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Sobolev's inequality for Orlicz-Sobolev spaces of variable exponents

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1 Introduction

Variable exponent spaces have been studied in many articles over the past decade; for a survey see [6, 22]. These investigations have dealt both with the spaces themselves, with related differential equations, and with applications.

Our aim in this note is to deal with Sobolev's inequality for Orlicz-Sobolev functions with $|\nabla u| \in L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Here p and q are variable exponents satisfying natural continuity conditions. For $q = 0$, there are many results for Sobolev's embeddings (see e.g. A. Almeida and S. Samko [1], B. Çekiç, R. Mashiyev and G. T. Alisoy [3], L. Diening [5], D. Edmunds and J. Rákosník [7, 8], V. Kokilashvili and S. Samko [15], S. Samko, E. Shargorodsky and B. Vakulov [23]). Also the case when p attains the value 1 in some parts of the domain is included in the results.

Our results obtained here will appear in the papers [14] and [19].

2 Variable exponents

Following Cruz-Uribe and Fiorenza [4], we consider more general variable exponents p and q on $\mathbb{R}^n$ satisfying:

(p1) $1 \leqslant p^- := \inf_{x\in\mathbb{R}^n} p(x) \leqslant \sup_{x\in\mathbb{R}^n} p(x) =: p^+ < \infty$;

(p2) $|p(x) - p(y)| \leqslant \dfrac{C}{\log(e + 1/|x-y|)}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$;

(p3) $|p(x) - p(y)| \leqslant \dfrac{C}{\log(e + |x|)}$ whenever $|y| \geqslant |x|/2$;

(q1) $-\infty < q^- := \inf_{x\in\mathbb{R}^n} q(x) \leqslant \sup_{x\in\mathbb{R}^n} q(x) =: q^+ < \infty$;

(q2) $|q(x)-q(y)| \leqslant \dfrac{C}{\log(e + \log(e + 1/|x-y|))}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.

Set

$$\Phi(x,t) = t^{p(x)}(\log(c_0 + t))^{q(x)},$$

where $c_0 \geqslant e$ is chosen such that

(Φ_1) $\Phi(x,\cdot)$ is convex on $[0,\infty)$ for fixed $x \in \mathbb{R}^n$.

In view of (Φ_1), $t^{-1}\Phi(x,t)$ is nondecreasing on $(0,\infty)$ for fixed $x \in \mathbb{R}^n$, that is,

(Φ_2)
$$s^{p(x)-1}(\log(e+s))^{q(x)} \leqslant t^{p(x)-1}(\log(e+t))^{q(x)}$$

whenever $0 < s < t$ and $x \in \mathbb{R}^n$.

REMARK 2.1. Note that (Φ_1) holds if there is a positive constant K such that

$$K(p(x)-1) + q(x) \geqslant 0. \tag{2.1}$$

We define the space $L^\Phi(\Omega)$ ($= L^{p(\cdot)}\log L^{q(\cdot)}(\Omega)$) to consist of all measurable functions f on an open set Ω with

$$\int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$. We define the norm

$$\|f\|_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}$$

for $f \in L^\Phi(\Omega)$. These spaces have been studied in [4, 18]. Note that $L^\Phi(\Omega)$ is a Musielak–Orlicz space [20]. In case $q \equiv 0$, $L^\Phi(\Omega)$ reduces to the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$.

Let $B(x,r)$ denote the open ball centered at x with radius r. For a locally integrable function f on $\mathbb{R}^n$, we consider the maximal function Mf defined by

$$Mf(x) := \sup_B f_B = \sup_B \frac{1}{|B|}\int_B |f(y)|dy,$$

where the supremum is taken over all balls $B = B(x,r)$ and $|B|$ denotes the volume of B.

REMARK 2.2. For $\alpha > 0$, consider

$$p_\alpha(x) = \begin{cases} p_0 & \text{when } x \leqslant 0, \\ p_0 + \frac{1}{(\log 1/x)^\alpha} & \text{when } 0 < x \leqslant r_0, \\ p_0 + \frac{1}{(\log 1/r_0)^\alpha} & \text{when } x \geqslant r_0, \end{cases}$$

where $p_0 > 1$ and $0 < r_0 < 1$ is chosen such that

$$|p_\alpha(s) - p_\alpha(t)| \leqslant \frac{1}{(\log 1/|s-t|)^\alpha} \qquad \text{whenever } |s-t| < r_0$$

(see [9, Example 2.1]). Note here that $p_\alpha(\cdot)$ satisfies the log-Hölder condition when $\alpha \geqslant 1$. We can show that

(i) if $\alpha \geqslant 1$, then M is bounded from $L^{p_\alpha(\cdot)}(\mathbb{R}^1)$ to $L^{p_\alpha(\cdot)}(\mathbb{R}^1)$; and

(ii) if $0 < \alpha, \beta < 1$, then M fails to be bounded from $L^{p_\alpha(\cdot)}(\mathbb{R}^1)$ to $L^{p_\beta(\cdot)}(\mathbb{R}^1)$.

To show (ii), consider

$$f(x) = \begin{cases} |x|^{-1/p_0}(\log 1/|x|)^{-2/p_0} & \text{when } -r_0 < x < 0, \\ 0 & \text{when } x \geqslant 0. \end{cases}$$

Then it suffices to see that

(i) $f \in L^{p_\alpha(\cdot)}(\mathbb{R}^1)$ for all $\alpha > 0$; and

(ii) $Mf \notin L^{p_\beta(\cdot)}(\mathbb{R}^1)$ for any $0 < \beta < 1$.

3 Weak type inequality of maximal functions

Our aim in this section is to prove a weak-type inequality for the maximal function.

The following lemma is an improvement of [18, Lemma 2.6].

LEMMA 3.1. *Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$. Set*

$$I := \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

and

$$J := \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi(y, f(y))\,dy.$$

Then

$$I \leqslant C\{J^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)} + 1\}.$$

Proof. By condition (Φ_2), we have for $K > 0$

$$I \leqslant K + \frac{C}{|B(x,r)|}\int_{B(x,r)} f(y)\left(\frac{f(y)}{K}\right)^{p(y)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K)}\right)^{q(y)} dy,$$

where the first term, K, represents the contribution to the integral of points where $f(y) < K$. If $J \leqslant 1$, then we take $K = 1$ and obtain

$$I \leqslant 1 + CJ \leqslant C.$$

Now suppose that $J \geqslant 1$ and set

$$K := CJ^{1/p(x)}(\log(c_0+J))^{-q(x)/p(x)}.$$

Note that $J^{C/\log(CJ^{1/n})} \leqslant C$ and $(\log(c_0+J))^{C/\log(\log(e+CJ^{1/n}))} \leqslant C$. Since we assumed that $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$, we conclude that

$$J \leqslant \frac{1}{|B(x,r)|}\int_{\mathbb{R}^n} \Phi(y, f(y))\, dy \leqslant \frac{1}{|B(x,r)|}.$$

Hence, by conditions (p2) and (q2), we obtain, for $y \in B(x,r)$, that

$$\begin{aligned} K^{-p(y)} &\leqslant \left\{CJ^{1/p(x)}(\log(c_0+J))^{-q(x)/p(x)}\right\}^{-p(x)+C/\log(1/r)} \\ &\leqslant \left\{CJ^{1/p(x)}(\log(c_0+J))^{-q(x)/p(x)}\right\}^{-p(x)+C/\log(CJ^{1/n})} \\ &\leqslant CJ^{-1}(\log(c_0+J))^{q(x)} \end{aligned}$$

and

$$\begin{aligned} (\log(c_0+K))^{-q(y)} &\leqslant \{C\log(c_0+J)\}^{-q(x)+C/\log(\log(e+1/r))} \\ &\leqslant \{C\log(c_0+J)\}^{-q(x)+C/\log(\log(e+CJ^{1/n}))} \\ &\leqslant C(\log(c_0+J))^{-q(x)}. \end{aligned}$$

Consequently it follows that

$$I \leqslant CJ^{1/p(x)}(\log(c_0+J))^{-q(x)/p(x)}.$$

Combining this with the estimate $I \leqslant C$ from the previous case yields the claim. □

In view of Lemma 3.1, for each bounded open set G in $\mathbb{R}^n$ we can find a positive constant C such that

$$\{Mf(x)\}^{p(x)} \leqslant C\{Mg(x)(\log(c_0+Mg(x)))^{-q(x)} + 1\}, \tag{3.1}$$

so that

$$\Phi(x, Mf(x)) \leqslant C\{Mg(x) + 1\} \tag{3.2}$$

for all $x \in G$ and $g(y) := \Phi(y, f(y))$, whenever f is a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$.

LEMMA 3.2. *Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$. If $J \leqslant 1$, then*

$$I = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy \leqslant C\{J^{1/p(x)} + (1+|x|)^{-n/p(x)}\}.$$

LEMMA 3.3. *Let f be a nonnegative measurable function on an open set G with $\|f\|_{L^\Phi(G)} \leqslant 1$. Set*

$$N(x) := Mg(x)^{1/p(x)}(\log(c_0 + Mg(x)))^{-q(x)/p(x)},$$

where $g(y) := \Phi(y, f(y))$. Then

$$\int_{E_t} \Phi(x,t)\ dx \leqslant C,$$

where $E_t := \{x \in G : N(x) > t, Mg(x) > C_1(1+|x|)^{-n}\}$ and $C_1 := |B(0,1/2)|^{-1}$.

We are now ready to give a weak-type estimate for the maximal function, which is an extension of [2, Theorem 1.6] and [12, Theorem 3.2].

THEOREM 3.4. *Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$. Then*

$$\int_{\{x\in\mathbb{R}^n : Mf(x)>t\}} \Phi(x,t)\ dx \leqslant C.$$

4 Weak type inequality for Riesz potentials

For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbb{R}^n$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Here it is natural to assume that

$$\int_{\mathbb{R}^n} (1+|y|)^{\alpha-n}|f(y)|\, dy < \infty, \tag{4.1}$$

which is equivalent to the condition that $I_\alpha|f| \not\equiv \infty$ (see [16, Theorem 1.1, Chapter 2]).

Our aim in this section is to establish weak-type estimates for Riesz potentials of functions in $L^{\Phi}(\mathbb{R}^n)$, when the exponent p satisfies

$$p^+ < n/\alpha.$$

Let $p^\sharp_\alpha(x)$ denote the Sobolev conjugate of $p(x)$, that is,

$$1/p^\sharp_\alpha(x) = 1/p(x) - \alpha/n.$$

LEMMA 4.1. *Suppose that* $p^+ < n/\alpha$. *If* f *is a nonnegative measurable function on* $\mathbb{R}^n$ *with* $\|f\|_{L^{\Phi}(\mathbb{R}^n)} \leqslant 1$, *then*

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \leqslant C\{r^{\alpha - n/p(x)} + (1+|x|)^{\alpha - n/p(x)}\}$$

for all $x \in \mathbb{R}^n$ *and* $r \geqslant 1/e$.

LEMMA 4.2. *Suppose that* $p^+ < n/\alpha$. *Let* f *be a nonnegative measurable function on* $\mathbb{R}^n$ *with* $\|f\|_{L^{\Phi}(\mathbb{R}^n)} \leqslant 1$. *Then*

$$\int_{B(x,1/e)\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \leqslant C\delta^{\alpha - n/p(x)}(\log(c_0 + 1/\delta))^{-q(x)/p(x)}$$

for all $x \in \mathbb{R}^n$ *and* $0 < \delta < 1/e$.

The next lemma is a generalization of [18, Theorem 2.8].

LEMMA 4.3. *Suppose that* $p^+ < n/\alpha$. *Let* $f \in L^{\Phi}(\mathbb{R}^n)$ *be nonnegative with* $\|f\|_{L^{\Phi}(\mathbb{R}^n)} \leqslant 1$. *Then*

$$I_\alpha f(x) \leqslant C\{Mf(x)^{p(x)/p^\sharp_\alpha(x)}(\log(c_0 + Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p^\sharp_\alpha(x)}\}.$$

Set

$$\Psi_\alpha(x,t) = \{t(\log(c_0+t))^{q(x)/p(x)}\}^{p^\sharp_\alpha(x)}.$$

Note from condition (Φ_1) that $\Psi_\alpha(x,\cdot)$ is convex on $(0,\infty)$ for each fixed $x \in \mathbb{R}^n$.

LEMMA 4.4. *Suppose that* $p^+ < n/\alpha$. *Let* f *be a nonnegative measurable function on an open set* G *with* $\|f\|_{L^{\Phi}(G)} \leqslant 1$. *Set*

$$N(x) := Mg(x)^{1/p^\sharp_\alpha(x)}(\log(c_0 + Mg(x)))^{-q(x)/p(x)},$$

where $g(y) := \Phi(y, f(y))$. *Then*

$$\int_{\tilde{E}_t} \Psi_\alpha(x,t)\ dx \leqslant C,$$

where $\tilde{E}_t := \{x \in G : N(x) > t, Mg(x) \geqslant C_1(1+|x|)^{-n}\}$ *and* $C_1 := |B(0,1/2)|^{-1}$.

Now we are ready to introduce the weak-type estimate for Riesz potentials, as an extension of [2, Theorem 1.9] and [12, Theorem 3.4].

THEOREM 4.5. *Suppose that* $p^+ < n/\alpha$. *Let* f *be a nonnegative measurable function on* $\mathbb{R}^n$ *with* $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$. *Then*

$$\int_{\{x\in\mathbb{R}^n : I_\alpha f(x)>t\}} \Psi_\alpha(x,t)\ dx \leqslant C.$$

REMARK 4.6. In view of [17], for each $\beta > 1$ one can find a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \{I_\alpha f(x)\}^{p^\sharp_\alpha(x)} (\log(e + I_\alpha f(x)))^{-\beta} (\log(e + I_\alpha f(x)^{-1}))^{-\beta}\ dx \leqslant C$$

whenever f is a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant 1$. This gives a supplement of O'Neil [21, Theorem 5.3].

5 Sobolev functions

Let us consider the generalized Orlicz-Sobolev space $W^{1,\Phi}(G)$ with the norm

$$\|u\|_{1,L^\Phi(G)} = \|u\|_{L^\Phi(G)} + \|\nabla u\|_{L^\Phi(G)} < \infty.$$

Further we denote by $W^{1,\Phi}_0(G)$ the closure of $C^\infty_0(G)$ in the space $W^{1,\Phi}(G)$ (cf. [10] for definitions of zero boundary value functions in the variable exponent context). To conclude the paper, we derive a Sobolev inequality for functions in $W^{1,\Phi}_0(G)$ as the application of Sobolev's weak type inequality for Riesz potentials of functions in $L^\Phi(G)$.

Let us begin with the following lemma:

LEMMA 5.1 (Corollary 2.3, [18]). *Set* $\kappa(y,t) := t(\log(e+t))^y$ *for* y *and* $t \geqslant 0$. *Then*

$$\kappa(y,at) \leqslant \tau(y,a)\kappa(y,t)$$

whenever $a, t > 0$, *where*

$$\tau(y,a) := a \max\left\{(C\log(e+a))^y, (C\log(e+a^{-1}))^{-y}\right\}.$$

Using the previous lemma we can derive a scaled version of the weak type estimate from the previous section which will be needed below.

LEMMA 5.2. *Suppose that $p^+ < n/\alpha$. Let $f \in L^\Phi(\mathbb{R}^n)$ be nonnegative with $\|f\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$. Then for every $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\int_{\{x\in\mathbb{R}^n : I_\alpha f(x)>t\}} \Psi_\alpha(x,t)\ dx \leqslant C\|f\|_{L^\Phi(\mathbb{R}^n)}^{(p_\alpha^\sharp)^- - \varepsilon},$$

for every $t > 0$.

LEMMA 5.3. *Suppose that $p^+ < \min\{n, (p_1^\sharp)^-\}$ and G is an open set. If $u \in W_0^{1,\Phi}(\mathbb{R}^n)$, then there exists a constant $c_1 > 0$ such that*

$$\|u\|_{L^{\Psi_1}(G)} \leqslant c_1 \|\nabla u\|_{L^\Phi(\mathbb{R}^n)}.$$

Proof. We may assume that $\|\nabla u\|_{L^\Phi(\mathbb{R}^n)} \leqslant 1$ and u is nonnegative. It follows from [16, Theorem 1.2, Chapter 6] that

$$|v(x)| \leqslant C(n) I_1 |\nabla v|(x)$$

for $v \in W_0^{1,1}(G)$ and almost every $x \in G$. For $u \in W_0^{1,\Phi}(G)$ and each integer j, we write $U_j = \{2^j < u(x) \leqslant 2^{j+1}\}$ and $v_j = \max\{0, \min\{u - 2^j, 2^j\}\}$. Since $v_j \in W_0^{1,1}(G)$ and $v_j(x) = 2^j$ for almost every $x \in U_{j+1}$, we have

$$I_1|\nabla v_j|(x) \geqslant C2^j$$

for almost every $x \in U_{j+1}$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \Psi_1(x, u(x))\, dx &\leqslant \sum_{j\in\mathbf{Z}} \int_{U_{j+1}} \Psi_1(x, u(x))\, dx \\ &\leqslant C \sum_{j\in\mathbf{Z}} \int_{U_{j+1}} \Psi_1(x, 2^{j+1})\, dx \\ &\leqslant C \sum_{j\in\mathbf{Z}} \int_{\{x\in U_{j+1} : I_1|\nabla v_j|(x) > C2^j\}} \Psi_1(x, C2^j)\, dx.\end{aligned}$$

Taking $r \in (p^+, (p_1^\sharp)^-)$, we obtain by Lemma 5.2 that

$$\begin{aligned}\sum_{j\in\mathbf{Z}} \int_{\{x\in U_{j+1} : I_1|\nabla v_j|(x) > C2^j\}} \Psi_1(x, C2^j)\, dx &\leqslant C \sum_{j\in\mathbf{Z}} \|\nabla v_j\|_{L^\Phi(\mathbb{R}^n)}^r \\ &\leqslant C \sum_{j\in\mathbf{Z}} \int_{U_j} \Phi(x, |\nabla u(x)|)\, dx \leqslant C,\end{aligned}$$

which completes the proof. □

Recall that $\Phi(x,t) = (t\log(c_0+t)^{q(x)/p(x)})^{p(x)}$ and $\Psi_\alpha(x,t) = \Phi(x,t)^{p^\sharp_\alpha(x)/p(x)} = (t\log(c_0+t)^{q(x)/p(x)})^{p^\sharp_\alpha(x)}$, where $p^\sharp_\alpha(x)$ denotes the Sobolev conjugate of $p(x)$, that is,

$$1/p^\sharp_\alpha(x) = 1/p(x) - \alpha/n.$$

The space $L^{\Psi_\alpha}(\Omega)$ is defined in the same manner as $L^\Phi(\Omega)$ (see Section 2).

THEOREM 5.4. *Let p and q satisfy the above conditions. If $p^+ < n$, then*

$$\|u\|_{L^{\Psi_1}(\Omega)} \leqslant c_1 \|\nabla u\|_{L^\Phi(\Omega)}$$

for every $u \in W_0^{1,\Phi}(\Omega)$.

This extends [11, Proposition 4.2(1)] and [13, Theorem 3.4] which dealt with the case $q \equiv 0$.

Proof of Theorem 5.4. We may split $\mathbb{R}^n$ into a finite number of cubes $\Omega_1, \ldots, \Omega_k$ and the complement of a cube, Ω_0, in such a way that $p^+_{\Omega_i} < (p^\sharp_1)^-_{\Omega_i}$ for each i. Then

$$\|u\|_{L^{\Psi_1}(\mathbb{R}^n)} \leqslant \sum_{i=0}^{k} \|u\|_{L^{\Psi_1}(\Omega_i)} \leqslant c_1 \sum_{i=0}^{k} \|\nabla u\|_{L^\Phi(\mathbb{R}^n)} = (k+1)c_1 \|\nabla u\|_{L^\Phi(\mathbb{R}^n)},$$

by the previous lemma. □

6 Variable exponents near Sobolev's exponent

In this section we assume that G is a bounded open set in $\mathbb{R}^n$. The results in this and next sections will appear in the paper by Y. Mizuta, T. Ohno and T. Shimomura [19].

Let p, q, $\Phi = \Phi(x,t)$ and $\Psi_\alpha = \Psi_\alpha(x,t)$ be as before.

THEOREM 6.1. *Suppose further*

$$1 < p^- \leqslant p(x) < n/\alpha$$

for $x \in G$. Then there exists a constant $c_1 > 0$ such that

$$\|\gamma_1^{-1} I_\alpha f\|_{L^{\Psi_\alpha}(G)} \leqslant c_1 \|f\|_{L^\Phi(G)}$$

for all $f \in L^\Phi(G)$, where

$$\gamma_1(x) = p^\sharp_\alpha(x)^{(q(x)+p(x)-1)/p(x)} (\log p^\sharp_\alpha(x))^{q(x)/p(x)}.$$

THEOREM 6.2. *Suppose further*

$$p(x) \geqslant n/\alpha \qquad \text{and} \qquad q(x) < p(x) - 1$$

for $x \in G$. *Then there exist constants* $c_1, c_2 > 0$ *such that*

$$\int_G \exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_3(x))^{p(x)/(p(x)-q(x)-1)}}\right) dx \leqslant c_2$$

for all nonnegative measurable functions f *on* G *with* $\|f\|_{L^\Phi(G)} \leqslant 1$, *where*

$$\gamma_3(x) = \gamma_2(x)^{-(p(x)-1)/p(x)}(\log(1/\gamma_2(x)))^{q(x)/p(x)}$$

with $\gamma_2(x) = \min\{p(x) - q(x) - 1, 1/2\}$.

THEOREM 6.3. *Suppose further*

$$p(x) \geqslant n/\alpha \qquad \text{and} \qquad q(x) \geqslant p(x) - 1$$

for $x \in \mathbb{R}^n$. *Then there exist constants* $c_1, c_2 > 0$ *such that*

$$\int_G \exp\left(\exp\left(\frac{I_\alpha f(x)^{p(x)/(p(x)-1)}}{c_1^{p(x)/(p(x)-1)}}\right)\right) dx \leqslant c_2$$

for all nonnegative measurable functions f *on* G *with* $\|f\|_{L^\Phi(G)} \leqslant 1$.

7 Continuity of Riesz potentials

THEOREM 7.1. *Suppose further*

$$p(x) \geqslant n/\alpha \qquad \text{and} \qquad q(x) > p(x) - 1$$

for $x \in \mathbb{R}^n$. *If* f *is a nonnegative measurable function on* G *with* $\|f\|_{L^\Phi(G)} \leqslant 1$, *then* $I_\alpha f(x)$ *is continuous and*

$$|I_\alpha f(z) - I_\alpha f(x)| \leqslant C\gamma_5(x)(\log(1/|z-x|))^{-(q(x)-p(x)+1)/p(x)}$$

as $z \to x$ *for each* $x \in G$, *where*

$$\gamma_5(x) = \gamma_4(x)^{-(p(x)-1)/p(x)}(\log(1/\gamma_4(x)))^{q(x)/p(x)}$$

with $\gamma_4(x) = \min\{q(x) - p(x) + 1, 1/2\}$.

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