数理解析研究所講究録
第 1679 巻 2010 年 44-48

On the Glauberman-Watanabe correspondence for p-blocks of a p-nilpotent group with a cyclic defect group

千葉大学 (Chiba university)
田阪文規 (Fuminori Tasaka)

1

Let p be a prime. Let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field k of characteristic p and having a quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article. We use the notation $^{-}$ for the reduction modulo $J(\mathcal{O})$. Let $\mathcal{R} \in \{\mathcal{K}, \mathcal{O}, k\}$. Below, for groups H_1 and H_2, an $\mathcal{R}[H_1 \times H_2]$-module X and an $(\mathcal{R}H_1, \mathcal{R}H_2)$-bimodule X will be identified in the usual way, namely $(h_1, h_2)\cdot x = h_1 \cdot x \cdot h_2^{-1}$ where $h_1 \in H_1$, $h_2 \in H_2$ and $x \in X$. For a common subgroup D of H_1 and H_2, denote by $\Delta D = \{(u, u) \mid u \in D\}$ a diagonal subgroup of $H_1 \times H_2$. Let $\mathcal{R}' \in \{\mathcal{O}, k\}$. For a p-group P, an $\mathcal{R}'$-free $\mathcal{R}'P$-module T is called an *endo-permutation* module if $\mathrm{End}_{\mathcal{R}'}(T)$ has an P-invariant $\mathcal{R}'$-basis ([1]).

Let q be a prime such that $q \neq p$. Let $S = <s>$ be a cyclic group of order q. Let $\mu \in \mathcal{O}$ be a fixed non-trivial q-th root of unity.

Let G be a finite group such that $q \nmid |G|$. Assume that S acts on G. Then with this action, we can consider the semi-direct product of G and S, denoted by GS. Denote by G^S the centralizer $C_G(S)$ of S in G. When q is odd, for $\theta \in \mathrm{Irr}(G)^S$, there is a unique extension $\hat{\theta} \in \mathrm{Irr}(GS)$ of θ, a unique character $\pi(G, S)(\theta) \in \mathrm{Irr}(G^S)$ and a unique sign ϵ_θ such that $\hat{\theta}(cs) = \epsilon_\theta \pi(G, S)(\theta)(c)$ where $c \in G^S$. When $q = 2$, for $\theta \in \mathrm{Irr}(G)^S$ and a chosen sign ϵ_θ, there is a unique extension $\hat{\theta} \in \mathrm{Irr}(GS)$ of θ and a unique character $\pi(G, S)(\theta) \in \mathrm{Irr}(G^S)$ such that $\hat{\theta}(cs) = \epsilon_\theta \pi(G, S)(\theta)(c)$ for $c \in G^S$. The character $\pi(G, S)(\theta)$ is called the *Glauberman correspondence* of θ, see [3]. For $t \in \mathbb{Z}$, let $\lambda^t\hat{\theta} \in \mathrm{Irr}(GS)$ be the extension of $\theta \in \mathrm{Irr}(G)^S$ such that $\lambda^t\hat{\theta}(gs) = \mu^t\hat{\theta}(gs)$ where $g \in G$.

Let b be an S-invariant (p-)block of G having an S-centralized defect group D. Denote by $w(b)$ the *Glauberman-Watanabe corresponding block* of b, that is, the block of G^S with a defect group D such that $\mathrm{Irr}(w(b)) = \{\pi(G, S)(\theta) \mid \theta \in \mathrm{Irr}(b) = \mathrm{Irr}(b)^S\}$. For $t \in \mathbb{Z}$, let $\hat{b}_t$ be the block of GS such that $\mathrm{Irr}(\hat{b}_t) = \{\lambda^t\hat{\theta} \mid \theta \in \mathrm{Irr}(b)\}$ (under appropriate choices of signs ϵ_θ when $q = 2$), and let e_t be the block of S corresponding to the representation of S determined by $s \mapsto \mu^t$. Let

$$b_r = \sum_{t=0}^{q-1} e_t \hat{b}_{t+r} \qquad \text{for } 0 \leqq r \leqq q-1. \tag{1}$$

Then $b = \sum_{r=0}^{q-1} b_r$ is an orthogonal idempotent decomposition of b in $(\mathcal{O}Gb)^{G^S}$ and so $b_r\mathcal{O}G$ is a direct summand of the $\mathcal{O}[G^S \times G]$-module $\mathcal{O}Gb$, and the following equation of the generalized characters of $G^S \times G$ holds, see [6] and [7]:

$$\chi_{b_0\mathcal{O}G} - \chi_{b_l\mathcal{O}G} = \sum_{\theta \in \mathrm{Irr}(b)} \epsilon_\theta \pi(G,S)(\theta) \otimes_{\mathcal{K}} \check{\theta} \qquad \text{for } 1 \leqq l \leqq q-1, \tag{2}$$

where $\chi_{b_r\mathcal{O}G}$ is a character corresponding to a $\mathcal{K}[G^S \times G]$-module $b_r\mathcal{K}G$ and $\check{\theta}$ is a $\mathcal{K}$-dual of θ. (Below, denote by $\check{b}$ the block containing $\check{\theta}$ for $\theta \in \mathrm{Irr}(b)$.) Equation (2) gives immediately the following Watanabe's result, see [9]:

The map determined by $\theta \mapsto \epsilon_\theta \pi(G,S)(\theta)$ where $\theta \in \mathrm{Irr}(b)$, induces a perfect isometry $\mathbb{Z}\mathrm{Irr}(b) \simeq \mathbb{Z}\mathrm{Irr}(w(b))$ between the Glauberman-Watanabe corresponding blocks.

and, as noted by Okuyama in [6], raised the following question:

Is the left hand side of equation (2) is a "shadow" of a complex of $(\mathcal{O}G^S w(b), \mathcal{O}Gb)$-bimodule which induces a derived equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$?

In fact, we have the following:

Theorem 1.1. *With the above notations, moreover assume that G is p-nilpotent and D is cyclic. Then there is a two term complex $C^\bullet$ of $(\mathcal{O}G^S w(b), \mathcal{O}Gb)$-bimodule satisfying the following:*

(1) $b_0\mathcal{O}G$ is in degree 0 and $b_l\mathcal{O}G$ is in degree 1 or -1.

(2) $C^\bullet$ induces a derived equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.

Further, $C^\bullet$ is quasi-isomorphic to a one term complex consisting of the bimodule M satisfying the following (M is in degree 0 if $\epsilon_b = 1$ and M is in degree 1 or -1 if $\epsilon_b = -1$ where $\epsilon_b = \epsilon_\theta$ for $\theta \in \mathrm{Irr}(b)$, which depends only on b):

(a) M induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.

(b) M has a vertex ΔD and an endo-permutation source.

$C^\bullet$ in Theorem 1.1 induces above Watanabe's perfect isometry, see the condition in Theoreme 1.1(1) and quation (2), and M in Theorem 1.1 induces the Glauberman correspondence of characters belonging to b and $w(b)$. The existence of M as in Theorem 1.1 is a particular case of the result of Harris-Linckelman for p-solvable case and of Watanabe for p-nilpotent blocks, see [5] and [10]. See also [4] for the existence of a derived equivalence between blocks with cyclic defect groups inducing prescribed perfect isometry.

2

Below, with the assumptions in Section 1, G and b are such that:

Condition 2.1. G is a p-nilpotent group with an S-centralized cyclic Sylow p-subgroup P of order p^α, that is, $G = KP = K \rtimes P$ where $K = O_{p'}(G)$. b is a P-invariant block of K, hence a block of G with a defect group $D = P$.

In fact, by the Fong's first reduction as described in [5, Section 5] and Theorem 2.2 and 2.3 below, Theorem 1.1 above can be shown.

Denote by P_i the unique subgroup of P with the order p^i for i such that $0 \leqq i \leqq \alpha$. Recall that the image $\mathrm{Br}_{P_i}(b)$ of the Brauer homomorphism Br_{P_i} of b is primitive in $Z\big(kC_K(P_i)\big)$ and hence is a block of $C_G(P_i) = C_K(P_i)P$, and let $\mathfrak{Br}_{P_i}(b)$ be the corresponding block over $\mathcal{O}$. Note that $b = \mathfrak{Br}_{P_0}(b)$. Idempotents $\mathfrak{Br}_{P_i}(b)_r \in \big(\mathcal{O}C_G(P_i)\mathfrak{Br}_{P_i}(b)\big)^{C_{G^S}(P_i)}$ (see (1) in Section 1) are defined similarly. Denote by M_i^j the unique trivial source $\mathcal{O}[C_{G^S}(P_i) \times C_G(P_i)]$-module in $w(\mathfrak{Br}_{P_i}(b)) \times \mathfrak{Br}_{P_i}(b)$ with vertex ΔP_j for j such that $0 \leqq j \leqq \alpha$. Let $M^j = M_0^j$. Let $\epsilon_{\mathfrak{Br}_{P_i}(b)} = \epsilon_{\chi_i}$ where $\chi_i \in \mathrm{Irr}(C_G(P_i) \mid \mathfrak{Br}_{P_i}(b))$. Note that $\epsilon_{\mathfrak{Br}_{P_i}(b)}$ depends only on $\mathfrak{Br}_{P_i}(b)$.

Theorem 2.2. *The following are equivalent for a fixed i where $0 \leqq i \leqq \alpha$:*

(1) $\epsilon_{\mathfrak{Br}_{P_h}(b)} = \epsilon_{\mathfrak{Br}_P(b)}$ for any h such taht $i \leqq h \leqq \alpha$.

(2) The unique simple $k(C_{K^S}(P_i) \times C_K(P_i))\Delta P$-module in $w(Br_{P_i}(b)) \times \check{Br}_{P_i}(b)$ is a trivial source module.

(3) M_i^α is a unique indecomposable direct summand of $\mathcal{O}C_G(P_i)\mathfrak{Br}_{P_i}(b){\downarrow}_{C_{G^S}(P_i) \times C_G(P_i)}$ with a multiplicity not divisible by q.

(4) (a) $\mathfrak{Br}_{P_i}(b)_0\mathcal{O}C_G(P_i) \simeq M_i^\alpha \oplus \mathfrak{Br}_{P_i}(b)_l\mathcal{O}C_G(P_i)$ if $\epsilon_{\mathfrak{Br}_P(b)} = 1$.
(b) $\mathfrak{Br}_{P_i}(b)_l\mathcal{O}C_G(P_i) \simeq M_i^\alpha \oplus \mathfrak{Br}_{P_i}(b)_0\mathcal{O}C_G(P_i)$ if $\epsilon_{\mathfrak{Br}_P(b)} = -1$.

(5) M_i^α induces a Morita equivalence between $\mathcal{O}C_G(P_i)\mathfrak{Br}_{P_i}(b)$ and $\mathcal{O}C_{G^S}(P_i)w(\mathfrak{Br}_{P_i}(b))$.

(6) $\mathcal{O}C_G(P_i)\mathfrak{Br}_{P_i}(b)$ and $\mathcal{O}C_{G^S}(P_i)w(\mathfrak{Br}_{P_i}(b))$ are Puig equivalent.

The conditions of Theorem 2.2 above always holds for $i = \alpha$. If the conditions of Theorem 2.2 holds for $i = 0$, that is, $\mathcal{O}Gb$ and $\mathcal{O}G^Sw(b)$ are Puig equivalent, then, by the conditions of Theorem 2.2(4) and (5), we can construct a desired two term complex $C^\bullet$ as in Theorem 1.1 with $M = M^\alpha$.

Below, we consider the case where $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ are not Puig equivalent. Then there is some β as in Theorem 2.3 below, see, for example, conditions of Theorem 2.2(1) and Thorem 2.3(1).

Since $(K^S \times K)\Delta P$ is p-nilpotent, sources of simple $k(K^S \times K)\Delta P$-modules are endo-permutation modules (Dade [2]). Since ΔP is cyclic, indecomposable endo-permutation $k\Delta P$-modules with vertex ΔP are the modules of the following form (Dade [2]):

$$\Omega^{a_0}_{\Delta P}\mathrm{Inf}^{\Delta P}_{\Delta(P/P_1)}\Omega^{a_1}_{\Delta(P/P_1)}\mathrm{Inf}^{\Delta(P/P_1)}_{\Delta(P/P_2)}\cdots\mathrm{Inf}^{\Delta(P/P_{\alpha-3})}_{\Delta(P/P_{\alpha-2})}\Omega^{a_{\alpha-2}}_{\Delta(P/P_{\alpha-2})}\mathrm{Inf}^{\Delta(P/P_{\alpha-2})}_{\Delta(P/P_{\alpha-1})}\Omega^{a_{\alpha-1}}_{\Delta(P/P_{\alpha-1})}(k),$$

where Ω means Heller translate and $a_i \in \{0, 1\}$.

Theorem 2.3. *Let β be such that $0 \leqq \beta \leqq \alpha - 1$. The following conditions on β are equivalent:*

(1) $\epsilon_{\mathfrak{Br}_{P_\beta}(b)} \neq \epsilon_{\mathfrak{Br}_P(b)}$ and $\epsilon_{\mathfrak{Br}_{P_h}(b)} = \epsilon_{\mathfrak{Br}_P(b)}$ for any h such that $\beta+1 \leqq h \leqq \alpha$.

(2) $a_\beta = 1$ and $a_h = 0$ for any h such that $\beta + 1 \leqq h \leqq \alpha$ where a_i's are 0 or 1 describing a source of the unique simple $k(K^S \times K)\Delta P$-module in $w(\bar{b}) \times \check{\bar{b}}$ as above (when $p = 2$, let $a_{\alpha-1} = 0$).

(3) $\mathcal{O}C_G(P_\beta)\mathfrak{Br}_{P_\beta}(b)$ and $\mathcal{O}C_{G^S}(P_\beta)w(\mathfrak{Br}_{P_\beta}(b))$ are not Puig equivalent and $\mathcal{O}C_G(P_h)\mathfrak{Br}_{P_h}(b)$ and $\mathcal{O}C_{G^S}(P_h)w(\mathfrak{Br}_{P_h}(b))$ are Puig equivalent for any h such that $\beta + 1 \leqq h \leqq \alpha$.

(4) The multiplicity of M^β in $\mathcal{O}Gb\downarrow_{G^S \times G}$ is not divisible by q.

(5) M^α and M^β are only indecomposable direct summands of $\mathcal{O}Gb\downarrow_{G^S \times G}$ with multiplicities not divisible by q.

(6) (a) $b_l\mathcal{O}G \oplus M^\alpha \simeq b_0\mathcal{O}G \oplus M^\beta$ if $\epsilon_{\mathfrak{Br}_P(b)} = 1$.
(b) $b_0\mathcal{O}G \oplus M^\alpha \simeq b_l\mathcal{O}G \oplus M^\beta$ if $\epsilon_{\mathfrak{Br}_P(b)} = -1$.

(7) (a) When $\epsilon_b\,\epsilon_{\mathfrak{Br}_P(b)}=-1$, there is an epimorphism $\Phi : M^\beta \twoheadrightarrow M^\alpha$ such that $N=\mathrm{Ker}\Phi$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.
(b) When $\epsilon_b\,\epsilon_{\mathfrak{Br}_P(b)}=1$, there is an epimorphism $\Phi : M^\alpha \twoheadrightarrow M^\beta$ such that $N=\mathrm{Ker}\Phi$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.

If $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ are not Puig equivalent, then, by the conditions of Thorem 2.3(6) and (7), we can construct a desired two term complex $C^\bullet$ as in Theorem 1.1 with $M = N$. Note that a source of $\overline{N}$ is a source of the unique simple $k(K^S \times K)\Delta P$-module in $w(\bar{b}) \times \check{\bar{b}}$, and an $\mathcal{O}$-lift of an endo-permutation module is an endo-permutation module.

In fact, a source of the module inducing the concerned Morita equivalence between $kG\bar{b}$ and $kG^S w(\bar{b})$ and "signs of the local blocks" $\epsilon_{\mathfrak{Br}_{P_i}(b)}$ are related as follows:

Proposition 2.4. *The following conditions on α numbers $a_i \in \{0,1\}$ $(0 \leqq i \leqq \alpha-1)$ are equivalent when p is odd:*

(1) A source of the unique simple $k(K^S \times K)\Delta P$-module in $w(\bar{b}) \times \check{\bar{b}}$ has the following form:

$$\Omega^{a_0}_{\Delta P}\mathrm{Inf}^{\Delta P}_{\Delta(P/P_1)}\Omega^{a_1}_{\Delta(P/P_1)}\mathrm{Inf}^{\Delta(P/P_1)}_{\Delta(P/P_2)}\cdots\mathrm{Inf}^{\Delta(P/P_{\alpha-3})}_{\Delta(P/P_{\alpha-2})}\Omega^{a_{\alpha-2}}_{\Delta(P/P_{\alpha-2})}\mathrm{Inf}^{\Delta(P/P_{\alpha-2})}_{\Delta(P/P_{\alpha-1})}\Omega^{a_{\alpha-1}}_{\Delta(P/P_{\alpha-1})}(k).$$

(2) $\epsilon_{\mathfrak{Br}_{P_i}(b)} = (-1)^{a_i}\epsilon_{\mathfrak{Br}_{P_{i+1}}(b)}$ for any i such that $0 \leqq i \leqq \alpha-1$.

Proposition 2.5. *The following conditions on $\alpha-1$ numbers $a_i \in \{0,1\}$ $(0 \leqq i \leqq \alpha-2)$ are equivalent when $p=2$:*

(1) A source of the unique simple $k(K^S \times K)\Delta P$-module in $w(\bar{b}) \times \check{\bar{b}}$ has the following form:

$$\Omega^{a_0}_{\Delta P}\mathrm{Inf}^{\Delta P}_{\Delta(P/P_1)}\Omega^{a_1}_{\Delta(P/P_1)}\mathrm{Inf}^{\Delta(P/P_1)}_{\Delta(P/P_2)}\cdots\mathrm{Inf}^{\Delta(P/P_{\alpha-3})}_{\Delta(P/P_{\alpha-2})}\Omega^{a_{\alpha-2}}_{\Delta(P/P_{\alpha-2})}(k).$$

(2) $\epsilon_{\mathfrak{Br}_{P_i}(b)} = (-1)^{a_i}\epsilon_{\mathfrak{Br}_{P_{i+1}}(b)}$ for any i such that $0 \leqq i \leqq \alpha-2$.

References

[1] E. C. Dade: *Endo-permutation modules over p-groups I,* Ann. of Math. **107** (1978), 459–494.

[2] E. C. Dade: *Endo-permutation modules over p-groups II,* Ann. of Math. **108** (1978), 317–346

[3] G. Glauberman: *Correspondence of characters for relatively prime operator groups,* Canad. J. Math. **20** (1968), 1465–1488.

[4] N. Kunugi: *A Remark on derived equivalences and perfect isometries,* 京都大学数理解析研究所講究録 **1466** (2006), 80–83.

[5] M. E. Harris, M. Linckelmann: *On the Glauberman and Watanabe correspondences for blocks of finite p-solvable groups,* Trans. Amer. Math. Soc. **354** (2002), 3435–3453.

[6] T. Okuyama: A talk at a Seminar in Ochanomizu University on 5 November, 2005.

[7] F. Tasaka: *On the isotypy induced by the Glauberman-Dade correspondence between blocks of finite groups,* J. Algebra **319** (2008), 2451–2470.

[8] F. Tasaka: *On the Glauberman-Watanabe correspondence of p-blocks of a p-nilpotent group with a cyclic Sylow p-subgroup,* preprint.

[9] A. Watanabe: *The Glauberman character correspondence and perfect isometries for blocks of finite groups,* J. Algebra **216** (1999), 548–565.

[10] A. Watanabe: *The Glauberman correspondent of a nilpotent block of a finite group,* Osaka J. Math **45** (2008), 869–875.