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LITTLEWOOD-RICHARDSON COEFFICIENTS AND EXTREMAL WEIGHT CRYSTALS

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ABSTRACT. We describe the tensor product of two extremal weight crystals of type $A_{+\infty}$ by constructing an explicit bijection between the connected components in the tensor product and a set of quadruples of Littlewood-Richardson tableaux.

1. INTRODUCTION

Let $\mathfrak{gl}_{>0}$ be the infinite rank affine Lie algebra of type $A_{+\infty}$ and $U_q(\mathfrak{gl}_{>0})$ its quantized enveloping algebra. For an integral weight Λ, there exists an integrable $U_q(\mathfrak{gl}_{>0})$-module called the *extremal weight module with extremal weight* Λ. The notion of extremal weight modules introduced by Kashiwara [5] is a generalization of integrable highest weight and lowest weight modules. An extremal weight module has a crystal base, which we call an *extremal weight crystal* for short, and two extremal weight crystals are isomorphic if their extremal weights are in the same Weyl group orbit.

Let $\mathscr{P}$ be the set of partitions. The Weyl group orbit of Λ is naturally in one-to-one correspondence with a pair of partitions $(\mu,\nu)\in\mathscr{P}^2$, where $(\mu,\emptyset)$ (resp. $(\emptyset,\nu)$) corresponds to a dominant (resp. anti-dominant) weight. Let us denote by $\mathcal{B}_{\mu,\nu}$ the extremal weight crystal with extremal weight corresponding to $(\mu,\nu)\in\mathscr{P}^2$.

In [9], it is shown that the tensor product of two extremal weight crystals is isomorphic to a finite disjoint union of extremal weight crystals and the Grothendieck ring associated with the category of $\mathfrak{gl}_{>0}$-crystals whose object is a finite union of extremal weight crystals, is isomorphic to the Weyl algebra of infinite rank. Using this characterization, it is shown that the multiplicity of $\mathcal{B}_{\zeta,\eta}$ in $\mathcal{B}_{\mu,\nu}\otimes\mathcal{B}_{\sigma,\tau}$ for (μ,ν), (σ,τ), $(\zeta,\eta)\in\mathscr{P}^2$ is

$$\sum_{\alpha,\beta,\gamma\in\mathscr{P}} c^{\zeta}_{\sigma\,\alpha}c^{\mu}_{\alpha\,\beta}c^{\tau}_{\beta\,\gamma}c^{\eta}_{\gamma\,\nu}, \tag{1.1}$$

which is a sum of products of four Littlewood-Richardson coefficients.

The main purpose of this note is to construct an explicit crystal isomorphism

$$\mathcal{B}_{\mu,\nu}\otimes\mathcal{B}_{\sigma,\tau}\xrightarrow{\sim}\bigsqcup_{(\zeta,\eta)\in\mathscr{P}^2}\ \bigsqcup_{\alpha,\beta,\gamma\in\mathscr{P}}\mathcal{B}_{\zeta,\eta}\times\mathbf{LR}^{\zeta}_{\sigma\,\alpha}\times\mathbf{LR}^{\mu}_{\alpha\,\beta}\times\mathbf{LR}^{\tau}_{\beta\,\gamma}\times\mathbf{LR}^{\eta}_{\gamma\,\nu}, \tag{1.2}$$

which gives a bijective proof of (1.1). Here $\mathbf{LR}^{\lambda}_{\mu\,\nu}$ denotes the set of Littlewood-Richardson tableaux of shape λ/μ with content ν for $\lambda,\mu,\nu\in\mathscr{P}$. We remark that the decomposition of $\mathcal{B}_{\mu,\nu}\otimes\mathcal{B}_{\sigma,\tau}$ is given in [9] by generalizing the insertion algorithm of Stembridge's rational

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tableaux [13, 14] for $\mathfrak{gl}_n$, but the associated recording tableaux which parameterize the connected components in $\mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau}$ do not imply (1.1) directly.

The multiplicity (1.1) has another representation theoretical interpretation, that is, it coincides with a generalization of Littlewood-Richardson coefficients introduced in [2], whose positivity is equivalent to the existence of a long exact sequence of 6 finite abelian p-groups with types $\sigma, \zeta, \mu, \tau, \eta, \nu$. The author would like to thank Alexander Yong for pointing out this connection.

This note is organized as follows. In Section 2, we recall briefly the notion of crystals and a combinatorial realization of $\mathcal{B}_{\mu,\nu}$. In Section 3, we review some combinatorics of Littlewood-Richardson tableaux and an insertion algorithm for $\mathcal{B}_{\mu,\nu}$. Finally, in Section 4, we construct the isomorphism (1.2).

2. Extremal weight crystals

2.1. Let $\mathfrak{gl}_{>0}$ denote the Lie algebra of complex matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with finitely many non-zero entries. Let E_{ij} be the elementary matrix with 1 at the i-th row and the j-th column and zero elsewhere. Then $\{ E_{ij} \,|\, i,j \geq 1 \}$ is a linear basis of $\mathfrak{gl}_{>0}$.

Let $\mathfrak{h} = \bigoplus_{i\geq 1} \mathbb{C}E_{ii}$ be the Cartan subalgebra of $\mathfrak{gl}_{>0}$ and $\langle \cdot, \cdot \rangle$ the natural pairing on $\mathfrak{h}^* \times \mathfrak{h}$. Let $\Pi^\vee = \{ h_i = E_{ii} - E_{i+1,i+1} \,|\, i \geq 1 \}$ be the set of simple coroots and $\Pi = \{ \alpha_i = \epsilon_i - \epsilon_{i+1} \,|\, i \geq 1 \}$ the set of simple roots of $\mathfrak{gl}_{>0}$, where $\epsilon_i \in \mathfrak{h}^*$ is determined by $\langle \epsilon_i, E_{jj} \rangle = \delta_{ij}$.

Let $P = \bigoplus_{i\geq 1} \mathbb{Z}\epsilon_i$ be the weight lattice of $\mathfrak{gl}_{>0}$ and $P_+ = \{ \Lambda \in P \,|\, \langle \Lambda, h_i \rangle \geq 0 \ \ (i \geq 1) \}$ the set of dominant integral weights. The map $\lambda = (\lambda_i)_{i\geq 1} \mapsto \omega_\lambda = \sum_{i\geq 1} \lambda_i \epsilon_i$ gives a bijection between $\mathscr{P}$ and P_+, where $\mathscr{P}$ denotes the set of partitions.

For $i \geq 1$, let r_i be the simple reflection given by $r_i(\Lambda) = \Lambda - \langle \Lambda, h_i \rangle \alpha_i$ for $\Lambda \in \mathfrak{h}^*$. Let W be the Weyl group of $\mathfrak{gl}_{>0}$, that is, the subgroup of $GL(\mathfrak{h}^*)$ generated by r_i for $i \geq 1$. Let P/W be the set of W-orbits in P. For $\Lambda = \sum_{i\geq 1} \Lambda_i \epsilon_i \in P$, let μ and ν be the partitions determined by $\{ \Lambda_i \,|\, \Lambda_i > 0 \}$ and $\{ -\Lambda_i \,|\, \Lambda_i < 0 \}$, respectively. Then the map $W\Lambda \mapsto (\mu, \nu)$ is a bijection from P/W to $\mathscr{P}^2$.

2.2. Let us recall briefly the notion of crystals based on [6]. A $\mathfrak{gl}_{>0}$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ ($i \in \mathbb{N}$) such that for $b \in B$

(1) $\varphi_i(b) = \langle \mathrm{wt}(b), h_i \rangle + \varepsilon_i(b)$,

(2) $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$, $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\widetilde{e}_i b \neq \mathbf{0}$,

(3) $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$, $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\widetilde{f}_i b \neq \mathbf{0}$,

(4) $\widetilde{f}_i b = b'$ if and only if $b = \widetilde{e}_i b'$ for $b, b' \in B$,

(5) $\widetilde{e}_i b = \widetilde{f}_i b = \mathbf{0}$ if $\varphi_i(b) = -\infty$,

where $\mathbf{0}$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$.

A crystal B is an $\mathbb{N}$-colored oriented graph where $b \xrightarrow{i} b'$ if and only if $b' = \widetilde{f}_i b$ for $b, b' \in B$ and $i \geq 1$. We say that B is *connected* if it is connected as a graph and *regular* if $\varepsilon_i(b) = \max\{\, k \mid \widetilde{e}_i^k b \neq \mathbf{0}\,\}$ and $\varphi_i(b) = \max\{\, k \mid \widetilde{f}_i^k b \neq \mathbf{0}\,\}$ for $b \in B$ and $i \geq 1$.

The *dual crystal* $B^\vee$ *of* B is defined to be the set $\{\, b^\vee \mid b \in B\,\}$ with

$$\begin{aligned}
&\mathrm{wt}(b^\vee) = -\mathrm{wt}(b),\\
&\varepsilon_i(b^\vee) = \varphi_i(b), \quad \varphi_i(b^\vee) = \varepsilon_i(b),\\
&\widetilde{e}_i(b^\vee) = \left(\widetilde{f}_i b\right)^\vee, \quad \widetilde{f}_i(b^\vee) = (\widetilde{e}_i b)^\vee,
\end{aligned}$$

for $b \in B$ and $i \geq 1$. Here we assume that $\mathbf{0}^\vee = \mathbf{0}$.

Let B_1 and B_2 be crystals. The *tensor product of* B_1 *and* B_2 is defined to be the set $B_1 \otimes B_2 = \{\, b_1 \otimes b_2 \mid b_i \in B_i \ (i = 1, 2)\,\}$ with

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i\rangle\},$$

$$\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i\rangle, \varphi_i(b_2)\},$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2),\\ b_1 \otimes \widetilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),\end{cases}$$

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1 \otimes \widetilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),\end{cases}$$

for $b_1 \otimes b_2 \in B_1 \otimes B_2$ and $i \geq 1$, where we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. Then $B_1 \otimes B_2$ is also a crystal.

A map $\psi : B_1 \longrightarrow B_2$ is called an *isomorphism of crystals* if it is a bijection, preserves wt, ε_i and φ_i and commutes with $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \geq 1)$, where we assume that $\psi(\mathbf{0}) = \mathbf{0}$. In this case, we say that B_1 *is isomorphic to* B_2 and write $B_1 \simeq B_2$. For example, $(B_1 \otimes B_2)^\vee \simeq B_2^\vee \otimes B_1^\vee$, where $(b_1 \otimes b_2)^\vee$ is mapped to $b_2^\vee \otimes b_1^\vee$.

For $b_i \in B_i$ $(i = 1, 2)$, we say that b_1 *is equivalent to* b_2, and write $b_1 \equiv b_2$ if there exists an isomorphism of crystals $C(b_1) \longrightarrow C(b_2)$ sending b_1 to b_2, where $C(b_i)$ denotes the connected component of B_i including b_i $(i = 1, 2)$.

2.3. We identify a partition with a Young diagram as usual (see [11]), where we enumerate rows and columns from the top and the left, respectively. Let $\mathcal{A}$ be a linearly ordered set. A tableau T obtained by filling a skew Young diagram λ/μ with entries in $\mathcal{A}$ is called a *semistandard tableau of shape* λ/μ if the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of all semistandard tableaux of shape λ/μ with entries in $\mathcal{A}$ (cf.[3, 11]).

For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, let $w(T)_{\rm col}$ (resp. $w(T)_{\rm row}$) denote the word obtained by reading the entries of T column by column (resp. row by row) from right to left (resp. top to bottom), and in each column (resp. row) from top to bottom (resp. right to left). For $a \in \mathcal{A}$, we denote by $(a \rightarrow T)$ (resp. $(T \leftarrow a)$) the tableau obtained by the Schensted column (resp. row) insertion (see for example [3, Appendix A.2]). For a finite word $w = w_1 \ldots w_r$ with letters in $\mathcal{A}$, we let $(w \rightarrow T) = (w_r \rightarrow (\cdots (w_1 \rightarrow T)\cdots))$ and $(T \leftarrow w) = ((\cdots (T \leftarrow w_1)\cdots) \leftarrow w_r)$. For semistandard tableaux S and T, we define $(T \rightarrow S)$ (resp. $(S \leftarrow T)$) to be $(w(T)_{\rm col} \rightarrow S)$ (resp. $S \leftarrow (w(T)_{\rm row})^{\rm rev})$ where $w^{\rm rev}$ is the reverse word of w.

We denote by $T^\vee$ the tableau obtained from T by 180°-rotation and replacing each entry t with $t^\vee$. Then $T^\vee$ is a semistandard tableau with entries in $\mathcal{A}^\vee$, where $\mathcal{A}^\vee = \{\, a^\vee \,|\, a \in \mathcal{A}\,\}$ and $a^\vee < b^\vee$ if and only if $b < a$ for $a, b \in \mathcal{A}$. Here we use the convention $(t^\vee)^\vee = t$ and hence $(T^\vee)^\vee = T$.

Let $\mathcal{A}$ be either $\mathbb{N}$ or $\mathbb{N}^\vee$ with the following regular crystal structures

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots,$$

$$\cdots \xrightarrow{3} 3^\vee \xrightarrow{2} 2^\vee \xrightarrow{1} 1^\vee,$$

where ${\rm wt}(k) = \epsilon_k$ and ${\rm wt}(k^\vee) = -\epsilon_k$ for $k \geq 1$. Then the set of all finite words with letters in $\mathcal{A}$ is a regular crystal, where we identify each word of length r with an element in $\mathcal{A}^{\otimes r} = \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (r times). Now, the injective image of $SST_{\mathcal{A}}(\lambda/\mu)$ in the set of finite words under the map $T \mapsto w(T)_{\rm col}$ (or $w(T)_{\rm row}$) together with $\{\mathbf{0}\}$ is invariant under $\widetilde{e}_i$, $\widetilde{f}_i$. Hence $SST_{\mathcal{A}}(\lambda/\mu)$ is a regular crystal [8]. Also, the row or column insertion is compatible with the crystal structure on tableaux in the following sense [10];

$$(a \rightarrow T) \equiv T \otimes a, \quad (T \leftarrow a) \equiv a \otimes T,$$

for $a \in \mathcal{A}$ and $T \in SST_{\mathcal{A}}(\lambda)$, and hence $(T \rightarrow S) \equiv S \otimes T$, $(S \leftarrow T) \equiv T \otimes S$ for $S \in SST_{\mathcal{A}}(\mu)$.

2.4. For $\Lambda \in P$, let $\mathbf{B}(\Lambda)$ be the crystal base of the extremal weight $U_q(\mathfrak{gl}_{>0})$-module with extremal weight Λ. Then $\mathbf{B}(\Lambda)$ is a regular crystal, and $\mathbf{B}(\Lambda) \simeq \mathbf{B}(w\Lambda)$ for $w \in W$. Moreover, if $\Lambda \in P_+$ (resp. $-\Lambda \in P_+$), then $\mathbf{B}(\Lambda)$ is isomorphic to the crystal base of the irreducible highest (resp. lowest) weight $U_q(\mathfrak{gl}_{>0})$-module with highest (resp. lowest) weight Λ (see [5, 7] for detailed exposition of extremal weight modules and their crystal bases).

Recall that for $\lambda \in \mathscr{P}$

$$\mathbf{B}(\omega_\lambda) \simeq SST_{\mathbb{N}}(\lambda), \quad \mathbf{B}(-\omega_\lambda) \simeq \mathbf{B}(\omega_\lambda)^\vee \simeq SST_{\mathbb{N}^\vee}(\lambda^\vee),$$

where $\lambda^\vee$ is the skew Young diagram obtained from $\lambda \in \mathscr{P}$ by 180°-rotation, and $SST_{\mathbb{N}}(\lambda)$ is connected with a unique highest weight element H_λ, where each i-th row is filled with i for $i \geq 1$ [8].

Now, for $\mu, \nu \in \mathscr{P}$, we define $\mathcal{B}_{\mu,\nu}$ to be the set of bitableaux (S, T) such that

(E1) $S \in SST_{\mathbf{N}}(\mu)$ and $T \in SST_{\mathbf{N}^\vee}(\nu^\vee)$,

(E2) for each $k \geq 1$,

$$s(k) + t(k) \leq k$$

where $s(k)$ is the number of entries in the left-most column of S no more than k, and $t(k)$ is the number of entries in the right-most column of T no less than $k^\vee$.

Since $\mathcal{B}_{\mu,\nu} \subset SST_{\mathbf{N}}(\mu) \otimes SST_{\mathbf{N}^\vee}(\nu^\vee)$, we can apply $\widetilde{e}_i, \widetilde{f}_i$ $(i \geq 1)$ on $\mathcal{B}_{\mu,\nu}$. Then $\mathcal{B}_{\mu,\nu} \cup \{\mathbf{0}\}$ is stable under $\widetilde{e}_i, \widetilde{f}_i$ $(i \geq 1)$ and hence a regular crystal. Moreover, we have the following [9, Theorem 3.5].

Theorem 2.1. *For $\mu, \nu \in \mathscr{P}$,*

(1) *$\mathcal{B}_{\mu,\nu}$ is connected,*

(2) *$\mathcal{B}_{\mu,\nu} \simeq \mathbf{B}(\Lambda)$, where $W\Lambda \in P/W$ corresponds to $(\mu, \nu) \in \mathscr{P}^2$.*

3. Insertion algorithm

3.1. For $\lambda, \mu, \nu \in \mathscr{P}$, let $\mathbf{LR}^\lambda_{\mu\nu}$ be the set of tableaux U in $SST_{\mathbf{N}}(\lambda/\mu)$ such that for $i \geq 1$

(LR1) the number of i's in U is ν_i,

(LR2) the number of i's in $w_1 \ldots w_k$ is no less than that of $i+1$'s in $w_1 \ldots w_k$ for $1 \leq k \leq r$, where $w(U)_{\text{col}} = w_1 \ldots w_r$.

We call $\mathbf{LR}^\lambda_{\mu\nu}$ the set of *Littlewood-Richardson tableaux of shape λ/μ with content ν* and put $c^\lambda_{\mu\nu} = |\mathbf{LR}^\lambda_{\mu\nu}|$ [11].

Suppose that $\mathcal{A}$ is a linearly ordered set. For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{A}}(\nu)$, let λ be the shape of $(T \rightarrow S)$ and $w(T)_{\text{col}} = w_1 \cdots w_r$. If w_i is in the kth row of T and inserted into $(w_{i-1} \rightarrow (\cdots (w_1 \rightarrow T)))$ to create a node in λ/μ, then let us fill the node with k. We denote the resulting tableau in $SST_{\mathbf{N}}(\lambda/\mu)$ by $(T \rightarrow S)_R$ and call it the *recording tableau of* $(T \rightarrow S)$. Then we have a bijection

$$SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{A}}(\nu) \stackrel{1-1}{\longleftrightarrow} \bigsqcup_{\lambda \in \mathscr{P}} SST_{\mathcal{A}}(\lambda) \times \mathbf{LR}^\lambda_{\mu\nu}, \tag{3.1}$$

where (S, T) corresponds to $((T \rightarrow S), (T \rightarrow S)_R)$ [15]. Moreover, if we assume that $\mathcal{A}$ is either $\mathbf{N}$ or $\mathbf{N}^\vee$, then the above bijection commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ $(i \geq 1)$ (cf.[4, 10]), where $\widetilde{e}_i$ and $\widetilde{f}_i$ act on the first component of $SST_{\mathcal{A}}(\lambda) \times \mathbf{LR}^\lambda_{\mu\nu}$. Summarizing, we have

Proposition 3.1. *Let $\mu, \nu \in \mathscr{P}$ be given.*

(1) *The map sending $S \otimes T$ to $((T \rightarrow S), (T \rightarrow S)_R)$ is an isomorphism of crystals*

$$SST_{\mathbf{N}}(\mu) \otimes SST_{\mathbf{N}}(\nu) \stackrel{\sim}{\longrightarrow} \bigsqcup_{\lambda \in \mathscr{P}} SST_{\mathbf{N}}(\lambda) \times \mathbf{LR}^\lambda_{\mu\nu}.$$

(2) *The map sending* $S \otimes T$ *to* $((S^\vee \to T^\vee)^\vee, (S^\vee \to T^\vee)_R)$ *is an isomorphism of crystals*

$$SST_{\mathbb{N}^\vee}(\mu^\vee) \otimes SST_{\mathbb{N}^\vee}(\nu^\vee) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathscr{P}} SST_{\mathbb{N}^\vee}(\lambda^\vee) \times \mathbf{LR}^\lambda_{\nu\,\mu}.$$

Remark 3.2. (1) Let $U \in SST_{\mathbb{N}}(\lambda/\mu)$ be given. Then as a crystal element, $U \in \mathbf{LR}^\lambda_{\mu\,\nu}$ if and only if $U \equiv H_\nu$.

(2) For $U \in \mathbf{LR}^\lambda_{\mu\,\nu}$, one may identify U with a unique $T \in SST_{\mathbb{N}}(\nu)$, say $\iota(U)$, such that the number of k's in the i-th row of T is equal to the number of i's in the k-th row of λ/μ for $i, k \geq 1$. Equivalently, $H_\mu \otimes \iota(U) \equiv H_\lambda$ [12].

3.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two linearly ordered sets. Let U be a tableau of shape λ/μ with entries in $\mathcal{A} \sqcup \mathcal{B}$, satisfying the following conditions;

(S1) if $u, u' \in \mathcal{X}$ are entries of U and u is northwest of u', then $u \leq u'$,

(S2) in each column of U, entries in $\mathcal{X}$ increase strictly from top to bottom,

where $\mathcal{X} = \mathcal{A}$ or $\mathcal{B}$, and we say that u is northwest of u' provided the row and column indices of u are no more than those of u'. Suppose that $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are two adjacent entries in U such that a is placed above or to the left of b. Interchanging a and b is called a *switching* if the resulting tableau still satisfies the conditions (S1) and (S2).

For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{B}}(\lambda/\mu)$, we denote by $S*T$ be the tableau in $SST_{\mathcal{A}\sqcup\mathcal{B}}(\lambda)$ obtained by gluing S and T. Let U be a tableau obtained from $S*T$ by applying switching procedures as far as possible. Then it is shown in [1, Theorems 2.2 and 3.1] that

(1) $U = T' * S'$, where $T' \in SST_{\mathcal{B}}(\nu)$ and $S' \in SST_{\mathcal{A}}(\lambda/\nu)$ for some ν,

(2) U is uniquely determined by S and T,

(3) when $\mathcal{A} = \mathbb{N}$, $S' \in \mathbf{LR}^\lambda_{\nu\,\mu}$ if and only if $S = H_\mu$.

Suppose that $\mathcal{A} = \mathbb{N}$ and $S = H_\mu$. Put

$$\jmath(T) = T', \qquad \jmath(T)_R = S'.$$

Then the map $T \mapsto (\jmath(T), \jmath(T)_R)$ gives a bijection [1]

$$SST_{\mathcal{B}}(\lambda/\mu) \overset{1-1}{\longleftrightarrow} \bigsqcup_{\nu \in \mathscr{P}} SST_{\mathcal{B}}(\nu) \times \mathbf{LR}^\lambda_{\nu\,\mu}. \tag{3.2}$$

If $\mathcal{B} = \mathbb{N}$, then the map $Q \mapsto \jmath(Q)_R$ restricts to a bijection from $\mathbf{LR}^\lambda_{\mu\,\nu}$ to $\mathbf{LR}^\lambda_{\nu\,\mu}$. Moreover, if $\mathcal{B}$ is either $\mathbb{N}$ or $\mathbb{N}^\vee$, then we can check that $T \equiv \jmath(T)$ and $\jmath(T)_R$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ $(i \geq 1)$. Hence we have the following.

Proposition 3.3. *Suppose that* $\mathcal{B}$ *is either* $\mathbb{N}$ *or* $\mathbb{N}^\vee$. *For a skew Young diagram* λ/μ, *we have an isomorphism of crystals*

$$SST_{\mathcal{B}}(\lambda/\mu) \xrightarrow{\sim} \bigsqcup_{\nu \in \mathscr{P}} SST_{\mathcal{B}}(\nu) \times \mathbf{LR}^\lambda_{\nu\,\mu},$$

where T *is mapped to* $(\jmath(T), \jmath(T)_R)$.

3.3. Let us review an insertion algorithm for extremal weight crystal elements [9].

3.3.1. Let $\mu, \nu \in \mathscr{P}$ be given. For $a \in \mathbb{N}$ and $(S,T) \in \mathcal{B}_{\mu,\nu}$, we define $(a \to (S,T))$ in the following way;

Suppose that S is empty and T is a single column tableau. Let (T', a') be the pair obtained by the following process;

(1) If T contains $a^\vee, (a+1)^\vee, \ldots, (b-1)^\vee$ but not $b^\vee$, then T' is the tableau obtained from T by replacing $a^\vee, (a+1)^\vee, \ldots, (b-1)^\vee$ with $(a+1)^\vee, (a+2)^\vee, \ldots, b^\vee$, and put $a' = b$.

(2) If T does not contain $a^\vee$, then leave T unchanged and put $a' = a$.

Now, we suppose that S and T are arbitrary.

(1) Apply the above process to the leftmost column of T with a.

(2) Repeat (1) with a' and the next column to the right.

(3) Continue this process to the right-most column of T to get a tableau T' and a''.

(4) Define

$$(a \to (S,T)) = ((a'' \to S), T').$$

Then $(a \to (S,T)) \in \mathcal{B}_{\sigma,\nu}$ for some $\sigma \in \mathscr{P}$ with $|\sigma/\mu| = 1$. For a finite word $w = w_1 \ldots w_r$ with letters in $\mathbb{N}$, we let $(w \to (S,T)) = (w_r \to (\cdots(w_1 \to (S,T))\cdots))$.

3.3.2. For $a \in \mathbb{N}$ and $(S,T) \in \mathcal{B}_{\mu,\nu}$, we define $((S,T) \leftarrow a^\vee)$ to be the pair (S', T') obtained in the following way;

(1) If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (E2) in Section 2.4, then put $S' = S$ and $T' = (T^\vee \leftarrow a)^\vee$.

(2) Otherwise, choose the smallest k such that a_k is bumped out of the k-th row in the row insertion of a into $T^\vee$ and the insertion of a_k into the $(k+1)$-th row violates the condition (E2) in Section 2.4.

(2-a) Stop the row insertion of a into $T^\vee$ when a_k is bumped out and let T' be the resulting tableau after taking $\vee$.

(2-b) Remove a_k in the left-most column of S, which necessarily exists, and then apply the *jeu de taquin* (see for example [3, Section 1.2]) to obtain a tableau S'.

In this case, $((S,T) \leftarrow a^\vee) \in \mathcal{B}_{\sigma,\tau}$, where either (1) $|\mu/\sigma| = 1$ and $\tau = \nu$, or (2) $\sigma = \mu$ and $|\tau/\nu| = 1$. For a finite word $w = w_1 \ldots w_r$ with letters in $\mathbb{N}^\vee$, we let $((S,T) \leftarrow w) = ((\cdots((S,T) \leftarrow w_1)\cdots) \leftarrow w_r)$.

3.3.3. Let $\mu, \nu, \sigma, \tau \in \mathscr{P}$ be given. For $(S,T) \in \mathcal{B}_{\mu,\nu}$ and $(S',T') \in \mathcal{B}_{\sigma,\tau}$, we define

$$((S',T') \to (S,T)) = ((w(S')_{\mathrm{col}} \to (S,T)) \leftarrow w(T')_{\mathrm{col}}).$$

Then $((S',T') \to (S,T)) \in \mathcal{B}_{\zeta,\eta}$ for some $(\zeta,\eta) \in \mathscr{P}^2$. Assume that $w(S')_{\mathrm{col}} = w_1 \dots w_s$ and $w(T')_{\mathrm{col}} = w_{s+1} \dots w_{s+t}$. For $1 \le i \le s+t$, let

$$(S^i,T^i) = \begin{cases} w_i \to (\cdots (w_1 \to (S,T))), & \text{if } 1 \le i \le s, \\ (((S^s,T^s) \leftarrow w_{s+1}) \cdots) \leftarrow w_i, & \text{if } s+1 \le i \le s+t, \end{cases}$$

and $(S^0,T^0) = (S,T)$. We define

$$((S',T') \to (S,T))_R = (U,V),$$

where (U,V) is the pair of tableaux with entries in $\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$ determined by the following process;

(1) U is of shape σ and V is of shape τ.
(2) Let $1 \le i \le s$. If w_i is inserted into (S^{i-1},T^{i-1}) to create a dot (or box) in the k-th row of the shape of S^{i-1}, then we fill the dot in σ corresponding to w_i with k.
(3) Let $s+1 \le i \le s+t$. If w_i is inserted into (S^{i-1},T^{i-1}) to create a dot in the k-th row (from the bottom) of the shape of T^{i-1}, then we fill the dot in τ corresponding to w_i with $-k$. If w_i is inserted into (S^{i-1},T^{i-1}) to remove a dot in the k-th row of the shape of S^{i-1}, then we fill the corresponding dot in τ with k.

We call $((S',T') \to (S,T))_R$ *the recording tableau of* $((S',T') \to (S,T))$. By [9, Theorem 4.10], we have the following.

Proposition 3.4. *Under the above hypothesis, we have*

(1) $((S',T') \to (S,T)) \equiv (S,T) \otimes (S',T')$,
(2) $((S',T') \to (S,T))_R \in SST_{\mathbb{N}}(\sigma) \times SST_{\mathbb{Z}^\times}(\tau)$,
(3) *the recording tableaux are constant on the connected component of* $\mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau}$ *including* $(S,T) \otimes (S',T')$,

where the linear ordering on $\mathbb{Z}^\times$ *is given by* $1 \prec 2 \prec 3 \prec \cdots \prec -3 \prec -2 \prec -1$.

Example 3.5. Consider

$$(S,T) = \left(\begin{matrix} 2 & 3 & 4 \\ 3 & 5 & \end{matrix} \ , \ \begin{matrix} 5^\vee & 5^\vee \\ 3^\vee & 2^\vee \end{matrix} \right), \quad (S',T') = \left(\begin{matrix} 3 & 3 \\ 5 & \end{matrix} \ , \ \begin{matrix} & 4^\vee \\ 3^\vee & 1^\vee \end{matrix} \right).$$

Since $w(S')_{\mathrm{col}} = 3\,3\,5$ and $w(T')_{\mathrm{col}} = 4^\vee\,1^\vee\,3^\vee$, we have

$$(w(S')_{\mathrm{col}} \to (S,T)) = \left(\begin{matrix} 2 & 3 & 3 & 4 \\ 3 & 5 & & \\ 4 & & & \\ 6 & & & \end{matrix} \ , \ \begin{matrix} 6^\vee & 5^\vee \\ 4^\vee & 2^\vee \end{matrix} \right)$$

and

$$((w(S')_{\mathrm{col}} \;\rightarrow\; (S,T)) \leftarrow w(T')_{\mathrm{col}}) = \left(\begin{array}{cccc} 2 & 3 & 3 & 4 \\ 3 & 5 & & \\ 4 & & & \end{array} , \begin{array}{ccc} & & 5^{\vee} \\ & 6^{\vee} & 4^{\vee} \\ 4^{\vee} & 3^{\vee} & 1^{\vee} \end{array} \right).$$

Hence,

$$((S',T') \rightarrow (S,T)) = \left(\begin{array}{cccc} 3 & 3 & 3 & 4 \\ 4 & 5 & & \\ 6 & & & \end{array} , \begin{array}{ccc} & & 5^{\vee} \\ & 6^{\vee} & 4^{\vee} \\ 4^{\vee} & 3^{\vee} & 1^{\vee} \end{array} \right),$$

$$((S',T') \rightarrow (S,T))_R = \left(\begin{array}{cc} 1 & 3 \\ 4 & \end{array} , \begin{array}{cc} 4 & -3 \\ -1 & \end{array} \right).$$

Remark 3.6. For $(U,V) \in SST_{\mathbf{N}}(\sigma) \times SST_{\mathbf{Z}^\times}(\tau)$, an equivalent condition for (U,V) to be a recording tableau, that is, $(U,V) = ((S',T') \rightarrow (S,T))_R$ for some $(S,T) \in \mathcal{B}_{\mu,\nu}$ and $(S',T') \in \mathcal{B}_{\sigma,\tau}$, can be found in [9, Section 4.3].

4. Main Theorem

To prove our main theorem, let us first describe the decompositions of $SST_{\mathbf{N}^\vee}(\nu^\vee) \otimes SST_{\mathbf{N}}(\mu)$ and $SST_{\mathbf{N}}(\mu) \otimes SST_{\mathbf{N}^\vee}(\nu^\vee)$ for $\mu, \nu \in \mathscr{P}$.

Proposition 4.1. *For $\mu,\nu \in \mathscr{P}$, we have an isomorphism of crystals*

$$SST_{\mathbf{N}^\vee}(\nu^\vee) \otimes SST_{\mathbf{N}}(\mu) \xrightarrow{\sim} \mathcal{B}_{\mu,\nu},$$

where $T \otimes S$ is mapped to $((S,\emptyset) \rightarrow (\emptyset,T))$.

Proof. For $T \otimes S \in SST_{\mathbf{N}^\vee}(\nu^\vee) \otimes SST_{\mathbf{N}}(\mu)$, it follows from Proposition 3.4 (2) that

(1) $((S,\emptyset) \rightarrow (\emptyset,T)) \in \mathcal{B}_{\mu,\nu}$,
(2) $((S,\emptyset) \rightarrow (\emptyset,T))_R = (H_\mu,\emptyset)$.

Therefore, we have a map

$$SST_{\mathbf{N}^\vee}(\nu^\vee) \otimes SST_{\mathbf{N}}(\mu) \longrightarrow \mathcal{B}_{\mu,\nu} \times \{\,(H_\mu,\emptyset)\,\}$$

sending $T \otimes S$ to $(((S,\emptyset) \rightarrow (\emptyset,T))\,, ((S,\emptyset) \rightarrow (\emptyset,T))_R)$. Since the insertion algorithm is reversible [9, Proposition 4.9], the above map is indeed a bijection and hence an isomorphism of crystals by Proposition 3.4 (1). $\square$

Next, suppose that $S \otimes T \in SST_{\mathbf{N}}(\mu) \otimes SST_{\mathbf{N}^\vee}(\nu^\vee)$ is given. Let $U^{>0}$ (resp. $U^{<0}$) be the subtableau in $((\emptyset,T) \rightarrow (S,\emptyset))_R$ consisting of positive (resp. negative) entries. We define

$$\theta(S \otimes T) = (\iota^{-1}(U^{>0}), \jmath(\jmath(U^{<0})_R)_R)$$

(see Remark 3.2 (2) and Section 3.2 (3.2)).

Proposition 4.2. *For $\mu,\nu \in \mathscr{P}$, we have an isomorphism of crystals*

$$SST_{\mathbb{N}}(\mu)\otimes SST_{\mathbb{N}^\vee}(\nu^\vee) \xrightarrow{\ \sim\ } \bigsqcup_{\lambda,\sigma,\tau\in\mathscr{P}} \mathcal{B}_{\sigma,\tau}\times \mathbf{LR}^{\mu}_{\sigma\lambda}\times \mathbf{LR}^{\nu}_{\lambda\tau},$$

where $S\otimes T$ is mapped to $(((\emptyset,T)\to(S,\emptyset)),\theta(S\otimes T))$.

Proof. For $S\otimes T\in SST_{\mathbb{N}}(\mu)\otimes SST_{\mathbb{N}^\vee}(\nu^\vee)$, suppose that $((\emptyset,T)\to(S,\emptyset))\in\mathcal{B}_{\sigma,\tau}$ for some $\sigma,\tau\in\mathscr{P}$.

First, note that $U^{>0}\in SST_{\mathbb{N}}(\lambda)$ for some $\lambda\subset\nu$. Then it is not difficult to check that $\iota^{-1}(U^{>0})\in\mathbf{LR}^{\mu}_{\sigma\lambda}$ (see Remark 3.2). Next, consider $U^{<0}\in SST_{\mathbb{Z}_{<0}}(\nu/\lambda)$. Then $(w(U^{<0})_{\rm col})^{\rm rev}$ satisfies (LR1) with respect to τ and (LR2), ignoring $-$ sign in each letter. Let L_τ be the tableau in $SST_{\mathbb{Z}_{<0}}(\tau)$, where the i-th entry from the bottom in each column is $-i$. Considering the Knuth equivalence on the set of words with letters in $\mathbb{Z}_{<0}$ (cf.[3]), we have $j(U^{<0})=L_\tau$ and $j(U^{<0})_R\in\mathbf{LR}^{\nu}_{\tau\lambda}$ by (3.2). So,we get $j(j(U^{<0})_R)_R\in\mathbf{LR}^{\nu}_{\lambda\tau}$.

Now, we have a map

$$SST_{\mathbb{N}}(\mu)\otimes SST_{\mathbb{N}^\vee}(\nu^\vee) \longrightarrow \bigsqcup_{\lambda,\sigma,\tau\in\mathscr{P}} \mathcal{B}_{\sigma,\tau}\times \mathbf{LR}^{\mu}_{\sigma\lambda}\times \mathbf{LR}^{\nu}_{\lambda\tau},$$

sending $S\otimes T$ to $(((\emptyset,T)\to(S,\emptyset)),\theta(S\otimes T))$. Since the insertion algorithm is reversible [9, Proposition 4.9], the above map is a bijection and therefore an isomorphism of crystals by Proposition 3.4 (1) and (3). □

Now, we are in a position to state our main result in this note.

Theorem 4.3. *For $(\mu,\nu),(\sigma,\tau)\in\mathscr{P}^2$, we have an isomorphism of crystals*

$$\mathcal{B}_{\mu,\nu}\otimes\mathcal{B}_{\sigma,\tau}\simeq \bigsqcup_{(\zeta,\eta)\in\mathscr{P}^2}\ \bigsqcup_{\alpha,\beta,\gamma\in\mathscr{P}} \mathcal{B}_{\zeta,\eta}\times\mathbf{LR}^{\zeta}_{\sigma\,\alpha}\times\mathbf{LR}^{\mu}_{\alpha\,\beta}\times\mathbf{LR}^{\tau}_{\beta\,\gamma}\times\mathbf{LR}^{\eta}_{\gamma\,\nu}.$$

Proof. Note that $\mathcal{B}_{\mu,\emptyset}=SST_{\mathbb{N}}(\mu)$ and $\mathcal{B}_{\emptyset,\nu}=SST_{\mathbb{N}^\vee}(\nu^\vee)$. Then as a crystal, we have

$$\begin{aligned}
&\mathcal{B}_{\mu,\nu}\otimes\mathcal{B}_{\sigma,\tau}\\
&\simeq \mathcal{B}_{\emptyset,\nu}\otimes\mathcal{B}_{\mu,\emptyset}\otimes\mathcal{B}_{\emptyset,\tau}\otimes\mathcal{B}_{\sigma,\emptyset} && \text{(by Proposition 4.1)}\\
&\simeq \bigsqcup_{\alpha,\beta,\gamma\in\mathscr{P}} (\mathcal{B}_{\emptyset,\nu}\otimes\mathcal{B}_{\alpha,\gamma}\otimes\mathcal{B}_{\sigma,\emptyset})\times\mathbf{LR}^{\mu}_{\alpha\beta}\times\mathbf{LR}^{\tau}_{\beta\gamma} && \text{(by Proposition 4.2)}\\
&\simeq \bigsqcup_{\alpha,\beta,\gamma\in\mathscr{P}} (\mathcal{B}_{\emptyset,\nu}\otimes\mathcal{B}_{\emptyset,\gamma}\otimes\mathcal{B}_{\alpha,\emptyset}\otimes\mathcal{B}_{\sigma,\emptyset})\times\mathbf{LR}^{\mu}_{\alpha\beta}\times\mathbf{LR}^{\tau}_{\beta\gamma}\\
&\simeq \bigsqcup_{(\zeta,\eta)\in\mathscr{P}^2}\ \bigsqcup_{\alpha,\beta,\gamma\in\mathscr{P}} \mathcal{B}_{\emptyset,\eta}\otimes\mathcal{B}_{\zeta,\emptyset}\times\mathbf{LR}^{\zeta}_{\alpha\sigma}\times\mathbf{LR}^{\eta}_{\gamma\nu}\times\mathbf{LR}^{\mu}_{\alpha\beta}\times\mathbf{LR}^{\tau}_{\beta\gamma} && \text{(by Proposition 3.1)}\\
&\simeq \bigsqcup_{(\zeta,\eta)\in\mathscr{P}^2}\ \bigsqcup_{\alpha,\beta,\gamma\in\mathscr{P}} \mathcal{B}_{\zeta,\eta}\times\mathbf{LR}^{\zeta}_{\sigma\,\alpha}\times\mathbf{LR}^{\mu}_{\alpha\,\beta}\times\mathbf{LR}^{\tau}_{\beta\,\gamma}\times\mathbf{LR}^{\eta}_{\gamma\,\nu} && \text{(by (3.2)).}
\end{aligned}$$

□

Corollary 4.4. *The multiplicity of $\mathcal{B}_{\zeta,\eta}$ in $\mathcal{B}_{\mu,\nu} \otimes \mathcal{B}_{\sigma,\tau}$ is given by*

$$\sum_{\alpha,\beta,\gamma\in\mathscr{P}} c^{\zeta}_{\sigma\,\alpha} c^{\mu}_{\alpha\,\beta} c^{\tau}_{\beta\,\gamma} c^{\eta}_{\gamma\,\nu}.$$

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