数理解析研究所講究録
第1693巻2010年45-56

Hölder continuity for some degenerate parabolic equation and its application

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1. Hölder continuity for some degenerate parabolic equation

We consider the following degenerate parabolic equation of the porous medium type:

$$
\left\{\begin{aligned} \partial_t u - \Delta u^\alpha &= \operatorname{div} f, \quad t>0,\ x\in\mathbb{R}^n,\\ u(0,x) &= u_0(x)\geq 0, \quad x\in\mathbb{R}^n,\end{aligned}\right. \tag{1.1}
$$

where $\alpha > 1$ is a constant, $f = f(t,x)$ is a given $\mathbb{R}^n$-valued function, $u_0 = u_0(x)$ is a given non-negative initial datum and $u = u(t,x)$ is a unknown function. It is well-known that a classical solution of (1.1) does not generally exist even if u_0 is smooth and $f \equiv 0$ (we will introduce the explicit solution of (1.1) with $f \equiv 0$ in the next section). For this reason, we introduce the notion of weak solutions of (1.1).

Definition 1.1 (weak solutions). Let u_0 be non-negative function in $L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ and let $f \in L^1((0,T)\times\mathbb{R}^n)$. Then a function $u : (0,\infty)\times\mathbb{R}^n \to [0,\infty)$ is a weak solution of (1.1) if there exists $T > 0$ such that u satisfies the following three conditions:

(i) $u(t,x) \geq 0$ for almost all $(t,x) \in (0,T)\times\mathbb{R}^n$;
(ii) $u \in L^\infty(0,T; L^1(\mathbb{R}^n)\cap L^\alpha(\mathbb{R}^n))$ and $\nabla u^\alpha \in L^2((0,T)\times\mathbb{R}^n)$;
(iii) u satisfies (1.1) in the sense of distributions, i.e. for all $\phi \in C^1([0,T);\ C_0^1(\mathbb{R}^n))$,

$$
\int_{\mathbb{R}^n} u(t)\phi(t)\,dx - \int_{\mathbb{R}^n} u_0\phi(0)\,dx + \int_0^t\int_{\mathbb{R}^n} u\partial_t\phi\,d\tau dx + \int_0^t\int_{\mathbb{R}^n} \nabla u^\alpha\cdot\nabla\phi\,d\tau dx = -\int_0^t\int_{\mathbb{R}^n} f(\tau,x)\cdot\nabla\phi\,d\tau dx
$$

for almost all $0 < t < T$.

The existence of the weak solutions of (1.1) is shown by Oleĭnik-Kalašinkov-Čžou [12] and J.-L. Lions [6] (cf. Ôtani [15]). Our first aim is to obtain the uniform Hölder continuity for weak solutions of (1.1). We will discuss the application on this aim in the next section.

Caffarelli-Friedman [4] firstly showed the uniform Hölder continuity for weak solutions of (1.1) with $f \equiv 0$. Their proof essentially relies on the Aronson-Benilan estimate [2], that is some pointwise estimate for the derivative of a solution derived from the comparison principle. However, for the general external forces f, the Aronson-Benilan type estimate is not generally known for (1.1). Furthermore, for an application, the comparison principle does not generally hold for non-local cases, like the degenerate Keller-Segel system or the degenerate drift-diffusion system. Therefore, it seems difficult to apply their method for (1.1).

This work is partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists and Grant-in-Aid for JSPS Fellows # 21-1281.

On the other hand, DiBenedetto-Friedman [5] (independently Wiegner [19], generalized by Misawa [10]) considered the following p-Laplace evolution equation:

$$(1.2)\qquad \begin{cases} \partial_t v - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \qquad\qquad v(0,x) = v_0(x), & x \in \mathbb{R}^n. \end{cases}$$

For $p > 2$, they showed the Hölder continuity for the gradient of solutions of (1.2). We remark that their proof does not rely on the comparison principle. If $n = 1$, then $u = |\nabla v|$ satisfies (1.1) with $f \equiv 0$ and $\alpha = p - 1$. Therefore, they gave a different proof of the Hölder continuity for solutions of the porous medium equation. In fact, DiBenedetto-Friedman showed the Hölder continuity for solutions of (1.1) with $f \equiv 0$ in any space dimension. They also studied (1.1) with the general external forces f, however they only observed the Hölder continuity for (1.1) if $f \in L^\infty(0,\infty; L^p(\mathbb{R}^n))$ with $p > n$ and did not give a proof. We try to extend their results, and we obtain Hölder estimates for the solutions of (1.1) with the external forces f belonging to some larger class than $L^\infty(0,\infty; L^p(\mathbb{R}^n))$.

Now, we introduce the weak L^p spaces.

Definition 1.2 (weak L^p spaces). Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p > 2$. Then a function f in Ω belongs to $L^p_w(\Omega)$ if $f \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\|f\|^2_{L^p_w(\Omega)} := \sup_{k \subset \Omega:\text{compact}} \frac{1}{|K|^{1-\frac{2}{p}}} \int_K |f|^2\,dx < \infty.$$

By the Hölder inequality, the L^p space is included by the weak L^p space. In fact, the weak L^p space is strictly larger than the L^p space. Indeed, $|x|^{-\frac{n}{p}}$ is belonging to $L^p_w(\mathbb{R}^n)$ but not belonging to $L^p(\mathbb{R}^n)$.

Remark. For our purpose, we only consider for the case of $p > 2$. In general, one can define the weak L^p space for the case of $p > 1$ as the similar way.

Remark. For $p > 2$, it is known that the weak L^p space and the Lorentz space $L^{p,\infty}$ are same spaces, where

$$L^{p,\infty}(\Omega) := \{f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|^p_{L^{p,\infty}(\Omega)} := \sup_{\lambda>0} \lambda^p |\{x \in \Omega : |f(x)| > \lambda\}| < \infty\}.$$

More precisely, for $f \in L^p_w(\Omega)$, we obtain

$$\frac{p-1}{p^{1+\frac{1}{p}}} \|f\|_{L^p_w(\Omega)} \le \|f\|_{L^{p,\infty}(\Omega)} \le \|f\|_{L^p_w(\Omega)}$$

(cf. Benilan-Brezis-Crandall [3]).

To state the main theorem, we give the following notation:

$$(1.3)\qquad A_p := \sup_{r \le 1,\, a \in \mathbb{R}^n} \|f\|_{L^\infty(0,\infty; L^p_w(B_r(a)))}.$$

where $B_r(a)$ is an open ball with radius r and center a.

Theorem 1.3 (Hölder estimates for solutions of (1.1)). *Let u be a bounded non-negative weak solution of* (1.1) *and let $n \ge 2$ for simplicity. Assume $A_p < \infty$ for some $p > n$, where A_p is given by* (1.3). *Then, there exist constants $0 < \gamma < 1$ and $C > 0$ depending only on n, α and p such that*

$$(1.4)\quad |u^\alpha(t,x) - u^\alpha(s,y)| \le C(\|u\|^\alpha_{L^\infty((0,\infty)\times\mathbb{R}^n)} + A_p)(\|u\|^{\frac{\gamma}{2}(\alpha-1)}_{L^\infty((0,\infty)\times\mathbb{R}^n)}|t-s|^{\frac{\gamma}{2}} + |x-y|^\gamma)$$

for $(t,x), (s,y) \in (0,\infty) \times \mathbb{R}^n$.

Remark. If $\alpha = 1$, then the estimate (1.4) is well-known for the heat equation. Indeed, the constants C and γ are stable as $\alpha \to 1$ in the sense that

$$\lim_{\alpha \to 1} C(\alpha) < \infty \text{ and } \lim_{\alpha \to 1} \gamma(\alpha) \in (0, 1].$$

Hence we can regard the estimates (1.4) as generalised Hölder estimates for solutions of the porous medium equation with the external force of divergence form.

2. Application for the Hölder estimate

We consider the Keller-Segel system of degenerate type:

$$\begin{cases} \partial_t u - \Delta u^\alpha + \operatorname{div}(u\nabla\psi) = 0, & t > 0\,, x \in \mathbb{R}^n, \\ -\Delta\psi + \psi = u, & t > 0\,, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $n \geq 3$. The notion of the weak solution of (2.1) is like as definition 1.1, where $\psi = (-\Delta + 1)^{-1}u$ is the Bessel potential of u (Sugiyama [16], Sugiyama-Kunii [17] or Ogawa [13] give more precise definition).

Sugiyama [16] and Sugiyama-Kunii [17] study the existence or nonexistence of the time global weak solution of (2.1). We roughly summarize that (For more precise statement, we refer to Sugiyama [16], Sugiyama-Kunii [17] and Ogawa [13]) if $\alpha \leq 2 - \frac{2}{n}$ and u_0 is enough small in some sense, then there exists a time global weak decaying solution, namely the solution goes to zero as $t \to \infty$; if $\alpha \leq 2 - \frac{2}{n}$ but u_0 is not small, then the corresponding solution blows up at a finite time; if $\alpha > 2 - \frac{2}{n}$, then for arbitrary u_0, there exists a time global weak solution. For the case of $\alpha \leq 2 - \frac{2}{n}$ and sufficiently small initial data, we consider the asymptotic stability of the decaying solution u of (2.1). Since the solution is small as t is large, the third term of (2.1) seems small. Therefore, the solution should be close to a solution of (1.1) with $f \equiv 0$ as t is enough large. Here we give the rigorous way of the preceding argument.

Definition 2.1 (the Barenblatt solution). For $\alpha > 1$, let $\sigma = n(\alpha - 1) + 2$. We define the Barenblatt solution U as

$$U(t,x) := (1+\sigma t)^{-\frac{n}{\sigma}} V\left(\frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}}\right), \quad V(y) := \left(A - \frac{\alpha - 1}{2\alpha}|y|^2\right)_+^{\frac{1}{\alpha-1}}, \tag{2.2}$$

where $A > 0$ is a constant satisfying $\|U(0)\|_1 = \|u_0\|_1$ and $(f)_+$ is the positive part of f.

The Barenblatt solution satisfies (1.1) with $f \equiv 0$ in the sense of distributions. Since the Barenblatt solution is not smooth, weak solutions of (1.1) are not generally smooth. We remark that, if $\alpha = 1$, then $-\frac{n}{\sigma}|_{\alpha=1} = -\frac{1}{2}$ and hence the decaying order of the Barenblatt solution is same as the Gauss kernel. Therefore the Barenblatt solution is like as the Gauss kernel for (1.1) with $f \equiv 0$.

To study the asymptotic stability, the following forward self-similar transform plays the important role:

$$s = \frac{1}{\sigma}\log(1+\sigma t), \qquad y = \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}},$$
$$v(s,y) = (1+\sigma t)^{\frac{n}{\sigma}} u(t,x), \qquad \phi(s,y) = (1+\sigma t)^{\frac{n}{\sigma}} \psi(t,x),$$

where $\sigma = n(\alpha - 1) + 2$. Then, (v, ϕ) satisfies the following equation:

$$\begin{cases} \partial_s v - \Delta_y v^\alpha = \operatorname{div}_y(yv - e^{-\kappa s} v \nabla_y \phi), & s > 0\,,\ y \in \mathbb{R}^n, \\ -e^{-2s}\Delta_y \phi + \phi = v, \quad s > 0\,,\ y \in \mathbb{R}^n, & \\ v(0, y) = u_0(y) \geq 0, & \end{cases} \tag{2.3}$$

where $\kappa = n(2 - \alpha)$.

Luckhaus-Sugiyama [8] showed

$$(1 + \sigma t)^{\frac{n}{\sigma}(1-\frac{1}{p})} \|u(t) - U(t)\|_p \to 0 \quad \text{as } t \to \infty$$

and the convergence rate in L^p space for $p > 1$. However, for $p = 1$, their results gave no information of the convergence rate in L^1 space. Ogawa [13] showed if $\alpha < 2 - \frac{2}{n}$, then

$$\|u(t) - U(t)\|_1 \leq C(1 + \sigma t)^{-\frac{\nu}{\sigma}}$$

and the algebraic convergence rate is shown. However, for the case of the critical exponent $\alpha = 2 - \frac{2}{n}$, we did not obtain the convergence rate in L^1 space since the uniform Hölder continuity for corresponding solution v of (2.3) was not clear. Using Theorem 1.3, we obtain the uniform Hölder continuity of v and hence we obtain the convergence rate for the solution of (2.1) in L^1 space for the case of the critical exponent.

Theorem 2.2 (asymptotic stability in L^1 space). *Let $\alpha = 2 - \frac{2}{n}$ and $n \geq 3$. Let initial data u_0 be enough small. Assume $|x|^\beta u_0 \in L^1(\mathbb{R}^n)$ for some $\beta > n$. Then there exist constants $C, \nu > 0$ such that*

$$\|u(t) - U(t)\|_1 \leq C(1 + \sigma t)^{-\nu}, \quad t > 0,$$

where the Barenblatt solution U satisfies $\|U(0)\|_1 = \|u_0\|_1$.

Remark. If the Hölder exponent of v is more larger, then we can obtain the order parameter ν more larger.

Remark. The moment boundedness $|x|^\beta u_0 \in L^1(\mathbb{R}^n)$ is the technical assumption. It seems possible to obtain the convergence rate for $\beta = 2$.

The forward self-similar transform has the following important property:

$$(1 + \sigma t)^{\frac{n}{\sigma}(1-\frac{1}{p})} \|u(t) - U(t)\|_{L^p(\mathbb{R}^n_x)} = \|v(s) - V\|_{L^p(\mathbb{R}^n_y)}, \tag{2.4}$$

where U is the Barenblatt solution and V is the self-similar profile given by (2.2). Therefore, once we obtain the convergence of the self-similar transform v to the self-similar profile V in L^p space, then we obtain the asymptotic stability for the solution of (2.1) in L^p space. From the point of view, we obtain

$$\|v(s) - V\|_1 \leq Ce^{-\nu s}, \quad s \geq 0$$

for some constants $C, \nu > 0$.

Outline of proof of Theorem 2.2. We give an only formal argument. Multiplying (2.3) by $|y|^\beta$ and integrating over $(0, T) \times \mathbb{R}^n$, we obtain

$$\sup_{0<s<\infty} \int_{\mathbb{R}^n} |y|^\beta v\, dy \Big|_s < \infty.$$

By the smallness of the initial data, we have $\|v\|_{L^\infty((0,\infty)\times\mathbb{R}^n)}$, $\|\nabla\phi\|_{L^\infty((0,\infty)\times\mathbb{R}^n)} < \infty$ and hence

$$\sup_{r\leq 1\,,\, a\in\mathbb{R}^n} \sup_{0<s<\infty} \|yv - e^{-\kappa s} v \nabla\phi\|_{L^\beta_w(B_r(a))} < \infty.$$

Applying Theorem 1.3, we find the uniform Hölder continuity of the rescaled solution v. Therefore we obtain

$$\|u\|_{\dot{C}^{0,\gamma}(\mathbb{R}^n)} = O(t^{-\frac{n}{\sigma}(1+\gamma)}) \quad \text{as } t \to \infty,$$

where $\|\cdot\|_{\dot{C}^{0,\gamma}(\mathbb{R}^n)}$ is the γ-Hölder semi-norm. The regularity of the solution is directly connected to the explicit decay rate of the solution. Therefore, we obtain the convergence rate for the solution of (2.1) in L^1 space for the case of the critical exponent. The authors will give more details of the proof by another paper [14]. □

3. Proof of Theorem 1.3

In this section, the same letter C gives the different constants. Before considering the Hölder continuity for the case of the degenerate type, we review the proof for the non-degenerate case $\alpha = 1$ with $f \equiv 0$. To show the Hölder continuity, the following weak Harnack inequality plays the important role (Moser [11], Aronson-Serrin [1], Trudinger [18]):

$$\|u\|_{L^2((T,2T)\times B_\rho)} \leq C \inf_{(3T,4T)\times B_\rho} u \quad \text{for } T > 0\,,\ r > 0. \tag{3.1}$$

For $\rho > 0$ and $(t_0, x_0) \in (0.\infty) \times \mathbb{R}^n$, we put an usual parabolic cylinder:

$$Q_\rho = Q_\rho(t_0, x_0) := (t_0 - \rho^2, t_0) \times B_\rho(x_0).$$

Then, using the weak Harnack inequality (3.1), we obtain

$$\operatorname*{osc}_{Q_\rho} u := \sup_{Q_\rho} u - \inf_{Q_\rho} u \leq C\rho^\gamma$$

for some $\gamma > 0$ and hence we obtain the γ-Hölder continuous of the solution. However, for the case of the degenerate type $\alpha > 1$, the weak Harnack inequality does not generally hold. Indeed, the Barenblatt solution has the compact support for all time hence the infimum of the Barenblatt solution may vanish. Therefore, we should obtain the Hölder continuity of the solution without using the weak Harnack inequality.

3.1. The case of $\alpha = 1$.

From now on, we replace u^α by u and we consider the following equation:

$$\partial_t u^{\frac{1}{\alpha}} - \Delta u = \operatorname{div} f. \tag{3.2}$$

For the sake of understanding the argument clearly, we first consider the case of $\alpha = 1$. Then we obtain the following proposition:

Proposition 3.1 (cf. Wu-Yin-Wang [20]). *Let u be a bounded solution of* (3.2) *with $\alpha = 1$ and let $\rho > 0$. Assume $\inf_{Q_\rho} u = 0$. Then, for all $0 < \varepsilon < \sup_{Q_\rho} u$ and $0 < \theta_0 < 1$, there exist constants $0 < r_0, \eta_0 < 1$ such that we obtain the following estimates:*

- *(lower bounds) Either if*

$$|Q_\rho \cap \{u \leq \varepsilon\}| \leq \theta_0 |Q_\rho|,$$

then

$$u \geq \eta_0 \quad \textit{in} \quad Q_{r_0\rho}\,;$$

- *(upper bounds) Otherwise, i.e.*

$$|Q_\rho \cap \{u \le \varepsilon\}| > \theta_0 |Q_\rho|,$$

then

$$u \le (1-\eta_0) \sup_{Q_\rho} u \quad \textit{in} \quad Q_{r_0\rho}.$$

Using Proposition 3.1, we obtain

$$\operatorname*{osc}_{Q_{r_0\rho}} u = \sup_{Q_{r_0\rho}} u - \inf_{Q_{r_0\rho}} \le (1-\eta_0) \operatorname*{osc}_{Q_\rho} u. \tag{3.3}$$

For $m \in \mathbb{N}$ and $\rho_0 > 0$, we put $\rho_m = r_0\rho_{m-1}$. Then using (3.3) inductively and considering $u - \inf_{Q_{\rho_m}} u$ if we need, we obtain

$$\operatorname*{osc}_{Q_{\rho_m}} u \le (1-\eta_0) \operatorname*{osc}_{Q_{\rho_{m-1}}} u \le \cdots \le (1-\eta_0)^m \operatorname*{osc}_{Q_{\rho_0}} u.$$

Let $\gamma > 0$ be a positive number satisfying $1 - \eta_0 \le r_0^\gamma$. Then we have

$$\operatorname*{osc}_{Q_{\rho_m}} u \le (r_0^m)^\gamma \operatorname*{osc}_{Q_{\rho_0}} u = \left(\frac{\rho_m}{\rho_0}\right)^\gamma \operatorname*{osc}_{Q_{\rho_0}} u.$$

Considering the interpolation, we find

$$\operatorname*{osc}_{Q_\rho} u \le C\rho^\gamma, \quad \rho > 0$$

and hence we obtain the Hölder continuity of the solution.

3.2. **The case of $\alpha > 1$.** Now, we apply the above argument for the case of $\alpha > 1$. For that purpose, we introduce the following modified parabolic cylinder: for $\rho, M > 0$ and $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^n$, we put

$$I_{\rho,M} = I_{\rho,M}(t_0) := \left(t_0 - \frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, t_0\right), \quad B_\rho := B_\rho(x_0) \quad \text{and}$$
$$Q_{\rho,M} = Q_{\rho,M}(t_0, x_0) := I_{\rho,M} \times B_\rho.$$

The modified parabolic cylinder is derived from the following invariant scaling for (3.2) with zero external force:

$$u_M(s,x) := \frac{1}{M} u(t,x), \quad t = \frac{s}{M^{1-\frac{1}{\alpha}}}.$$

In other words, for the quasi-linear case, we should change the scaling if we control the value of the solution. Using the modified parabolic cylinder, we obtain the following alternative lemma:

Lemma 3.2 (alternative lemma). *There exist constants $0 < \theta_0, \eta_0, r_0 < 1$ and $\delta_0 > 0$ depending only on n, α, p such that for all $\rho, \omega, M > 0$ satisfying*

$$\rho^{1-\frac{n}{p}} \le \delta_0 \omega \|f\|^{-1}_{L^\infty(I_{\rho,M}; L^p_w(B_\rho))}, \tag{A1}$$

$$\sup_{Q_{\rho,M}} u \le M \le 3 \sup_{Q_{\rho,M}} u \tag{A2}$$

and

$$\frac{3}{4}\omega \le \operatorname*{osc}_{Q_{\rho,M}} u \le \omega, \tag{A3}$$

we obtain the following estimates:

- *(lower bounds) Either if*

$$\left|Q_{\rho,M} \cap \left\{u < \inf_{Q_{\rho,M}} u + \frac{\omega}{2}\right\}\right| \le \theta_0 |Q_{\rho,M}|,$$

then

$$u(t,x) \ge \inf_{Q_{\rho,M}} u + \eta_0\omega \quad \textit{for } (t,x) \in Q_{r_0\rho,M} \,;$$

- *(upper bounds) Otherwise, i.e.*

$$\left|Q_{\rho,M} \cap \left\{u < \inf_{Q_{\rho,M}} u + \frac{\omega}{2}\right\}\right| \le \theta_0 |Q_{\rho,M}|,$$

then

$$u(t,x) \le \sup_{Q_{\rho,M}} u - \eta_0\omega \quad \textit{for } (t,x) \in Q_{r_0\rho,M}.$$

In either cases, we obtain

$$\operatorname*{osc}_{Q_{r_0\rho,M}} u = \sup_{Q_{r_0\rho,M}} u - \inf_{Q_{r_0\rho,M}} u \le \operatorname*{osc}_{Q_{\rho,M}} u - \eta_0\omega \le (1-\eta_0)\omega$$

and hence we obtain the better oscillation estimates.

Remark. Among the assumptions (A1), (A2) and (A3), the most important assumption is (A2). From the scaling argument, we would take $M = \sup_{Q_{\rho,M}} u$, however this is impossible since the modified parabolic cylinder depends on M. Nevertheless, by the assumption (A2), we can regard M as the supremum of u in $Q_{\rho,M}$ and we obtain the lower bounds or the upper bounds not depending on the value of the solution.

Using the alternative lemma repeatedly, we obtain the following lemma:

Lemma 3.3 (Iteration). *Let* $\rho_0 > 0$ *be a number satisfying* (A1). *Then, there exist constants* $0 < r_1, \eta_1 < 1$ *depending only on* n, α, p *and a decreasing sequence* $\{M_m\}_{m=0}^{\infty}$ *such that*

$$\begin{cases} \operatorname*{osc}_{Q_{\rho_m,M_m}} u \le (1-\eta_1) \operatorname*{osc}_{Q_{\rho_{m-1},M_{m-1}}} u, \\ \qquad \rho_m = r_1 \rho_{m-1} \end{cases}$$

for all $m \in \mathbb{N} \cup \{0\}$.

By Lemma 3.3, we obtain the Hölder continuity of the solution. Calculating the sequence $\{M_m\}_{m=0}^{\infty}$ and using the assumption (A1), we have quantitative Hölder estimates of the solution.

Now, we compare the iteration argument for the case of $\alpha = 1$ with the argument for the case of $\alpha > 1$. For the case of $\alpha = 1$, the diffusive coefficient is uniformly for the value of u. Therefore, a ratio of the radius to the square of the height of the parabolic cylinder is a constant in the iteration argument. On the other hand, for the case of $\alpha > 1$, the diffusive coefficient may vanish. To obtain the oscillation estimates not depending on the value of u, we may change the ratio of the radius to the square of the height of the parabolic cylinder (see figure 1).

3.3. **Proof of the lower bounds.** Now, we explain how to prove the alternative lemma (Lemma 3.2). In this report, we only consider the lower bounds of the solutions. We hereafter write

$$\mu^+ := \sup_{Q_{\rho,M}} u \quad \text{and} \quad \mu^- := \inf_{Q_{\rho,M}} u.$$

To show the lower bounds, the following Caccioppoli estimates play the important role:

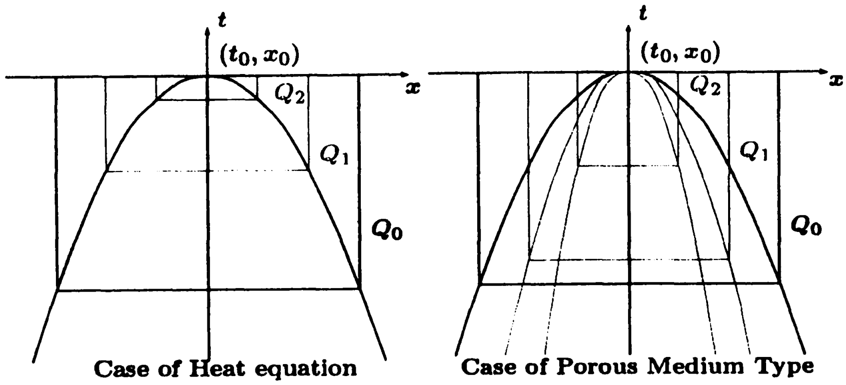

FIGURE 1. Iterating parabolic cylinders

Lemma 3.4 (Caccioppoli estimates for the negative part of solutions). *Let $\eta = \eta(t,x)$ be in $C^\infty(I_{\rho,M}; C_0^\infty(B_\rho))$ satisfing $\eta(t_0 - \frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, x) = 0$ for all $x \in B_\rho$. Assume the inequality* (A3). *Then for all $\mu^- < k < \mu^- + \frac{\omega}{2}$, there exists a constant $C > 0$ depending only on α such that*

$$(3.4)\quad \sup_{t\in I_{\rho,M}} \int_{B_\rho} (u-k)_-^2 \eta^2\, dx\Big|_t + (\mu^+)^{1-\frac{1}{\alpha}} \iint_{Q_{\rho,M}} |\nabla (u-k)_-|^2 \eta^2\, dtdx$$
$$\le C\Big\{\omega \iint_{Q_{\rho,M}} (u-k)_- \eta \partial_t \eta\, dtdx + (\mu^+)^{1-\frac{1}{\alpha}} \iint_{Q_{\rho,M}} (u-k)_-^2 |\nabla \eta|^2\, dtdx$$
$$+ (\mu^+)^{1-\frac{1}{\alpha}} \|f\|^2_{L^\infty(I_{\rho,M}; L^p_w(B_\rho))} \int_{I_{\rho,M}} |B_\rho \cap \{u(t) < k\}|^{1-\frac{2}{p}}\, dt\Big\},$$

where $(f)_-$ is thectj negative part of f.

Proof. Consider the test function $-(u-k)_-\eta^2$ in (3.2). Then

$$\frac{1}{\alpha} \iint_{Q_{\rho,M}} \partial_t \Big(\int_0^{(u-k)_-} (k-\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi\Big) \eta^2\, dtdx + \iint_{Q_{\rho,M}} \nabla (u-k)_- \cdot \nabla\{(u-k)_-\eta^2\}\, dtdx$$
$$= \iint_{Q_{\rho,M}} f \cdot \nabla\{(u-k)_-\eta^2\}\, dtdx.$$

By the integration by parts and the Young inequality, we have

$$(3.5)\quad \frac{1}{\alpha} \sup_{t\in I_{\rho,M}} \int_{B_\rho} \Big(\int_0^{(u-k)_-} (k-\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi\Big) \eta^2\, dx + \frac{1}{4} \iint_{Q_{\rho,M}} |\nabla (u-k)_-|^2 \eta^2\, dtdx$$
$$\le \frac{1}{\alpha} \iint_{Q_{\rho,M}} \Big(\int_0^{(u-k)_-} (k-\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi\Big) \partial_t \eta^2\, dtdx + 3 \iint_{Q_{\rho,M}} (u-k)_-^2 |\nabla \eta|^2\, dtdx$$
$$+ 2 \iint_{Q_{\rho,M} \cap \{u<k\}} |f|^2 \eta^2\, dtdx.$$

First, we estimate the 1st term of the left-hand side of (3.5). Since (A3) and $k \le \mu^- + \frac{\omega}{2} \le \mu^+ - \operatorname{osc}_{Q_{\rho,M}} u + \frac{\omega}{2} \le \mu^+$, we have

$$\int_0^{(u-k)_-} (k-\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi \ge (\mu^+)^{\frac{1}{\alpha}-1} (u-k)_-^2$$

and hence

$$(3.6)\quad \frac{1}{2\alpha}\sup_{t\in I_{\rho,M}}\int_{B_\rho}(u-k)_-^2\eta^2\,dx+\frac{1}{4}(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}|\nabla(u-k)_-|^2\eta^2\,dtdx$$
$$\leq\frac{1}{\alpha}(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}\left(\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\right)\partial_t\eta^2\,dtdx$$
$$+3(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}(u-k)_-^2|\nabla\eta|^2\,dtdx+2(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}\cap\{u<k\}}|f|^2\eta^2\,dtdx.$$

Now, we explain how to treat the external force f. By the definition of the weak L^p space, we have

$$\iint_{Q_{\rho,M}\cap\{u<k\}}|f|^2\eta^2\,dz=\int_{I_{\rho,M}}dt\int_{B_\rho\cap\{u(t)<k\}}|f|^2\,dx$$
$$\leq\int_{I_{\rho,M}}\|f(t)\|^2_{L^p_w(B_\rho)}|B_\rho\cap\{u(t)<k\}|^{1-\frac{2}{p}}\,dt$$

and hence

$$(3.7)\quad 2(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}\cap\{u<k\}}|f|^2\eta^2\,dtdx$$
$$\leq 2(\mu^+)^{1-\frac{1}{\alpha}}\|f\|^2_{L^\infty(I_{\rho,M};L^p_w(B_\rho))}\int_{I_{\rho,M}}|B_\rho\cap\{u(t)<k\}|^{1-\frac{2}{p}}\,dt.$$

To estimate the first term of the right-hand side of (3.6), we consider the case of either $\mu^-\leq\frac{1}{2}\mu^+$ or $\mu^->\frac{1}{2}\mu^+$. In either cases, we obtain

$$(3.8)\quad (\mu^+)^{1-\frac{1}{\alpha}}\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\leq(\mu^+)^{1-\frac{1}{\alpha}}\left[\left(\mu^-+\frac{\omega}{2}\right)^{\frac{1}{\alpha}}-(\mu^-)^{\frac{1}{\alpha}}\right](u-k)_-$$
$$\leq C(\alpha,n)\omega(u-k)_-.$$

Substituting (3.7) and (3.8) for (3.6), we obtain (3.4).

□

Proof of Lemma 3.2 (the lower bounds). We consider the scale transform:

$$s=M^{1-\frac{1}{\alpha}}t,\quad \tilde{u}(s,x)=u(t,x),\quad \tilde{\eta}(s,x)=\eta(t,x)\quad\text{and}\quad \tilde{f}(s,x)=f(t,x).$$

We rewrite the Caccioppoli estimates (3.4) as follows:

$$(3.9)\quad \sup_{t\in I_\rho}\int_{B_\rho}(\tilde{u}-k)_-^2\tilde{\eta}^2\,dx+\frac{(\mu^+)^{1-\frac{1}{\alpha}}}{M^{1-\frac{1}{\alpha}}}\iint_{Q_\rho}|\nabla(\tilde{u}-k)_-|^2\tilde{\eta}^2\,dsdx$$
$$\leq C(\alpha)\left\{\omega\iint_{Q_\rho}(\tilde{u}-k)_-\partial_s\tilde{\eta}^2\,dsdx+\frac{(\mu^+)^{1-\frac{1}{\alpha}}}{M^{1-\frac{1}{\alpha}}}\iint_{Q_\rho}(\tilde{u}-k)_-^2|\nabla\tilde{\eta}|^2\,dsdx\right.$$
$$\left.+\frac{(\mu^+)^{1-\frac{1}{\alpha}}}{M^{1-\frac{1}{\alpha}}}\|\tilde{f}\|^2_{L^\infty(I_\rho;L^p_w(B_\rho))}\left(\int_{I_\rho}|B_\rho\cap\{\tilde{u}(s)<k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,ds\right)^{\frac{2}{q'}}\right\},$$

where $I_\rho=I_\rho(t_0):=(t_0-\rho^2,t_0)$.

We take $\gamma_0, p_*, q_* > 0$ as

$$\gamma_0 = 1 - \frac{n}{p}, \quad 1 = \frac{2}{q_*}\left(1 + \frac{2\gamma_0}{n}\right) \quad \text{and } 1 - \frac{2}{p} = \frac{q_*}{p_*}.$$

We remark that $\frac{2}{q_*} + \frac{n}{p_*} = \frac{n}{2}$.

Now, we apply the De Giorgi-Nash-Moser iteration. For $i \in \mathbb{N}$, we take $\rho = \rho_i$, $k = k_i$, $\tilde{\eta} = \tilde{\eta}_i$ satisfying $\tilde{\eta}_i \equiv 1$ on $Q_{\rho_{i+1}}$ and

$$k_i = \mu^- + \frac{1}{4}\omega + \frac{1}{2^{i+1}}\omega, \qquad \rho_i = \frac{1}{2}\rho + \frac{1}{2^{i+1}}\rho,$$

$$Y_i := \frac{|Q_{\rho_i} \cap \{\tilde{u} < k_i\}|}{|Q_\rho|}, \qquad Z_i = \frac{\rho^2}{|Q_\rho|}\left(\int_{I_{\rho_i}} |B_{\rho_i} \cap \{\tilde{u}(s) < k_i\}|^{\frac{q_*}{p_*}} \, ds\right)^{\frac{2}{q_*}},$$

$$|\nabla \tilde{\eta}_i| \le \frac{2}{\rho_i - \rho_{i+1}} \le \frac{8 \cdot 2^i}{\rho}, \quad \partial_s \tilde{\eta}_i \le \frac{2}{\rho_i^2 - \rho_{i+1}^2} \le \frac{16 \cdot 2^{2i}}{3\rho^2}.$$

Then, by using (A2) and $(\tilde{u} - k_i)_- \le \frac{\omega}{2}$, we rewrite (3.9) as

$$\begin{aligned} &\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^\infty(I_{\rho_i}; L^2(B_{\rho_i})) \cap L^2(I_{\rho_i}; H^1(B_{\rho_i}))} \\ &\qquad \le C(\alpha) \frac{\omega^2 |Q_\rho|}{\rho^2} \left\{ 2^{2i} Y_i + \|\tilde{f}\|^2_{L^\infty(I_\rho; L^p_w(B_\rho))} \omega^{-2} \left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1+\frac{2\gamma_0}{n}} \right\}. \end{aligned}$$

Using the Ladyženskaja inequality (cf. Ladyženskaja-Solonnikov-Ural'ceva, [7, Section 3 in Chapter II (pp. 74)]) and the Hölder inequality, we have

$$\begin{aligned} &\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^2(Q_{\rho_i})} \\ &\qquad \le C(\alpha, n) \omega^2 |Q_\rho| Y_i^{\frac{2}{n+2}} \left\{ 2^{2i} Y_i + \|\tilde{f}\|^2_{L^\infty(I_\rho; L^p_w(B_\rho))} \omega^{-2} \left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1+\frac{2\gamma_0}{n}} \right\} \end{aligned}$$

and

$$\begin{aligned} &\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^{q_*}(I_{\rho_i}; L^{p_*}(B_{\rho_i}))} \\ &\qquad \le C(\alpha, n) \frac{\omega^2 |Q_\rho|}{\rho^2} \left\{ 2^{2i} Y_i + \|\tilde{f}\|^2_{L^\infty(I_\rho; L^p_w(B_\rho))} \omega^{-2} \left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1+\frac{2\gamma_0}{n}} \right\}. \end{aligned}$$

Since

$$\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^2(Q_{\rho_i})} \ge \|(\tilde{u} - k_i)_-\|^2_{L^2(Q_{\rho_{i+1}} \cap \{\tilde{u} < k_{i+1}\})} \ge \frac{\omega^2}{64 \cdot 2^{2i}} |Q_\rho| Y_{i+1}$$

and

$$\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^{q_*}(I_{\rho_i}; L^{p_*}(B_{\rho_i}))} \ge \|(\tilde{u} - k_i)_- \chi_{\{\tilde{u} < k_{i+1}\}}\|^2_{L^{q_*}(I_{\rho_i}; L^{p_*}(B_{\rho_i})))} \ge \frac{\omega^2}{64 \cdot 2^{2i}} \frac{|Q_\rho|}{\rho^2} Z_{i+1},$$

we obtain

$$Y_{i+1} \le C(\alpha, n) \left\{ 2^{4i} Y_i^{1+\frac{2}{n+2}} + 2^{2i} \|\tilde{f}\|^2_{L^\infty(I_\rho; L^p_w(B_\rho))} \omega^{-2} \left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Y_i^{\frac{2}{n+2}} Z_i^{1+\frac{2\gamma_0}{n}} \right\}$$

and

$$Z_{i+1} \leq C(\alpha, n)\left\{2^{4i}Y_i + 2^{2i}\|\tilde{f}\|^2_{L^\infty(I_\rho;L^p_w(B_\rho))}\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1+\frac{2\gamma_0}{n}}\right\}.$$

Using $\frac{q_*}{p_*} \leq 1$ and the Hölder inequality, we obtain

$$Z_0 \leq C(n,p,q)Y_0^{\frac{2}{p_*}}.$$

Let $\rho^{\gamma_0} \leq \omega^{-1}\|\tilde{f}\|^{-1}_{L^\infty(I_\rho;L^q_w(B_\rho))}$. Then, by the well-known result for the recurcive inequality (cf. Ladyženskaja-Solonnikov-Ural'ceva, [7, Lemma 5.7 in Chapter II (pp.96)]), there exists $0 < \theta_0 = \theta_0(n, \alpha, p, q) < 1$ such that if $Y_0 \leq \theta_0$, then $Y_i \longrightarrow 0$ as $i \longrightarrow \infty$, i.e.

$$\tilde{u}(s,x) > \mu^- + \frac{\omega}{4} \quad \text{a.a. } (s,x) \in Q_{\frac{\rho}{2}}.$$

□

Remark. In the proof of the lower bounds, we essentially consider the upper bounds of $v_- = (u-k)_-$. Then the function v_- satisfies

$$\partial_t v_- - \operatorname{div}(\alpha(k - v_-)^{\alpha-1}\nabla v_-) = -\operatorname{div} f.$$

The function v_- also satisfies the degenerate parabolic equation. To overcome this difficulty, we take θ_0 sufficiently small.

On the other hand, to show the upper bounds of the solution. we consider the upper bounds of $v_+ := (u-k)_+$. Then the function v_+ satisfies

$$\partial_t v_+ - \operatorname{div}(\alpha(v_+ + k)^{\alpha-1}\nabla v_+) = \operatorname{div} f.$$

Therefore we can regard v_+ as a solution of the uniformly parabolic equations. To show the upper bounds of v_+, we apply the Bernstein estimates, the Poincare inequality and the hole filling argument for v_+. These arguments are like as the arguments of the regularity theory for solutions of the uniformly parabolic equations. The full detailed discussion will be published elsewhere [9].

Acknowledgement

One of the authors M. Mizuno would like to express the deepest gratitude to Professor Toyohiko Aiki for giving me an opportunity to talk in the RIMS workshop "Nonlinear evolution equations and mathematical modeling" in October 20–23, 2009.

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