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On the Heat Equation in a Half-Space with a Nonlinear Boundary Condition

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1 Introduction

This is an abbreviated version of the forthcoming paper [12].

In this paper, we consider the heat equation in the half space of $\mathbf{R}^N$ with a nonlinear boundary condition,

$$(1.1)\qquad \begin{cases} \partial_t u = \Delta u, & x \in \Omega,\ \ t > 0, \\ \partial_\nu u = u^p, & x \in \partial\Omega,\ \ t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \end{cases}$$

where $\Omega = \{x = (x', x_N) \in \mathbf{R}^N : x_N > 0\}$, $N \geq 2$, $\partial_t = \partial/\partial t$, $\partial_\nu = -\partial/\partial x_N$, and $p > 1$. In this paper we assume that

$$(1.2)\qquad \phi \in X \equiv \left\{ f \in L^\infty(\Omega) \cap L^2\left(\Omega, e^{|x|^2/4}dx\right) \ :\ f \geq 0 \text{ in } \Omega \right\},$$

$$(1.3)\qquad 1 + 1/N < p, \quad (N-2)p < N,$$

and give a classification of the large time behaviors of the nonnegative global solutions of (1.1).

The nonlinear boundary value problem (1.1) can be physically interpreted as a nonlinear radiation law, and has been studied in many papers (see [2], [4], [5], [7], [8], [12], [14], [17], and references therein). However, for the large time behaviors of the solutions of (1.1) in unbounded domains, there are only a few papers even if $\Omega = \mathbf{R}^N_+$. Among others, in [2], Deng, Fila, and Levine proved that, if $1 < p \leq 1 + 1/N$, then there does not exist non-trivial global solutions of (1.1). Furthermore they proved that, if $p > 1 + 1/N$, then, for some "small" initial data ϕ, there exists a non-trivial global solution of (1.1) satisfying

$$\|u(t)\|_{L^\infty(\Omega)} = O(t^{-1/2(p-1)}) \qquad \text{as} \qquad t \to \infty.$$

Recently, in [14], the second author of this paper proved that there exists a positive constant δ with the following property:

$$\text{(1.4)} \qquad \begin{array}{l} \text{if } \|\phi\|_{L^1(\Omega)}\|\phi\|_{L^\infty(\Omega)}^{N(p-1)-1} < \delta, \text{ then there exists a global solution } u \text{ of (1.1)} \\ \text{such that } \|u(t)\|_{L^q(\Omega)} = O(t^{-(N/2)(1-1/q)}) \text{ as } t \to \infty \text{ for any } q \in [1,\infty]. \end{array}$$

Furthermore he proved that there exists the limit

$$\text{(1.5)} \qquad c_* = 2\lim_{t\to\infty}\int_\Omega u(x,t)dx = 2\left(\int_\Omega u(x,0)dx + \int_0^\infty\int_{\partial\Omega} u(x,t)^p d\sigma dt\right)$$

such that

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{q})}\|u(t) - c_* g(t)\|_{L^q(\Omega)} = 0$$

for any $q \in [1,\infty]$, where $g(x,t) = (4\pi t)^{-N/2}\exp(-|x|^2/4t)$. (See also Proposition 2.1.)

On the other hand, for the Cauchy problem of the semilinear heat equation,

$$\text{(1.6)} \qquad \partial_t u = \Delta u + u^p \quad \text{in} \quad \mathbf{R}^N \times (0,\infty), \quad u(x,0) = \lambda\varphi \quad \text{in} \quad \mathbf{R}^N,$$

in [15], Kawanago gave a classification of the large time behaviors of the global solutions. He proved that, if $p > 1 + 2/N$ and $(N-2)p < N+2$, for any $\varphi \in X \setminus \{0\}$, there exists a positive constant λ_φ such that

(a) if $0 < \lambda < \lambda_\varphi$, then the solution u of (1.6) exists globally in time and $\|u(t)\|_{L^\infty(\mathbf{R}^N)} \asymp t^{-\frac{N}{2}}$ as $t \to \infty$;

(b) if $\lambda = \lambda_\varphi$, then the solution u of (1.6) exists globally in time and $\|u(t)\|_{L^\infty(\mathbf{R}^N)} \asymp t^{-\frac{1}{p-1}}$ as $t \to \infty$;

(c) if $\lambda > \lambda_\varphi$, then the solution u of (1.6) does not exist globally in time, and blows-up in a finite time, that is, $\limsup_{t\to T_M-0}\|u(t)\|_{L^\infty(\mathbf{R}^N)} = \infty$ for some $T_M > 0$.

(See also [13].) Furthermore he proved that there exists a positive constant $\delta' > 0$ with the following property:

$$\text{(1.7)} \qquad \begin{array}{l} \text{if } \|\phi\|_{L^{N(p-1)/2}(\mathbf{R}^N)} < \delta', \text{ then there exists a global solution } u \text{ of (1.6)} \\ \text{such that } \|u(t)\|_{L^q(\Omega)} = O(t^{-(N/2)(1-1/q)}) \text{ as } t \to \infty \text{ for any } q \in [1,\infty]. \end{array}$$

The property (1.7) plays an important role of proving the existence of λ_φ.

In this paper, by following the strategy in [13] and [15], we study the nonlinear boundary problem (1.1) under the conditions (1.2) and (1.3), and give a classification of the large time behaviors of the nonnegative global solutions. Furthermore we improve the result of [14], and give an optimal estimate of the $L^q(\Omega)$ distance from the solution u to its asymptotic profile (see Theorem 1.2-(i)). For our problem (1.1), it seems difficult to apply the arguments in [13] and [15] directly because of the nonlinearity on the boundary $\partial\Omega$ and the unboundedness of the boundary $\partial\Omega$. In order to overcome this difficulty, we first prove the Hölder continuity of the solutions of the parabolic equations under a Robin boundary condition. Next we construct approximate solutions to the problem (1.1), and obtain uniform Hölder estimates for the approximate solutions. Then we can obtain Hölder estimates of the solution u of (1.1). Furthermore, by using the standard regularity theorems for parabolic equations, we can modify the argument in [6] and [15], and obtain global bounds for the global solutions of (1.1) (see also Remark 3.1). Moreover, by using the property (1.4), instead of (1.7), we can follow the strategy in [13] and [15], and obtain the similar classification of the large time behaviors of the global solutions of (1.1) as in [15] (see Theorem 1.1 and Theorem 1.2-(ii), (iii)).

Next we give the definition of the solution of (1.1).

Definition 1.1 *Let $\tau > 0$ and $u \in C(\overline{\Omega} \times (0,\tau)) \cap L^\infty(0,\sigma : L^\infty(\Omega))$ for all $\sigma \in (0,\tau)$. Then the function u is a solution of (1.1) in $\Omega \times [0,\tau)$ if u satisfies*

$$u(x,t) = \int_\Omega G(x,y,t)\phi(y)dy + \int_0^t \int_{\partial\Omega} G(x,y,t-s)u(y,s)^p d\sigma_y ds$$

for any $(x,t) \in \Omega \times (0,\tau)$. Here $d\sigma$ is the $(N-1)$ dimensional Lebesgue measure on $\partial\Omega = \mathbf{R}^{N-1}$ and $G = G(x,y,t)$ is the Green function for the heat equation on Ω with the homogeneous Neumann boundary condition, that is,

$$(1.8) \quad G(x,y,t) = (4\pi t)^{-\frac{N}{2}} \left[\exp\left(-\frac{|x-y|^2}{4t}\right) + \exp\left(-\frac{|x-y_*|^2}{4t}\right) \right], \quad x,y \in \Omega,\ t > 0,$$

where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \Omega$.

Then, for any nonnegative initial data $\phi \in L^\infty(\Omega)$, the problem (1.1) has a unique classical solution (see Lemma 2.5), and

$$(1.9) \qquad T_M(\phi) = \sup\{\tau \in (0,\infty) : u \text{ is a solution of (1.1) in } \Omega \times (0,\tau)\}$$

can be defined. In particular, if $T_M(\phi) < \infty$, then $\limsup_{t\to T_M(\phi)-0} \|u(t)\|_{L^\infty(\Omega)} = \infty$ (see Lemma 2.5-(ii)), and we call $T_M(\phi)$ the blow-up time of the solution u. Furthermore, under the conditions (1.2) and (1.3), we can define the following energy functional for the solution u,

$$F[u](t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_{\partial\Omega} u^{p+1} d\sigma$$

for any $t \in (0, T_M(\phi))$ (see Lemma 3.2).

We introduce some notation. Let

$$\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}, \qquad |||\cdot||| \equiv \|\cdot\|_\infty + \|\cdot\|_{L^2(\Omega, e^{|x|^2/4}dx)},$$

where $q \in [1,\infty]$. Then, by (1.2), the set X is a closed cone of the Banach space with the norm $|||\cdot|||$. We put

$$K = \{\phi \in X \,:\, T_M(\phi) = \infty\}, \qquad B = X \setminus K = \{\phi \in X \,:\, T_M(\phi) < \infty\},$$

and denote by $\operatorname{Int} K$ and ∂K the interior and the boundary of K in X, respectively.

Now we are ready to state the main results of this paper.

Theorem 1.1 ([12]) *Assume the condition* (1.3). *Then there holds the following:*
(i) *the set K is a unbounded closed convex set in X such that $0 \in \operatorname{Int} K$;*
(ii) *for any $\varphi \in X \setminus \{0\}$, there exists a positive constant λ_φ such that*

$$\lambda\varphi \in \begin{cases} \operatorname{Int} K & \textit{if} \quad \lambda \in (0, \lambda_\varphi), \\ \partial K & \textit{if} \quad \lambda = \lambda_\varphi, \\ B & \textit{if} \quad \lambda > \lambda_\varphi; \end{cases}$$

(iii) *the unit sphere $\mathbf{S}$ in X and ∂K are homeomorphic by the map $\mathbf{S} \ni \varphi \to \lambda_\varphi \varphi \in \partial K$.*

Theorem 1.2 ([12]) *Let u be a solution of* (1.1) *under the condition* (1.3). *Then there holds the following:*
(i) *if $\phi \in \operatorname{Int} K \setminus \{0\}$, then*

$$\|u(t)\|_q \asymp t^{-\frac{N}{2}(1-\frac{1}{q})} \quad as \quad t \to \infty \tag{1.10}$$

for any $q \in [1,\infty]$. Furthermore there exist the limit c_* given in (1.5) *and a constant C such that*

$$t^{\frac{N}{2}(1-\frac{1}{q})}\|u(t) - c_* g(t)\|_q \le Ct^{-\frac{1}{2}} + Ct^{-\frac{N}{2}(p-1-\frac{1}{N})}, \qquad t \ge 1, \tag{1.11}$$

for any $q \in [1,\infty]$;
(ii) *if $\phi \in \partial K$, then $\|u(t)\|_\infty \asymp t^{-1/2(p-1)}$ as $t \to \infty$;*
(iii) *if $\phi \in B$, then $\lim_{t \to T_M(\phi)-0} F[u](t) = -\infty$.*

Remark 1.1 (i) *Consider the Cauchy problem* (1.6) *under the conditions $p > 1+2/N$ and $(N-2)p < N+2$. Then there holds the similar classification of the large time behaviors of the global solutions as in Theorems* 1.1 *and* 1.2 *(see* [15]*). Furthermore, for the problem* (1.6)*, there also holds the similar estimate to* (1.11) *(see* [11] *and Proposition* 20.13 *in* [17]*).*
(ii) *In this paper we treat only the case $N \ge 2$, but can prove Theorems* 1.1 *and* 1.2 *for the case $N = 1$ with minor modifications.*

The rest of this paper is organized as follows: In Section 2 we consider the parabolic equations with a Robin boundary condition, and give the Harnack inequality and the

Hölder continuity of the solutions. Furthermore we prove the existence and the uniqueness of the solutions of (1.1), and give some properties of the solutions. In Section 3 we introduce a rescaled function $w = w(y,s)$ of u and its energy functional $E[w](s)$, and study the large time behavior of w. Furthermore we give a global bound for the function w by using the Sobolev trace inequality, the Hölder estimates given in Section 2, and the regularity theorems for parabolic equations. The proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5, respectively.

2 Preliminaries

In this section we consider parabolic equations with a Robin boundary condition, and give the Harnack inequality and the Hölder continuity of the solutions. Furthermore we give some preliminary results on the problem (1.1).

2.1 Parabolic equations with a Robin boundary condition

Let $\Omega = \mathbf{R}^N_+$. We consider the following parabolic equation with a Robin boundary condition,

$$\begin{cases} \partial_t v = \Delta v + b(x,t)\cdot\nabla v + V(x,t)v & \text{in} \quad D_+ \times (-1,1), \\ \partial_\nu v = \Gamma(x,t)v \quad \text{on} \quad \partial' D_+ \times (-1,1) & \text{if} \quad \partial' D_+ \neq \emptyset. \end{cases} \tag{2.1}$$

Here D is a smooth domain in $\mathbf{R}^N$ such that $D \cap \Omega \neq \emptyset$ and

$$D_+ = D \cap \Omega, \qquad \partial' D_+ = \partial D_+ \cap \partial\Omega, \qquad \partial_\nu = -\partial/\partial x_N.$$

In this subsection we assume that there exists a constant $r > \max\{N/2, 1\}$ such that

$$\begin{aligned} &b = (b_1, \dots, b_N) \in L^\infty(-1,1 : L^{2r}(D_+ : \mathbf{R}^N)), \\ &V \in L^\infty(-1,1 : L^r(D_+)), \qquad \Gamma \in L^\infty(-1,1 : L^{2r-1}(\partial' D_+)), \end{aligned} \tag{2.2}$$

and put

$$\Phi(b,V,\Gamma) \equiv \|b\|_{L^\infty(-1,1:L^{2r}(D_+:\mathbf{R}^N))} + \|V\|_{L^\infty(-1,1:L^r(D_+))} + \|\Gamma\|_{L^\infty(-1,1:L^{2r-1}(\partial' D_+))}.$$

We first give the definition of the solution of (2.1).

Definition 2.1 *Let $v \in L^\infty((-1,1) : L^2(D_+)) \cap L^2((-1,1) : H^1(D_+))$. Then the function v is said to be a solution of* (2.1) *if v satisfies*

$$\begin{aligned} &\int_{D_+} v(x,t)\varphi(x,t)dx\Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_{\partial' D_+} \Gamma(x,t)v(x,t)\varphi(x,t)d\sigma dt \\ &\quad + \int_{t_1}^{t_2}\int_{D_+}\Big[- v\partial_t\varphi + \nabla v\cdot\nabla\varphi - b(x,t)\cdot\nabla v\varphi - V(x,t)v\varphi\Big]dxdt = 0 \end{aligned}$$

for all $\varphi \in C_0^\infty(D \times (-1,1))$ *and almost all* $t_1, t_2 \in (-1,1)$.

We first obtain the following lemma by using the Sobolev trace inequality (see Theorem 5.22 in [1]) and Lemma A.3 in [10].

Lemma 2.1 *Let* $x_0 \in \Omega$ *and put* $D = B(x_0, 1)$ *and* $Q = D_+ \times (-1,1)$. *Then, for any* $\beta > 0$, *there exists a positive constant* C *such that*

$$\int_{-1}^{1}\int_{\partial' D_+} |\Gamma|\varphi^2 d\sigma dt + \iint_Q [|b|^2 + |V|]\varphi^2 dxdt \le \beta \iint_Q |\nabla\varphi|^2 dxdt + C\iint_Q \varphi^2 dxdt$$

for all $\varphi \in L^2((-1,1) : H_0^1(D))$. *Here* C *depends only on* N, r, *and* $\Phi(b, V, \Gamma)$.

By Lemma 2.1, we can apply the arguments in [19] (see also Appendix of [10]) directly, and obtain the following lemma on the Harnack inequality for the solutions of (2.1).

Lemma 2.2 *Let* $x_0 \in \Omega$ *and put* $D = B(x_0, 1)$. *Let* v *be a nonnegative solution of* (2.1) *in* $Q = D_+ \times (-1,1)$ *under the condition* (2.2). *Then there exists a positive constant* C_1 *such that*

$$\sup_{Q^-} v \le C_1 \inf_{Q^+} v,$$

where

$$Q^+ = \left[\Omega \cap B\left(x_0, \frac{1}{2}\right)\right] \times \left(\frac{1}{4}, \frac{3}{4}\right), \qquad Q^- = \left[\Omega \cap B\left(x_0, \frac{1}{2}\right)\right] \times \left(-\frac{3}{4}, -\frac{1}{4}\right).$$

Furthermore, let w *be a nonnegative solution of*

$$\begin{cases} \partial_t w = \Delta w + b(x,t)\cdot\nabla w + V(x,t)w + f & \text{in} \quad D_+ \times (-1,1), \\ \partial_\nu w = \Gamma(x,t)w + g & \text{on} \quad \partial D'_+ \times (-1,1) \quad \text{if} \quad \partial D'_+ \neq \emptyset, \end{cases}$$

where $f \in L^\infty((-1,1) : L^r(D_+))$ *and* $g \in L^\infty((-1,1) : L^{2r-1}(\partial' D_+))$. *Then there exists a positive constant* C_2 *such that*

$$\sup_{Q^-}(w + E) \le C_2 \inf_{Q^+}(w + E),$$

where $E = \|f\|_{L^\infty((-1,1):L^r(D_+))} + \|g\|_{L^\infty((-1,1):L^{2r-1}(\partial' D_+))}$. *Here the constants* C_1 *and* C_2 *depend only on* N, r, *and* $\Phi(b, V, \Gamma)$.

By Lemma 2.2, we apply the same arguments as in [18] and [19] (see also [9]) to the problem (2.1), and have the following lemma, which gives the Hölder continuity of the solutions of (2.1).

Lemma 2.3 *Let* $x_0 \in \Omega$ *and put* $D = B(x_0, 1)$. *Assume* (2.2). *Let* v *be a solution of* (2.1) *in* $D_+ \times (-1,1)$ *such that* $M \equiv \|v\|_{L^\infty(D_+ \times (-1,1))} < \infty$. *Then there exist positive constants* C *and* $\alpha \in (0,1)$ *such that*

$$\|v\|_{C^{\alpha,\alpha/2}(Q')} \le C,$$

where $Q' = [\Omega \cap B(x_0, 1/2)] \times (-1/4, 1/4)$. *Here the constants* C *and* α *depend only on* N, r, $\Phi(b, V, \Gamma)$, *and* M.

2.2 Preliminary results for the problem (1.1)

In this subsection we give some preliminary results on the problem (1.1). We first give the uniqueness of the solution of (1.1).

Lemma 2.4 *Let $i = 1, 2$, $\tau > 0$, and u_i be a solution of* (1.1) *in* $\Omega \times [0, \tau)$ *with* $\phi = \phi_i \in L^\infty(\Omega)$. *Then, for any* $\sigma \in (0, \tau)$, *there exists a constant C such that*

$$\sup_{0<t\leq\sigma} \|u_1(t) - u_2(t)\|_\infty \leq C\|\phi_1 - \phi_2\|_\infty.$$

Here the constant C depends on $\|u_1\|_{L^\infty(\Omega\times(0,\sigma))}$ *and* $\|u_2\|_{L^\infty(\Omega\times(0,\sigma))}$.

Next we obtain the following lemma by Lemmas 2.3, 2.4, the comparison principle, the regularity estimates (see [16]), and approximate solutions to the problem (1.1).

Lemma 2.5 *Let* $\phi \in L^\infty(\Omega)$. *Then there holds the following:*
(i) *there exists a unique solution of* (1.1) *in* $\Omega \times [0, \tau)$ *for some* $\tau > 0$. *In particular, there exists a constant* τ_0 *depending only on* N, p, *and* $\|\phi\|_\infty$, *such that* $0 < \tau_0 < \tau$ *and*

$$\sup_{0<t\leq\tau_0} \|u(t)\|_\infty \leq 2\|\phi\|_\infty;$$

(ii) *let u be a solution of* (1.1) *in* $\Omega \times [0, \tau)$ *for some* $\tau > 0$. *Then u satisfies* (1.1) *in the classical sense for all* $(x, t) \in \overline{\Omega} \times (0, \tau)$. *Furthermore, if*

$$\limsup_{t\to\tau-0} \|u(t)\|_\infty < \infty,$$

then there exists a solution U of (1.1) *in* $\Omega \times [0, \tau')$ *for some* $\tau' > \tau$ *such that* $U(x, t) = u(x, t)$ *in* $\Omega \times (0, \tau)$.

Proof. See the proof of Lemma 2.6 in [12]. □

In what follows, we write

$$(S(t)\phi)(x) = u(x, t), \qquad (x, t) \in \Omega \times (0, T_M(\phi)),$$

for simplicity. Here $T_M(\phi)$ is the constant defined by (1.9). Then we have the following two lemmas.

Lemma 2.6 *Let* ϕ_1, $\phi_2 \in L^\infty(\Omega)$. *Then, for any* $0 < \sigma < T_M(\phi_1)$ *and* $\epsilon > 0$, *there exists a positive constant δ such that, if* $\|\phi_1 - \phi_2\|_\infty \leq \delta$, *then*

$$T_M(\phi_2) > \sigma, \qquad \sup_{0<t\leq\sigma} \|S(t)\phi_1 - S(t)\phi_2\|_\infty < \epsilon.$$

Lemma 2.7 *Let ϕ_1, $\phi_2 \in L^\infty(\Omega) \cap L^1(\Omega)$. Then, for any $0 < \sigma < \min\{T_M(\phi_1), T_M(\phi_2)\}$ and $\epsilon > 0$, there exists a positive constant δ such that, if $\|\phi_1 - \phi_2\|_1 \le \delta$, then*

$$\sup_{0<t\le\sigma} \|S(t)\phi_1 - S(t)\phi_2\|_1 < \epsilon.$$

Finally we recall the following proposition given in [14].

Proposition 2.1 (See Theorem 1.1 in [14].) *Assume the conditions* (1.2) *and* (1.3). *Then there exists a positive constant δ with the following property: if the initial data ϕ satisfies*

$$\|\phi\|_1 \|\phi\|_\infty^{N(p-1)-1} < \delta,$$

then there exists a solution u of (1.1) *in $\Omega \times (0,\infty)$ such that*

$$\sup_{t>0} t^{\frac{1}{2q}+\frac{N}{2}(1-\frac{1}{q})}\|u(t)\|_{L^q(\partial\Omega)} + \sup_{t>0}(1+t)^{\frac{N}{2}(1-\frac{1}{q})}\|u(t)\|_q < \infty, \tag{2.3}$$

for any $q \in [1,\infty]$. Furthermore there exists the limit c_* given in (1.5) *such that*

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{q})}\|u(t) - c_* g(t)\|_q = 0, \quad q \in [1,\infty]. \tag{2.4}$$

3 Upper estimates of the solutions

Let $u = S(t)\phi$ be the solution of (1.1) under the conditions (1.2) and (1.3). Put

$$w(y,s) = (1+t)^{\frac{1}{2(p-1)}} u(x,t), \quad y = (1+t)^{-\frac{1}{2}}x, \quad s = \log(1+t). \tag{3.1}$$

We write $w = \mathcal{S}(s)\phi$. Then the function w satisfies

$$\begin{cases} \partial_s w = Lw + \kappa w & \text{in} \quad \Omega \times (0, S_M), \\ \partial_\nu w = w^p & \text{on} \quad \partial\Omega \times (0, S_M), \quad w(y,0) = \phi(y) \quad \text{in} \quad \Omega, \end{cases} \tag{3.2}$$

where $\kappa = 1/2(p-1)$ and $S_M = \log(1 + T_M(\phi))$. Here

$$Lw = \Delta w + \frac{y}{2}\cdot\nabla w = \frac{1}{\rho}\mathrm{div}(\rho\nabla w), \qquad \rho(y) = e^{|y|^2/4}.$$

In this section we give some upper estimates of the function $w(s)$. In what follows, we write $\|\cdot\| = \|\cdot\|_{L^2(\Omega,\rho dy)}$ for simplicity.

We first recall the following lemma on the eigenvalue problem for the operator L. (See also [3] and [13].)

Lemma 3.1 *Consider the eigenvalue problem*

$$(3.3) \qquad -L\varphi = \lambda\varphi \quad in \quad \Omega, \quad \partial_\nu\varphi = 0 \quad on \quad \partial\Omega, \quad \varphi \in H^1(\Omega, \rho dy).$$

Let $\{\lambda_i\}_{i=0}^{\infty}$ *be the eigenvalues of the problem* (3.3) *such that* $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$. *Then*

$$\lambda_i = \frac{N+i}{2}, \qquad i = 0, 1, 2 \dots .$$

The eigenspace corresponding to λ_0 *is spanned by* $\varphi_0(y) = c_0 e^{-|y|^2/4}$, *and the eigenspace corresponding to* λ_1 *is spanned by* $\varphi_i(y) = c_1 y_i e^{-|y|^2/4}$ $(i = 1, \dots, N-1)$, *where* c_0 *and* c_1 *are constants to be chosen such that* $\|\varphi_0\| = 1$ *and* $\|\varphi_1\| = \cdots = \|\varphi_{N-1}\| = 1$. *Furthermore*

$$\begin{aligned}
\lambda_0 = \frac{N}{2} &= \inf\left\{\int_\Omega |\nabla f|^2 \rho dy \ : \ f \in H^1(\Omega, \rho dy), \ \|f\| = 1\right\}, \\
\lambda_1 = \frac{N+1}{2} &= \inf\left\{\int_\Omega |\nabla f|^2 \rho dy \ : \ f \in H^1(\Omega, \rho dy), \ \|f\| = 1, \ (f, \varphi_0) = 0\right\}, \\
\lambda_2 = \frac{N+2}{2} &= \inf\left\{\int_\Omega |\nabla f|^2 \rho dy \ : \ f \in H^1(\Omega, \rho dy), \ \|f\| = 1, \right. \\
&\qquad\qquad \left. (f, \varphi_i) = 0 \quad for \quad i = 0, 1, \dots, N-1\right\}.
\end{aligned}$$

Next we have the following lemma by using the trace inequality in the space $H^1(\Omega, \rho dy)$ (see Lemma 3.2 in [12]).

Lemma 3.2 *Let* u *be the solution of* (1.1) *under the conditions* (1.2) *and* (1.3) *and* w *the function defined by* (3.1). *Then, for any* $0 \le S_1 < S_2 < S_M$, *there exists a constant* C *such that*

$$\|w(s)\|^2 + (s - S_1)\|\nabla w(s)\|^2 + (s - S_1)\int_{\partial\Omega} w(s)^{p+1} \rho d\sigma \le C\|w(S_1)\|^2, \quad S_1 < s < S_2.$$

Here C *depends only on* N, p, $S_2 - S_1$, *and* $M' \equiv \|w\|_{L^\infty(\Omega \times (S_1, S_2))} < \infty$.

By Lemma 3.2, we can define the energy functional $E[w](s)$ of w,

$$(3.4) \qquad E[w](s) = \int_\Omega \left[\frac{1}{2}|\nabla w|^2 - \frac{\kappa}{2}|w|^2\right] \rho dy - \frac{1}{p+1}\int_{\partial\Omega} w^{p+1} \rho d\sigma$$

for all $s \in (0, S_M)$. Then, by Lemma 3.1 and (3.2), we can apply the same arguments as in Lemma 2.3 of [13], Proposition 3.1-(i), (ii), (iii), and (iv) of [13], and obtain the following lemma.

Lemma 3.3 *Assume the same conditions as in Lemma* 3.2. *Then* $E[w](s)$ *is a non-increasing function in* $(0, S_M)$ *with the following properties:*
(i) *if there exists* $s_0 \geq 0$ *such that* $E[w](s_0) \leq 0$ *and* $w(s_0) \not\equiv 0$, *then* $T_M(\phi) < \infty$;
(ii) *if* $\phi \in K$, *then*

$$E[w](s) > 0, \qquad s > 0. \tag{3.5}$$

Furthermore, for any $s > 0$, *there exists a constant* C *depending only on* N, p, *and* $E[w](s)$ *such that*

$$\sup_{\tau > s} \|w(\tau)\|^2 + \int_s^\infty \|\partial_\tau w(\tau)\|^2 d\tau + \sup_{\tau \geq s} \int_\tau^{\tau+1} \|\nabla w(\eta)\|^4 d\eta \leq C.$$

Next, by using Lemma 2.3, we modify the argument in [6], and obtain the following lemma (see also Remark 3.1).

Lemma 3.4 *Assume the same conditions as in Lemma* 3.2 *and* $\phi \in K$. *Furthermore assume that*

$$\begin{gathered} \int_0^S \|\partial_s w\|^2 ds \leq l < \infty, \\ \sup_{0<s<S} \int_s^{s+1} \|\nabla w(\tau)\|^4 d\tau \leq l' < \infty, \\ \|w\|_{L^\infty(\partial\Omega\times(s_0,S))} = \|w\|_{L^\infty(\partial\Omega\times(0,S))}, \end{gathered} \tag{3.6}$$

for some $0 < s_0 < S$ *and positive constants* l *and* l'. *Then there exists a constant* A *such that* $\|w\|_{L^\infty(\partial\Omega\times(0,S))} \leq A$. *Here the constant* A *depends only on* N, p, s_0, l, l', *and* $\|\phi\|_\infty$ *and is independent of* w *and* S.

Remark 3.1 *For the Cauchy problem* (1.6), *the similar result to Lemma* 3.4 *is given in Lemma* 3 *in* [15], *without any conditions such as* (3.6). *The proof is based on the argument in the proof of Lemma* 1 *in* [15], *and the details of the proof are omitted. However the proof of Lemma* 3 *in* [15] *seems not to be clear. In our proof of Lemma* 3.4, *we obtain a contradiction by using the condition* (3.6). *See the proof of Lemma* 3.5 *in* [12].

By Lemmas 3.2–3.4, we can obtain a global bound for the global solutions of (1.1).

Lemma 3.5 *Assume the condition* (1.3). *Let* $\phi \in K$ *and* u *be a solution of* (1.1). *Then there exists a constant* C *depending only on* N, p, $\|\phi\|_\infty$, *and* $\|\phi\|$, *such that* $\|w\|_{L^\infty(\Omega\times(0,\infty))} \leq C$.

4 Behaviors of global solutions

In this section we study the large time behaviors of global solutions of (1.1), and prove Theorem 1.1. Put

$$
\begin{aligned}
H &= \left\{\phi \in K : \sup_{t\geq 1} t^{\frac{N}{2}(1-\frac{1}{q})}\|S(t)\phi\|_q < \infty \text{ for all } q \in [1,\infty]\right\},\\
S &= \Big\{f \in L^\infty(\Omega) \cap H^1(\Omega, \rho dy) \cap C(\overline{\Omega}) :\\
&\qquad\qquad f \text{ satisfies } Lf + \kappa f = 0 \text{ in } \Omega,\ f > 0 \text{ in } \Omega,\ \partial_\nu f = f^p \text{ on } \partial\Omega\Big\}.
\end{aligned}
$$

Lemma 4.1 *Assume the condition* (1.3). *Then*
(i) *K is a unbounded closed convex set;*
(ii) *H is an open set in X such that $0 \in H \subset \operatorname{Int} K$;*
(iii) *Let $\phi \in H$ and $u = S(t)\phi$. Then there hold* (2.3) *and* (2.4).
(iv) *Let f, $g \in S$ such that $f \geq g$ in Ω. Then $f = g$ in Ω.*

Proof. We first prove Lemma 4.1-(i). By Lemma 3.5, we see that K is a closed set in X. By Proposition 2.1, we see that K is a unbounded set in X such that $0 \in \operatorname{Int} H \subset \operatorname{Int} K$. Furthermore the convexity of K is proved by the comparison principle and the convexity of the nonlinear term u^p on the boundary $\partial\Omega$, and the proof of Lemma 4.1-(i) is complete.

Next we prove Lemma 4.1-(ii) and (iii). Let $\phi \in H$, $\tilde{\phi} \in X$, $u = S(t)\phi$, and $\tilde{u} = S(t)\tilde{\phi}$. Let δ be the constant given in Proposition 2.1. By $\phi \in H$, we have

$$\lim_{t\to\infty} \|u(t)\|_1 \|u(t)\|_\infty^{N(p-1)-1} = 0.$$

So there exists a constant T such that

$$\|u(T)\|_1 \|u(T)\|_\infty^{N(p-1)-1} < \delta/2.$$

Then, by Proposition 2.1, we have the statement of Lemma 4.1-(iii). Furthermore, by Lemmas 2.6 and 2.7, there exists a positive constant ϵ such that, if $|||\phi - \tilde{\phi}||| < \epsilon$, then

$$\|\tilde{u}(T)\|_1 \|\tilde{u}(T)\|_\infty^{N(p-1)-1} < \delta.$$

Therefore, by using Proposition 2.1 again, we have $\tilde{\phi} \in H$, and see $H = \operatorname{Int} H \subset \operatorname{Int} K$; thus the proof of Lemma 4.1-(ii) is complete.

Next we prove Lemma 4.1-(iv). Let f, $g \in S$ such that $f \geq g$ in Ω. Then we have

$$
\begin{aligned}
\int_\Omega \rho \nabla f \cdot \nabla g dy - \int_{\partial\Omega} f^p g \rho d\sigma &= \kappa \int_\Omega f g dy,\\
\int_\Omega \rho \nabla f \cdot \nabla g dy - \int_{\partial\Omega} g^p f \rho d\sigma &= \kappa \int_\Omega f g \rho dy.
\end{aligned}
$$

These imply that

$$\int_{\partial\Omega}(f^{p-1}-g^{p-1})fg\rho d\sigma=0,$$

that is, $f=g$ on $\partial\Omega$. Therefore the function $w=f-g$ satisfies

$$Lw+\kappa w=0 \quad \text{in} \quad \Omega, \qquad \partial_\nu w=w=0 \quad \text{on} \quad \partial\Omega.$$

This together with Lemma 3.1 implies that

$$\kappa\int_\Omega w^2\rho dy=\int_\Omega|\nabla w|^2\rho dy\geq\frac{N}{2}\int_\Omega w^2\rho dy.$$

Then, since $\kappa=1/2(p-1)<N/2$, we see that $w=0$ in Ω. Therefore we have $f=g$ in Ω, and obtain Lemma 4.1-(iv); thus the proof of Lemma 4.1 is complete. □

Lemma 4.2 *Assume the condition* (1.3). *Let* $\phi\in K$ *and* u *be a solution of* (1.1). *Then the* ω*-limit set of* w *in* X, $\omega(\phi)=\bigcap_{s>0}\overline{\{w(\tau)\ :\ \tau\geq s\}}^X$, *is a compact set in* X *such that* $\omega(\phi)\subset S\cup\{0\}$.

Proof. By Lemmas 3.2 and 3.5, there exists a constant C_1 such that

$$\|w(s)\|^2+\|\nabla w(s)\|^2\leq C_1 \tag{4.1}$$

for all $s\geq 1$. By Lemmas 2.3 and 3.5, there exists a constant $\alpha\in(0,1)$ such that $\|w\|_{C^{\alpha,\alpha/2}(\mathcal{K}\times(1,\infty))}<\infty$ for any compact set $\mathcal{K}\subset\overline{\Omega}$. Furthermore, by Theorem 10.1 in Chapter 4 of [16], we have

$$\|w\|_{C^{2+\alpha,1+\alpha/2}(\mathcal{K}'\times(2,\infty))}<\infty \tag{4.2}$$

for any compact set $\mathcal{K}'\subset\overline{\Omega}$. Then, by Lemma 3.3, (4.1), and (4.2), we can apply the same argument as in the proof of Proposition 5 in [15] to the function w, and obtain the conclusion of Lemma 4.2. □

Lemma 4.3 *Assume the condition* (1.3). *Let* $\varphi\in X$ *and put* $\lambda_K=\sup\{\lambda>0\ :\ \lambda\varphi\in K\}$. *Then* $\lambda_K\in(0,\infty)$ *and* $\lambda\varphi\in K$ *if and only if* $\lambda\leq\lambda_K$.

Proof. By Lemma 4.1 and the comparison principle, it suffices to prove $\lambda_K<\infty$. The proof is by contradiction. We assume that there exists a function $\varphi\in X\setminus\{0\}$ such that $\lambda\varphi\in K$ for all $\lambda>0$. By the positivity of the nontrivial nonnegative solutions of the heat equation, there exists a function $\psi\in C^\infty(\overline{\Omega})\setminus\{0\}$ such that $\operatorname{supp}\psi\subset\overline{\Omega}\cap B(0,1)$, $\inf_{\partial\Omega\cap B(0,1/2)}\psi(x)>0$, and

$$0\leq\psi(x)\leq\int_\Omega G(x,y,1)\varphi(y)dy, \qquad x\in\Omega.$$

Then, by the comparison principle, we have

$$[S(1)(\lambda\varphi)](x) \geq \lambda \int_\Omega G(x,y,1)\varphi(y)dy \geq \lambda\psi(x), \qquad x \in \Omega,$$

and obtain

$$[S(t+1)(\lambda\varphi)](x) \geq [S(t)(\lambda\psi)](x), \qquad x \in \Omega. \tag{4.3}$$

On the other hand, by (3.4), there exists a constant $\lambda' > 0$ such that

$$E[\lambda\psi](s) \leq \frac{\lambda^2}{2}\int_\Omega |\nabla\psi|^2 \rho dy - \lambda^{p+1}\frac{1}{p+1}\int_{\partial\Omega} \psi^{p+1}\rho d\sigma_y < 0$$

for all $\lambda \geq \lambda'$. This together with (3.5) implies that $\lambda'\psi \notin K$. Therefore, by (4.3), we have $\lambda'\varphi \notin K$, which is a contradiction. Therefore we see $\lambda_K < \infty$, and the proof of Lemma 4.3 is complete. □

Lemma 4.4 *Assume the condition* (1.3). *Let* $\phi \in K \setminus H$ *and* $w = \mathcal{S}(s)\phi$. *Then* $\omega(\phi) \subset S$ *and* $\liminf_{s\to\infty} \|w(s)\|_\infty > 0$.

Proof. Let $\phi \in K \setminus H$, $u(t) = S(t)\phi$, and $w(s) = \mathcal{S}(s)\phi$. Let δ be the constant given in Proposition 2.1. If $\|u(t)\|_1 \|u(t)\|_\infty^{N(p-1)-1} < \delta$ for some $t > 0$, then $\phi \in H \subset \operatorname{Int} K$. So, by $\phi \notin H$, we have

$$\|u(t)\|_1 \|u(t)\|_\infty^{N(p-1)-1} \geq \delta, \qquad t \geq 0.$$

This implies that

$$\|w(s)\|_1 \|w(s)\|_\infty^{N(p-1)-1} \geq \delta, \qquad s \geq 0.$$

Therefore, by Lemma 4.2, we have $\omega(\phi) \subset S$. Furthermore, if $\liminf_{s\to\infty} \|w(s)\|_\infty = 0$, then we have $0 \in \omega(\phi) \subset S$, which contradicts the definition of S. So we have $\liminf_{s\to\infty} \|w(s)\|_\infty > 0$, and the proof of Lemma 4.4 is complete. □

Lemma 4.5 *Assume the condition* (1.3). *Let* $\varphi \in X \setminus \{0\}$ *and put* $\lambda_H = \sup\{\lambda > 0 : \lambda\varphi \in H\}$. *Then* $\lambda\varphi \in H$ *if and only if* $\lambda < \lambda_H$. *Furthermore* $\lambda_H = \lambda_K$ *and* $\operatorname{Int} K = H$.

Proof. By Lemma 4.1-(ii) and the comparison principle, we see that $\lambda\varphi \in H$ if and only if $\lambda < \lambda_H$. In particular, since $H \subset K$, by Lemma 4.1-(i), we have

$$\lambda_H\varphi,\ \lambda_K\varphi \in K \setminus H \qquad \text{and} \qquad \lambda_K \geq \lambda_H. \tag{4.4}$$

Then the function $(\lambda_K/\lambda_H)\mathcal{S}(s)(\lambda_H\varphi)$ is a subsolution of (3.2) with the initial data $\lambda_K\varphi$, and by the comparison principle, we have

$$[\mathcal{S}(s)(\lambda_H\varphi)](y) \leq (\lambda_K/\lambda_H)[\mathcal{S}(s)(\lambda_H\varphi)](y) \leq [\mathcal{S}(s)(\lambda_K\varphi)](y)$$

for all $(y,s) \in \Omega \times (0,\infty)$. Therefore, by Lemma 4.4 and (4.4), there exist functions $f \in \omega(\lambda_H\varphi) \subset S$ and $g \in \omega(\lambda_K\varphi) \subset S$ such that

$$0 < f(y) \le (\lambda_K/\lambda_H)f(y) \le g(y), \qquad y \in \Omega.$$

Then, by Lemma 4.1-(iv), we have $f = g$ in Ω, and obtain $\lambda_K = \lambda_H$.

By Lemma 4.1-(ii), we have $H \subset \operatorname{Int} K$. It remains to prove $\operatorname{Int} K \subset H$. Let $\varphi \in \operatorname{Int} K$. Then there exists a constant $\lambda > 1$ such that $\lambda\varphi \in K$, that is, $1 < \lambda_K$. This together with $\lambda_H = \lambda_K$ implies $1 < \lambda_H$, and $\varphi = 1 \cdot \varphi \in H$. So we have $\operatorname{Int} K \subset H$, and the proof of Lemma 4.5 is complete. □

Proof of Theorem 1.1. By Lemma 4.1, we see that K is a unbounded closed convex set in X such that $0 \in \operatorname{Int} K$. By Lemmas 4.4 and 4.5, we obtain Theorem 1.1-(ii). Furthermore, by the same argument as in [15], we see that the unit sphere $\mathbf{S}$ in X and ∂K are homeomorphic, and the proof of Theorem 1.1 is complete. □

5 Proof of Theorem 1.2

Proof of Theorem 1.2-(ii) and (iii). By Theorem 1.1, we have $\partial K = K \setminus H$, and by Lemmas 3.5 and 4.4, if $\phi \in \partial K$, then

$$0 < \liminf_{s\to\infty} \|w(s)\|_\infty \le \limsup_{s\to\infty} \|w(s)\|_\infty < \infty.$$

This implies Theorem 1.2-(ii). Furthermore, by applying the similar arguments as in [6] and Proposition 2 in [15] to the solution u and its energy $F[u](t)$, we can prove Theorem 1.2-(iii) (see also Lemma 3.3). □

Proof of Theorem 1.2-(i). Let $\phi \in \operatorname{Int} K \setminus \{0\}$. By Lemma 4.5, we have $\phi \in H$, and by Lemma 4.1-(iii), we obtain (1.10). It remains to prove (1.11). Put

$$z(y,s) = (1+t)^{\frac{N}{2}} u(x,t), \quad y = (1+t)^{-\frac{1}{2}}x, \quad s = \log(1+t).$$

Then z satisfies

$$\begin{cases} \partial_s z = Lz + \dfrac{N}{2} z & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu z = e^{-ks} z^p & \text{on} \quad \partial\Omega \times (0,\infty), \quad z(y,0) = \phi(y) \quad \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

where $k = (N/2)(p - 1 - 1/N) > 0$. By (2.3), we have

$$\sup_{s>0} \|z(s)\|_\infty < \infty. \tag{5.2}$$

By Lemma 3.1, (5.1), (5.2), and the trace inequality in the space $H^1(\Omega, \rho dy)$, we have

$$\sup_{s>0} \|z(s)\|^2 < \infty. \tag{5.3}$$

Furthermore, since $w(s) = e^{\alpha s} z(s)$ with $\alpha = 1/2(p-1) - N/2$, by Lemma 3.2 and (5.3), we have

$$\sup_{s \geq 1} \|\nabla z(s)\| < \infty. \tag{5.4}$$

Then, by (5.2)–(5.4) and the trace inequality in the space $H^1(\Omega, \rho dy)$, we have

$$\sup_{s \geq 1} \int_{\partial\Omega} z(y,s)^\alpha \rho d\sigma \leq \sup_{s \geq 1} \|z(s)\|_\infty^{\alpha-2} \int_{\partial\Omega} z(y,s)^2 \rho d\sigma < \infty, \quad \alpha \geq 2. \tag{5.5}$$

Let φ_i $(i = 0, 1, \ldots, N-1)$ be functions given in Lemma 3.1. Put

$$\tilde{z}(y,s) = z(y,s) - \sum_{i=0}^{N-1} a_i(s) \varphi_i(y), \qquad s > 0, \tag{5.6}$$

where $a_i(s) = (z(s), \varphi_i)$ for $i \in \{0, 1, \ldots, N-1\}$. Then

$$(\tilde{z}(s), \varphi_i) = (\tilde{z}(s), L\varphi_i) = 0, \qquad s > 0, \tag{5.7}$$

for $i \in \{0, 1, \ldots, N-1\}$, and by Lemma 3.1, we have

$$\int_\Omega |\nabla \tilde{z}(y,s)|^2 \rho dy \geq \frac{N+2}{2} \int_\Omega |\tilde{z}(y,s)|^2 \rho dy. \tag{5.8}$$

Furthermore, by Lemma 3.1, (5.1), (5.3) (5.8), we have the following lemma.

Lemma 5.1 *Assume the same conditions as in Theorem 1.2 and $\phi \in \mathrm{Int}\, K$. Then*
(i) *there exists a constant C_1 such that $\|\tilde{z}(s)\| \leq C_1 e^{-k's}$ for all $s > 0$, where $k' = \min\{k, 1/2\}$;*
(ii) *there exists a constant C_2 such that $\|\nabla \tilde{z}(s)\| \leq C_2 e^{-\frac{k''}{4}s}$ for all $s \geq 2$, where $k'' = \min\{k, 1/4\}$;*
(iii) *for any $i = 1, \ldots, N-1$, there hold*

$$|a_i(s)|,\ |a_i'(s)| = \begin{cases} O(e^{-\frac{s}{2}}) & \text{if } k > 1/2, \\ O(se^{-\frac{s}{2}}) & \text{if } k = 1/2, \\ O(e^{-ks}) & \text{if } 0 < k < 1/2, \end{cases}$$

for all $s \geq 1$. Furthermore there holds

$$|a_0(s) - c_0 c_*|,\ |a_0'(s)| = O(e^{-ks}), \qquad s \geq 1.$$

Now we are ready to complete the proof of the inequality (1.11). By Lemma 5.1-(i) and (iii), there exists a constant C_1 such that

$$(5.9)\qquad \begin{aligned}\|z(s)-c_0c_*\varphi_0\| &\le \|\tilde{z}(s)\|+|a_0(s)-c_*c_0|+\sum_{i=1}^{N-1}|a_i(s)|\\ &\le \begin{cases} C_1e^{-ks}+C_1e^{-\frac{s}{2}} & \text{if } k\neq 1/2,\\ C_1(1+s)e^{-\frac{s}{2}} & \text{if } k=1/2,\end{cases}\end{aligned}$$

for all $s\ge 1$. Then there exists a constant C_2 such that

$$(5.10)\qquad \|u(t)-c_*g(t)\|_1\le \begin{cases} C_2t^{-k}+C_2t^{-\frac{1}{2}} & \text{if } k\neq 1/2,\\ C_2\log(1+t)t^{-\frac{1}{2}} & \text{if } k=1/2,\end{cases}$$

for all $t\ge e-1$.

On the other hand, by (1.1), we have

$$(5.11)\qquad \begin{aligned}&u(x,2t)-c_*g(x,2t)\\ &=\int_\Omega G(x,y,t)[u(y,t)-c_*g(y,t)]dy+\int_t^{2t}\int_{\partial\Omega}G(x,y,2t-s)u(y,s)^pd\sigma_yds\end{aligned}$$

for all $x\in\Omega$ and $t>0$. Then, by (1.8), (1.10) with $q=\infty$, and (5.11), there exist constants C_3 and C_4 such that

$$(5.12)\qquad \begin{aligned}t^{\frac{N}{2}}\|u(2t)-c_*g(2t)\|_\infty &\le C_3\|u(t)-c_*g(t)\|_1+C_3t^{\frac{N}{2}}\int_t^{2t}(2t-s)^{-\frac{1}{2}}\|u(s)\|_\infty^pds\\ &\le C_3\|u(t)-c_*g(t)\|_1+C_4t^{-k}\end{aligned}$$

for all $t>0$. Therefore, by (5.10) and (5.12), for any $q\in[1,\infty]$, we have

$$t^{\frac{N}{2}(1-\frac{1}{q})}\|u(t)-c_*g(t)\|_q\le \begin{cases} C_5t^{-k}+C_5t^{-\frac{1}{2}} & \text{if } k\neq 1/2,\\ C_5\log(1+t)t^{-\frac{1}{2}} & \text{if } k=1/2,\end{cases}$$

for all $t\ge e-1$, where C_5 is a constant independent of q. This together with (2.3) implies the inequality (1.11) for the case $k\neq 1/2$, and the proof of Theorem 1.2-(i) for the case $k\neq 1/2$ is complete.

It remains to prove the inequality (1.11) for the case $k=1/2$. Let $k=1/2$. Since

$$\int_{\partial\Omega}\varphi_0^p(y)\varphi_i(y)\rho d\sigma=0,\qquad i\in\{1,\dots,N-1\},$$

by Lemma 5.1-(iii), (5.2), and (5.6), there exist constants C_6 and C_7 such that

$$(5.13)\qquad \begin{aligned}&\left|\int_{\partial\Omega}z(s)^p\varphi_i\rho d\sigma\right|=\left|\int_{\partial\Omega}[z(s)^p-(c_*c_0\varphi_0)^p]\varphi_i\rho d\sigma\right|\\ &\le C_6\int_{\partial\Omega}|z(s)-c_0c_*\varphi_0||\varphi_i|\rho d\sigma\le C_6\int_{\partial\Omega}|\tilde{z}(s)||\varphi_i|\rho d\sigma+C_7se^{-\frac{s}{2}}\end{aligned}$$

for all $s > 0$. Furthermore, by Lemma 5.1-(i), (ii), the trace inequality in the space $H^1(\Omega, \rho dy)$, and the Hölder inequality, there exist constants C_8 and C_9 such that

$$\int_{\partial\Omega} |\tilde{z}(s)||\varphi_i|\rho d\sigma \leq \left(\int_{\partial\Omega} |\tilde{z}(s)|^2 \rho d\sigma\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |\varphi_i|^2 \rho d\sigma\right)^{\frac{1}{2}} \leq C_8 \|\tilde{z}(s)\|_{H^1(\Omega,\rho dy)} \leq C_9 e^{-\frac{k''}{4}s}$$

for all $s \geq 2$. This together with (5.13) implies that

$$\left|\frac{d}{ds}a_i(s) + \frac{1}{2}a_i(s)\right| \leq C_9 e^{-ks-\frac{k''}{4}s}$$

for all $s \geq 2$ and $i = 1, \ldots, N-1$. Then we can improve the inequality (5.9), and have

$$\|z(s) - c_0 c_* \varphi_0\|_1 \leq C_{10} e^{-\frac{s}{2}}, \quad s \geq 2.$$

for some constant C_{10}. Therefore, by the same argument as in the inequality (1.11) for the case $k \neq 1/2$, we have the inequality (1.11) for the case $k = 1/2$, and the proof of Theorem 1.2-(i) is complete; thus the proof of Theorem 1.2 is complete. □

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