf{R} \times S^2$ under the group G of Euclidean motions on $\mathbf{R}^3$ compatible with the double fibration $\mathbf{R}^3 \leftarrow T_1\mathbf{R}^3 \to \mathbf{R} \times S^2$. Note that G is not compact. In this sense, there is no dual Euclidean geometry: Duality in the level of Euclidean geometry is not straightforward, compared with projective geometry.

We are interested in the interaction between singularity and geometry. In our topic of this paper, local geometry of surface-curve provides a global effect to the singularity of the envelope. In fact we give the exact formula for the distance between the swallowtail tangent point on the surface-boundary and the swallowtail point on the boundary-envelope.

Proposition 1.9 *The distance d between the swallowtail tangent point on the surface-boundary and the swallowtail point on the boundary-envelope is given by*

$$d = \left| \frac{\kappa_2\sqrt{\kappa_2^2+\kappa_3^2}}{\kappa_2(\kappa_3' + \kappa_1\kappa_2) + \kappa_3(-\kappa_2' + \kappa_1\kappa_2)} \right|.$$

Remark 1.10 If the denominator of the above formula vanishes, then the formula reads $d = \infty$, and, in fact, the envelope-swallowtail lies at infinity. If $\kappa_2 = 0$, then the formula reads $d = 0$, and, in fact, the non-generic coincidence of an osculating-tangent point and a swallowtail-tangent point occurs, and the envelope-swallowtail coincides with the swallowtail-tangent point.

In §2, we give the background for the basic results Theorems 1.1 and 1.3. In §3, we explain on the Euclidean characterizations of osculating-tangent points and swallowtail-tangent points. For detailed proofs, consult [10].

2 Projective geometry of front-boundaries.

It is known that a generic front with boundary has B_3-singularity. by the theory of boundary singularities, which tells us the diffeomorphism type of a generic front with boundary [2]. See also [17][18][22]. However we wish to know more, the projective geometry of boundaries, $\gamma = \pi_1(\Gamma)$ and $\widehat{\gamma} = \pi_2(\Gamma)$.

A C^∞ space curve $\gamma : \mathbf{R} \to \mathbf{R}P^3$ is called of finite type at $t = t_0 \in \mathbf{R}$, if for each system of affine coordinates, the $3 \times \infty$ matrix

$$(\gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(\ell)}(t_0), \ldots)$$

is of rank 3. Then there exists a unique sequence (a_1, a_2, a_3), called the type, of positive integers with $a_1 < a_2 < a_3$ such that, for some system of affine coordinates centered at $\gamma(t_0)$, γ is expressed as

$$\begin{cases} X_1(t) &= (t-t_0)^{a_1} + o((t-t_0)^{a_1}), \\ X_2(t) &= (t-t_0)^{a_2} + o((t-t_0)^{a_2}), \\ X_3(t) &= (t-t_0)^{a_3} + o((t-t_0)^{a_3}). \end{cases}$$

A point of γ of type $(1,2,3)$ is called an ordinary point. Otherwise, it is called a special point of γ. Special points are isolated on a space curve of finite type.

Theorem 2.1 (O.P. Scherbak): *A generic space curve γ in $\mathbf{R}P^3$ is of type $(1,2,3)$ or $(1,2,4)$ at each point.*

We call a curve *S-generic* if it is of finite type of type $(1,2,3)$ or $(1,2,4)$ at any point.

A Legendre surface with boundary $(L,\Gamma) \subset M$ produces a triple of Legendre surfaces (L, L_1, L_2) in M:

$$L_1 = \{([x],[y]) \mid [x] \in \pi_1(\Gamma), [y] \text{ is a tangent plane to } \pi_1(\Gamma) \text{ at } [x]\}$$

the projective conormal bundle of the space curve $\pi_1(\Gamma)$.

$$L_2 = \{([x],[y]) \mid [y] \in \pi_2(\Gamma), [x] \text{ is a tangent plane to } \pi_2(\Gamma) \text{ at } [y]\}$$

the projective conormal bundle of the space curve $\pi_2(\Gamma)$.

The dual surface of the curve $\pi_1(\Gamma)$ is defined as $\pi_2(L_1)$. The dual surface of the curve $\pi_2(\Gamma)$ is defined as $\pi_1(L_2)$.

$\pi_1(L), \pi_1(L_2) \subset \mathbf{R}P^3$ $\pi_2(L), \pi_2(L_1) \subset \mathbf{R}P^{3*}$.

The osculating planes to a space curve γ form a dual curve γ^* of the curve γ in the dual space.

Theorem 2.2 *(Duality Theorem, Arnol'd, Scherbak):*
(1) The dual curve γ^* to a curve-germ γ of finite type (a_1,a_2,a_3) is a curve-germ of finite type (a_3-a_1, a_3-a_2, a_3).
(2) The dual surface to a curve-germ γ of finite type is the tangent developable of the dual curve γ^* of γ.

Theorem 2.3 *If γ is of type $(1,2,3)$, then γ^* is of type $(1,2,3)$, and the dual surface is diffeomorphic to the cuspidal edge. If γ is of type $(1,2,4)$, then γ^* is of type $(2,3,4)$, and the dual surface is diffeomorphic to the swallowtail.*

A tangent developable of γ is a surface ruled by tangent lines to γ.

Lemma 2.4 *If γ is of type $(1,2,3)$, then γ^* is of type $(1,2,3)$, and the dual surface is diffeomorphic to the cuspidal edge. If γ is of type $(1,2,4)$, then γ^* is of type $(2,3,4)$, and the dual surface is diffeomorphic to the swallowtail.*

For the proof, consult the survey paper [9] on the singularities of tangent developables. We also remark

Lemma 2.5 *The dual surface of a space curve-germ γ of finite type is diffeomorphic to the cuspidal edge (resp. the swallowtail) if and only if the type of γ is equal to $(1,2,3)$ (resp. $(1,2,4)$).*

Note that the type of $\hat{\gamma}^*$ is $(1,2,3)$ (resp. $(2,3,4)$) if and only if $\hat{\gamma}$ is of type $(1,2,3)$ (resp. $(1,2,4)$).

3 Euclidean geometry of surface-boundaries.

Let $S \subset \mathbf{R}^3$ be a cooriented immersed surface with boundary γ.

The 1-st fundamental form $I : TS \to \mathbf{R}$ is defined by $I(v) := g_{\mathrm{Eu}}(v,v) = \|v\|^2$. The 2-nd fundamental form $II : TS \to \mathbf{R}$ is defined by $II(v) := -g_{\mathrm{Eu}}(v, \nabla_v \boldsymbol{n})$, where $\boldsymbol{n} : S \to T\mathbf{R}^3$ is the unit normal to S. Then we have $(I, II) : TS \to \mathbf{R}^2$, which determines the surface with boundary essentially. In fact, the fundamental theorem of surface theory with boundary claims that the right equivalence of (I, II) implies Euclidean right-left-equivalence: If $\exists\ \varphi : (S, \gamma, p) \to (S', \gamma', p')$ diffeomorphism-germ, such that

$$
\begin{array}{ccc}
(TS, T_pS) & (\mathrm{I}, \mathrm{II}) & \\
 & \searrow & \\
\varphi_* \downarrow & & \mathbf{R}^2 \\
 & \nearrow & \\
(TS', T_{p'}S') & (\mathrm{I}, \mathrm{II}) &
\end{array}
$$

commutes. Then there exists an Euclidean motion $E : (\mathbf{R}^3, p) \to (\mathbf{R}^3, p')$ such that $E \circ (S, \gamma) = (S', \gamma') \circ \varphi$.

Set $G = \mathrm{Euclid}(\mathbf{R}^3) \subset GL(4, \mathbf{R})$, the group of Euclidean motions on $\mathbf{R}^3$. We consider Maurer-Cartan form of G

$$
\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega_1^2 & -\omega_1^3 \\ \omega^2 & \omega_1^2 & 0 & -\omega_2^3 \\ \omega^3 & \omega_1^3 & \omega_2^3 & 0 \end{pmatrix}
$$

For a surface with boundary, we have the adopted moving frame $\tilde{\gamma} = (\gamma, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) : \mathbf{R} \to G$ by $\boldsymbol{e}_1 = \gamma'$, the differentiation by arc-length parameter, $\boldsymbol{e}_2$, the inner normal to γ, and $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{n}$. which is different from the Frenet-Serre frame.

The structure equation is given by

$$
d(\gamma(s), \boldsymbol{e}_1(s), \boldsymbol{e}_2(s), \boldsymbol{e}_3(s)) = (\gamma(s), \boldsymbol{e}_1(s), \boldsymbol{e}_2(s), \boldsymbol{e}_3(s))\tilde{\gamma}^*\omega.
$$

Thus we have

$$
d(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \begin{pmatrix} 0 & -\kappa_1 & -\kappa_2 \\ \kappa_1 & 0 & -\kappa_3 \\ \kappa_2 & \kappa_3 & 0 \end{pmatrix} ds
$$

Namely we have

$$
\begin{cases} \boldsymbol{e}_1' & = & \kappa_1 \boldsymbol{e}_2 + \kappa_2 \boldsymbol{e}_3 \\ \boldsymbol{e}_2' & = & -\kappa_1 \boldsymbol{e}_1 + \kappa_3 \boldsymbol{e}_3 \\ \boldsymbol{e}_3' & = & -\kappa_2 \boldsymbol{e}_1 - \kappa_3 \boldsymbol{e}_2 \end{cases}
$$

See, for instance, [11].

Note that $\kappa_1 = \boldsymbol{e}_2 \cdot \gamma''$, $\kappa_2 = \boldsymbol{n} \cdot \gamma''$ and that $\kappa_3 = II(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

Remark 3.1 The curvature κ and the torsion τ of γ as a space curve is related to κ_1, κ_2 and κ_3 by

$$
\kappa^2 = \kappa_1^2 + \kappa_2^2, \qquad \tau = \kappa_3 + \left(\frac{\kappa}{\kappa_1}\right)\left(\frac{\kappa_2}{\kappa}\right)' = \kappa_3 - \left(\frac{\kappa}{\kappa_2}\right)\left(\frac{\kappa_1}{\kappa}\right)' = \kappa_3 + \frac{\kappa_1\kappa_2' - \kappa_2\kappa_1'}{\kappa_1^2 + \kappa_2^2},
$$

for the arc-length differential, provided $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$. Moreover it can be shown that for any space curve γ with curvature κ and τ and given any three functions κ_1, κ_2 and κ_3 on the curve satisfying the above relations. Then there exists a surface S with boundary γ such that the three invariants coincide with the given κ_1, κ_2 and κ_3.

Outline of Proof of Theorem 1.6. The dual-boundary $\widehat{\gamma}$ is given by $(-\gamma\cdot\boldsymbol{n},\boldsymbol{n}) : (\mathbf{R},0) \to S^2\times\mathbf{R}$ which is immersed in $\mathbf{R}P^{3*}$. To see the type of $\widehat{\gamma}$ we examine the 4×5 matrix

$$\begin{pmatrix} \boldsymbol{n} & \boldsymbol{n}' & \boldsymbol{n}'' & \boldsymbol{n}''' & \boldsymbol{n}'''' \\ -\gamma\cdot\boldsymbol{n} & (-\gamma\cdot\boldsymbol{n})' & (-\gamma\cdot\boldsymbol{n})'' & (-\gamma\cdot\boldsymbol{n})''' & (-\gamma\cdot\boldsymbol{n})'''' \end{pmatrix}.$$

The dual surface to a space curve $\widehat{\gamma}(t)$ at $t=t_0$ is diffeomorphic to the cuspidal edge if and only if

$$\det(\widehat{\gamma}',\widehat{\gamma}'',\widehat{\gamma}''')\neq 0,$$

at $t=t_0$. It is diffeomorphic to the swallowtail at $t=t_1$ if and only if

$$\operatorname{rank}(\widehat{\gamma}',\widehat{\gamma}'')=2,\quad \operatorname{rank}(\widehat{\gamma}',\widehat{\gamma}'',\widehat{\gamma}''')=2,\quad \operatorname{rank}(\widehat{\gamma}',\widehat{\gamma}'',\widehat{\gamma}'''')=3,$$

at $t=t_1$. Then using the structure equation, we have the criteria in Theorem 1.6.

Remark 3.2 The criteria is obtained also by using the criterion of swallowtail found in [12].

References

[1] Vladimir Igorevich Arnol'd, *Catastrophe theory,* Third edition. Springer-Verlag, Berlin, 1992.

[2] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of differentiable maps. Volume I: The classification of critical points, caustics and wave fronts,* Monographs in Mathematics **82**, Boston-Basel-Stuttgart, Birkhäuser (1986).

[3] James William (Bill) Bruce, *Envelopes, duality and contact structures,* Proc. Symp. Pure Math., **40–1** (1983), 195–202.

[4] James William (Bill) Bruce, Peter J. Giblin, *Projections of surfaces with boundary,* Proc. London Math. Soc., **60–2** (1990), 392–416.

[5] John P. Cleave, *The form of the tangent-developable at points of zero torsion on space curves,* Math. Proc. Cambridge Philos. Soc. **88–3** (1980), 403–407.

[6] Goo Ishikawa, Determinacy of the envelope of the osculating hyperplanes to a curve, Bull. London Math. Soc., 25(1993), pp.603–610.

[7] Goo Ishikawa, Developable of a curve and determinacy relative to osculation-type, Quart. J. Math. Oxford, Vol.46 (1995), 437–451.

[8] Goo Ishikawa, *Topological classification of the tangent developables of space curves,* Journal of London Math. Soc., 62–2 (2000), 583–598.

[9] Goo Ishikawa, *Singularities of developable surfaces,* London Mathematical Society Lecture Notes Series, **263** (1999), pp. 403–418.

[10] Goo Ishikawa, *Geometry of surfaces with boundaries from singularity viewpoint,* Preprint (2009).

[11] T.A. Ivey and J.M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems,* American Mathematical Society, 2003.

[12] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic 3-space*, Pacific J. of Math. **221** (2005), 303–351.

[13] David Mond, *On the tangent developable of a space curve,* Math. Proc. Cambridge Philos. Soc. **91–3** (1982), 351–355.

[14] David Mond, *Singularities of the tangent developable surface of a space curve,* Quart. J. Math. Oxford Ser. (2) **40–157** (1989), 79–91.

[15] J.J. Nuño Ballesteros, Osamu Saeki, *Singular surfaces in 3-manifolds, the tangent developable of a space curve and the dual of an immersed surface in 3-space,* Real and complex singularities (São Carlos, 1994), Pitman Res. Notes Math. Ser., 333, Longman, Harlow (1995), pp. 49–64.

[16] Kentaro Saji, Masaki Umehara and Kotaro Yamada, *A_k singularities of wave fronts,* to appear in Mathematical Proceedings Cambridge Philosophical Society. (Corollary 2.5)

[17] Inna G. Scherbak, *Boundary fronts and caustics and their metamorphoses,* Singularities (Lille, 1991), London Math. Soc. Lecture Note Series, **201**, Cambridge Univ. Press (1994).

[18] Inna G. Scherbak, *Boundary singularities and focal sets of surfaces with boundary in* $\mathbf{R}^3$*,* Singularity theory (Trieste, 1991), World Sci. Publ., River Edge, NJ, (1995), pp. 635–645.

[19] O.P. Scherbak, *Projective dual space curves and Legendre singularities,* Trudy Tbiliss Univ., 232–233 (1982), 280–336.

[20] O.P. Scherbak, *Wavefront and reflection groups,* Russian Math. Surveys, 43–3 (1988), 149–194.

[21] René Thom, *Sur la théorie des enveloppes,* Journ. de Math., **41–2** (1962), 177–192.

[22] Takaharu Tsukada, *Reticular Legendrian singularities,* Asian J. Math., **5–1** (2001), 109–127.

[23] C.T.C.(Terry) Wall, *Geometric properties of generic differentiable manifolds,* in Geometry and Topology, (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math., **597**, Springer, Berlin (1977).