数理解析研究所講究録
第 1705 巻 2010 年 51-62

Hilbert Space Representations of Quantum Phase Spaces with General Degrees of Freedom

Asao Arai (新井朝雄)
Department of Mathematics, Hokkaido University
Sapporo, Hokkaido 060-0810
Japan
E-mail: arai@math.sci.hokudai.ac.jp

Abstract

For each integer $n \geq 2$ and a parameter $\Lambda = (\theta, \eta)$ with θ and η being $n \times n$ real anti-symmetric matrices, a quantum phase space (QPS) (or a non-commutative phase space) with n degrees of freedom, denoted $\mathrm{QPS}_n(\Lambda)$, is defined, where θ and η are parameters measuring non-commutativity of the QPS. Some results on Hilbert space representations of $\mathrm{QPS}_n(\Lambda)$ are reported.

Keywords: Quantum phase space; non-commutative phase space; canonical commutation relations; quantum deformation.

Mathematics Subject Classification 2000: 81D05, 81R60, 47L60, 47N50

1 Introduction

As is well-known, one of the fundamental principles in von Neumann's axiomatic quantum mechanics is that a subset of physical quantities of a quantum system with n external degrees of freedom ($n \in \mathbb{N}$) are constructed from a self-adjoint representation of the canonical commutation relations (CCR) with n degrees of freedom, which is given by a triple $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)$ consisting of a complex Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D}$ of $\mathcal{H}$ and a set $\{Q_j, P_j\}_{j=1}^n$ of self-adjoint operators on $\mathcal{H}$ satisfying (i) $\mathcal{D} \subset \cap_{j,k=1}^n D(Q_jQ_k) \cap D(P_jP_k) \cap D(Q_jP_k) \cap D(P_kQ_j)$, where, for a linear operator A on a Hilbert space, $D(A)$ denotes the domain of A; (ii) (CCR)

$$[Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \tag{1.1}$$

$$[Q_j, P_k] = i\delta_{jk}, \quad j, k = 1, \cdots, n, \tag{1.2}$$

on $\mathcal{D}$, where $[X, Y] := XY - YX$, i is the imaginary unit and δ_{jk} is the Kronecker delta. If Q_j and P_j ($j = 1, \cdots, n$) are not necessarily self-adjoint, but symmetric, then the triple $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)$ is called a symmetric representation of the CCR with n degrees of freedom. This class of representations of CCR also plays important roles, e.g., in the theory of time operators ([1, 2, 3], [5, 6], [12]).

In commutation relations (1.1) and (1.2), non-commutativity is imposed only between Q_j and P_j $(j = 1, \cdots, n)$. But, from a general mathematical point of view, it may be natural to extend non-commutativity to Q_j's and P_j's too. This idea leads us to a general concept of a quantum phase space (QPS) or a non-commutative phase space[1]. In this paper we propose one of possible QPS's and report some results on Hilbert space representations of it (for more details, see [4]). In addition, we remark that non-commutative extensions of CCR have already been discussed in connection with quantum theory on non-commutative space-times (e.g., [7, 8, 9, 15]), non-commutative spaces (e.g., [10, 11]) and non-commutative phase spaces (e.g., [13, 14, 16, 17]). But it seems that representation theoretic investigations on non-commutative extensions of CCR have not yet been fully developed.

2 Hilbert Space Representations of a QPS

Let $n \in \mathbb{N}$ with $n \geq 2$. To define a QPS with n degrees of freedom, we take two $n \times n$ real anti-symmetric matrices $\theta = (\theta_{jk})_{j,k=1,\cdots,n}$ and $\eta = (\eta_{jk})_{j,k=1,\cdots,n}$. Then we introduce an algebra generated by $2n$ elements $\hat{Q}_j, \hat{P}_j (j = 1, \cdots, n)$ and a unit element I obeying deformed CCR with n degrees of freedom

$$[\hat{Q}_j, \hat{Q}_k] = i\theta_{jk}I, \tag{2.1}$$

$$[\hat{P}_j, \hat{P}_k] = i\eta_{jk}I, \tag{2.2}$$

$$[\hat{Q}_j, \hat{P}_k] = i\delta_{jk}I, \quad j, k = 1, \cdots, n, \tag{2.3}$$

We call this algebra the QPS or the *non-commutative phase space with n degrees of freedom* and parameter

$$\Lambda := (\eta, \theta). \tag{2.4}$$

We denote it by $\mathrm{QPS}_n(\Lambda)$.

It is obvious that $\hat{Q}_j$ and $\hat{Q}_k$ (resp. $\hat{P}_j$ and $\hat{P}_k$) with $j \neq k$ do not commute if and only if $\theta_{jk} \neq 0$ (resp. $\eta_{jk} \neq 0$). Hence the parameter Λ "measures" the non-commutativity of $\hat{Q}_j$'s and $\hat{P}_j$'s respectively. Moreover $\mathrm{QPS}_n(\Lambda)$ in the case $\theta = \eta = 0$ reduces to the algebra of the CCR with n degrees of freedom. Hence $\mathrm{QPS}_n(\Lambda)$ can be regarded as a deformation of the algebra of the CCR with n degrees of freedom.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ (linear in the second variable) and norm $\|\cdot\|$. Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ and $\hat{Q}_j, \hat{P}_j$ be symmetric operators on $\mathcal{H}$.

Definition 2.1 We say that the triple $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ is a representation (on $\mathcal{H}$) of the algebra $\mathrm{QPS}_n(\Lambda)$ if $\mathcal{D} \subset \cap_{j,k=1}^n D(\hat{Q}_j\hat{Q}_k) \cap D(\hat{P}_j\hat{P}_k) \cap D(\hat{Q}_j\hat{P}_k) \cap D(\hat{P}_j\hat{Q}_k)$ and it satisfy (2.1)–(2.3) on $\mathcal{D}$ with I being the identity on $\mathcal{H}$ (we sometimes omit the identity I below).

[1] Note that the components x_j and p_j $(j = 1, \cdots, n)$ of each element $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ in the classical phase space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ can be regarded as multiplication operators acting in $L^2(\mathbb{R}^{2n})$. They form a commutative algebra.

If all $\hat{Q}_j$ and $\hat{P}_j$ $(j = 1, \cdots, n)$ are self-adjoint, we say that the representation $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ is self-adjoint.

In every representation $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ of $\mathrm{QPS}_n(\Lambda)$, we have commutation relations (2.1)–(2.3) on $\mathcal{D}$. Hence the following Heisenberg uncertainty relations follow: for all $\psi \in \mathcal{D}$ with $\|\psi\| = 1$ and $j, k = 1, \cdots, n$,

$$(\Delta \hat{Q}_j)_\psi (\Delta \hat{Q}_k)_\psi \geq \frac{1}{2}|\theta_{jk}|, \tag{2.5}$$

$$(\Delta \hat{P}_j)_\psi (\Delta \hat{P}_k)_\psi \geq \frac{1}{2}|\eta_{jk}|, \tag{2.6}$$

$$(\Delta \hat{Q}_j)_\psi (\Delta \hat{P}_k)_\psi \geq \frac{1}{2}|\delta_{jk}|, \tag{2.7}$$

where, for a symmetric operator A and a vector $\psi \in D(A)$ with $\|\psi\| = 1$,

$$(\Delta A)_\psi := \|(A - \langle \psi, A\psi \rangle)\psi\|,$$

the uncertainty of A in the vector state ψ.

3 A Class of Self-Adjoint Representations of $\mathrm{QPS}_n(\Lambda)$ on $L^2(\mathbb{R}^n)$

In this section, we show that there exist self-adjoint representations of $\mathrm{QPS}_n(\Lambda)$ on $L^2(\mathbb{R}^n)$. This is done by using the Schrödinger representation of the CCR with n degrees of freedom.

We denote by $C_0^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

Let $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{q_j, p_j\}_{j=1}^n\right)$ be the Schrödinger representation of the CCR with n degrees of freedom, namely, q_j is the multiplication operator by the jth variable x_j on $L^2(\mathbb{R}^n)$ and $p_j := -iD_j$ with D_j being the generalized partial differential operator in x_j on $L^2(\mathbb{R}^n)$, so that

$$[q_j, p_k] = i\delta_{jk}, \tag{3.1}$$

$$[q_j, q_k] = 0, \quad [p_j, p_k] = 0, \quad j, k = 1, \cdots, n, \tag{3.2}$$

on the subspace $C_0^\infty(\mathbb{R}^n)$.

Lemma 3.1 *For all $a_j, b_j \in \mathbb{R}, j = 1, \cdots, n$, $\sum_{j=1}^n (a_j p_j + b_j q_j)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.*

For an n-tuple $L = (L_1, \cdots, L_n)$ of linear operators $L_j, j = 1, \cdots, n$, on a Hilbert space and an $n \times n$ matrix $A = (A_{jk})_{j,k=1,\cdots,n}$, we define the n-tuple $AL = ((AL)_1, \cdots, (AL)_n)$ of linear operators by

$$(AL)_j := \sum_{k=1}^n A_{jk} L_k. \tag{3.3}$$

We say that the parameter $\Lambda = (\theta, \eta)$ is *normal* if there exist $n \times n$ real matrices A, B, C and D satisfying

$$A\,{}^{t}D - B\,{}^{t}C = I_n, \tag{3.4}$$

$$A\,{}^{t}B - B\,{}^{t}A = \theta, \tag{3.5}$$

$$C\,{}^{t}D - D\,{}^{t}C = \eta, \tag{3.6}$$

where I_n is the $n \times n$ unit matrix and ${}^{t}A$ denotes the transposed matrix of A.

For a normal parameter Λ with (3.4)–(3.6), we can define a $(2n) \times (2n)$ matrix:

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.7}$$

Let

$$K(\Lambda) := \begin{pmatrix} \theta & I_n \\ -I_n & \eta \end{pmatrix}, \quad J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{3.8}$$

Then we have

$$G J_n {}^{t}G = K(\Lambda). \tag{3.9}$$

Conversely, if a $(2n) \times (2n)$ real matrix G of the form (3.7) satisfies (3.9), then A, B, C and D obey relations (3.4)–(3.6).

Thus Λ is normal if and only if there exists a $(2n) \times (2n)$ real matrix G satisfying (3.9). In that case, we call G a generating matrix of Λ.

We remark that, for a normal parameter Λ, its generating matrices are not unique. For example, if G is a generating matrix of Λ, then, for all orthogonal matrix M commuting with $K(\Lambda)$, MG is a generating matrix of Λ too.

Suppose that Λ is normal with (3.4)–(3.6). We set

$$\mathbf{q} = (q_1, \cdots, q_n), \quad \mathbf{p} = (p_1, \cdots, p_n) \tag{3.10}$$

and define

$$\hat{\mathbf{q}} := A\mathbf{q} + B\mathbf{p}, \quad \hat{\mathbf{p}} := C\mathbf{q} + D\mathbf{p}. \tag{3.11}$$

Then, by Lemma 3.1, the operators $\hat{q}_j$ and $\hat{p}_j$ $(j = 1, \cdots, n)$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Hence their closures $\bar{\hat{q}}_j$ and $\bar{\hat{p}}_j$ are self-adjoint[2]. Moreover, we have the following result:

Theorem 3.2 *The set* $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1,\cdots,n}\right)$ *is a self-adjoint representation of* $\mathrm{QPS}_n(\Lambda)$.

We call the representation $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1,\cdots,n}\right)$ *the quasi-Schrödinger representation* of $\mathrm{QPS}_n(\Lambda)$ with generating matrix G of the form (3.7).

---

[2]For a closable linear operator T, we denote its closure by $\bar{T}$.

Remark 3.3 One can write

$$\begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \\ \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} = G \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \tag{3.12}$$

on $\cap_{j=1}^{n} D(q_j) \cap D(p_j)$. Equation (3.9) is rewritten as follows:

$$G J_n {}^t G = J_n + \delta(\Lambda) \tag{3.13}$$

with

$$\delta(\Lambda) := \begin{pmatrix} \theta & 0 \\ 0 & \eta \end{pmatrix}. \tag{3.14}$$

Hence ${}^tG$ is symplectic if and only if $\delta(\Lambda) = 0$ (i.e., $\theta = \eta = 0$). Therefore the matrix $\delta(\Lambda)$ represents a difference from the symplectic relation. Note that the diagonal element θ (resp. η) of $\delta(\Lambda)$ gives the non-commutativity of $\hat{q}_j$'s (resp. $\hat{p}_k$'s) $(j, k = 1, \cdots, n)$.

3.1 The Schrödinger representation of QPS

It may be interesting to consider a special case of Λ. Let $a \geq 0, b \geq 0$ be constants and

$$\xi := \frac{1}{\sqrt{1 + \dfrac{ab}{4}}}. \tag{3.15}$$

Let γ be an $n \times n$ real anti-symmetric matrix satisfying

$$\gamma^2 = -I_n. \tag{3.16}$$

Then the parameter

$$\Lambda_{\rm S} := (\xi^2 a\gamma, \xi^2 b\gamma) \quad (\text{the case } \theta = \xi^2 a\gamma, \eta = \xi^2 b\gamma) \tag{3.17}$$

is normal, since the matirix

$$G_{\rm S} := \begin{pmatrix} \xi I_n & -\frac{1}{2}\xi a\gamma \\ \frac{1}{2}\xi b\gamma & \xi I_n, \end{pmatrix} \tag{3.18}$$

is a generating matrix of $\Lambda_{\rm S}$, as is easily checked. We denote $\bar{q}_j$ and $\bar{p}_j$ in the present case by $\hat{q}_j^{(\rm S)}$ and $\hat{p}_j^{(\rm S)}$ respectively:

$$\hat{q}_j^{(\rm S)} := \overline{\xi\left(q_j - \frac{1}{2}a(\gamma p)_j\right)}, \quad \hat{p}_j^{(\rm S)} := \overline{\xi\left(p_j + \frac{1}{2}b(\gamma q)_j\right)}, \quad j = 1, \cdots, n. \tag{3.19}$$

We call this self-adjoint representation $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\hat{q}_j^{(\rm S)}, \hat{p}_j^{(\rm S)}\}_{j=1,\cdots,n}\right)$ of $\mathrm{QPS}_n(\Lambda_{\rm S})$ the *Schrödinger representation of* $\mathrm{QPS}_n(\Lambda_{\rm S})$.

3.2 Reconstruction of the Schrödinger representation of the CCR with n degrees of freedom

In this subsection, we consider reconstruction of q_j and p_j in terms of $\hat{q}_j$ and $\hat{p}_j$. By (3.12), this problem may be reduced by the invertibility of the matrix G. From this point of view, we introduce a class of parameters Λ.

We say that Λ is *regular* if it is normal and has an invertible generating matrix. It follows from (3.9) that, if Λ is regular, then every generating matrix of Λ is invertible.

The next lemma characterizes the regularity of Λ:

Lemma 3.4 *Let Λ be normal with a generating matrix G given by (3.7). Then Λ is regular if and only if $I_n + \theta\eta$ and $I_n + \eta\theta$ are invertible. In that case, G is invertible and*

$$ {}^{t}(G^{-1})J_nG^{-1} = -\begin{pmatrix} (I_n+\eta\theta)^{-1}\eta & -(I_n+\eta\theta)^{-1} \\ (I_n+\theta\eta)^{-1} & (I_n+\theta\eta)^{-1}\theta \end{pmatrix}. \tag{3.20} $$

Let Λ be regular with a generating matrix G. Then we can write

$$ G^{-1} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}, \tag{3.21} $$

where F_1, F_2, F_3 and F_4 are $n \times n$ real matrices.

Let

$$ \hat{\mathbf{q}} := (\hat{q}_1, \cdots, \hat{q}_n), \quad \hat{\mathbf{p}} := (\hat{p}_1, \cdots, \hat{p}_n). \tag{3.22} $$

Theorem 3.5 *The following equations hold:*

$$ \mathbf{q} = F_1\hat{\mathbf{q}} + F_2\hat{\mathbf{p}}, \quad \mathbf{p} = F_3\hat{\mathbf{q}} + F_4\hat{\mathbf{p}}. \tag{3.23} $$

on $\cap_{j=1}^n D(q_j) \cap D(p_j)$.

Theorem 3.5 also implies relations of matrix elements of G^{-1}:

Corollary 3.6

$$ F_1\theta{}^{t}F_1 + F_2\eta{}^{t}F_2 + F_1{}^{t}F_2 - F_2{}^{t}F_1 = 0, \tag{3.24} $$

$$ F_3\theta{}^{t}F_3 + F_4\eta{}^{t}F_4 + F_3{}^{t}F_4 - F_4{}^{t}F_3 = 0, \tag{3.25} $$

$$ F_1\theta{}^{t}F_3 + F_2\eta{}^{t}F_4 + F_1{}^{t}F_4 - F_2{}^{t}F_3 = I_n. \tag{3.26} $$

We now apply Theorem 3.5 to the Schrödinger representation $\{\hat{q}_j^{(\mathrm{S})}, \hat{p}_j^{(\mathrm{S})}\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda_{\mathrm{S}})$:

Corollary 3.7 *Let a, b, ξ and γ be as in Subsection 3.1. Suppose that*

$$ \chi := 1 - \frac{1}{4}ab \neq 0. \tag{3.27} $$

Then

$$q_j = \frac{1}{\xi\chi}\left(\hat{q}_j^{(\mathrm{S})} + \frac{1}{2}a(\gamma\hat{p}^{(\mathrm{S})})_j\right), \tag{3.28}$$

$$p_j = \frac{1}{\xi\chi}\left(\hat{p}_j^{(\mathrm{S})} - \frac{1}{2}b(\gamma\hat{q}^{(\mathrm{S})})_j\right), \quad j = 1, \cdots, n, \tag{3.29}$$

on $C_0^\infty(\mathbb{R}^n)$.

4 General Correspondence Between a Representation of $\mathrm{QPS}_n(\Lambda)$ and a Representation of the CCR with n Degrees of Freedom

4.1 Construction of a representation of $\mathrm{QPS}_n(\Lambda)$ from a representation of the CCR with n degrees of freedom

The contents in Section 2 suggest a general method to construct a representation of $\mathrm{QPS}_n(\Lambda)$ from a representation of the CCR with n degrees of freedom.

Let $\left(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n\right)$ be a representation of the CCR with n degrees of freedom, namely, $\mathcal{H}$ is a Hilbert space, $\mathcal{D}$ is a dense subspace of $\mathcal{H}$ and Q_j and P_j $(j = 1, \cdots, n)$ are symmetric operators on $\mathcal{H}$ such that $\mathcal{D} \subset \cap_{j,k=1}^n D(Q_jQ_k) \cap D(P_jP_k) \cap D(Q_jP_k) \cap D(P_kQ_j)$ and $\{Q_j, P_j\}_{j=1}^n$ obeys the CCR with n degrees of freedom on $\mathcal{D}$: for $j, k = 1, \cdots, n$,

$$[Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad [Q_j, P_k] = i\delta_{jk} \tag{4.1}$$

on $\mathcal{D}$. Let

$$\mathbf{Q} = (Q_1, \cdots, Q_n), \quad \mathbf{P} = (P_1, \cdots, P_n).$$

Let Λ be normal and A, B, C, D be $n \times n$ real matrices obeying (3.4)–(3.6). By an analogy with (3.11), we define the n-tuples

$$\hat{\mathbf{Q}} := (\hat{Q}_1, \cdots, \hat{Q}_n), \tag{4.2}$$

and

$$\hat{\mathbf{P}} := (\hat{P}_1, \cdots, \hat{P}_n), \tag{4.3}$$

by

$$\hat{\mathbf{Q}} := A\mathbf{Q} + B\mathbf{P}, \quad \hat{\mathbf{P}} := C\mathbf{Q} + D\mathbf{P}. \tag{4.4}$$

Theorem 4.1 *The set* $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ *defined by (4.4) is a representation of* $\mathrm{QPS}_n(\Lambda)$.

We remark that the representation $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ of $\mathrm{QPS}_n(\Lambda)$ is not necessarily self-adjoint even in the case where all Q_j and P_j $(j = 1, \cdots, n)$ are self-adjoint.

As in the case of quasi-Schrödinger representations of $\mathrm{QPS}_n(\Lambda)$ discussed in Section 2, we have the following fact:

Theorem 4.2 *Let Λ be regular with generating matrix G given by (3.7) and F_1, F_2, F_3 and F_4 be as in (3.21). Then*

$$\mathbf{Q} = F_1\hat{\mathbf{Q}} + F_2\hat{\mathbf{P}}, \tag{4.5}$$

$$\mathbf{P} = F_3\hat{\mathbf{Q}} + F_4\hat{\mathbf{P}}. \tag{4.6}$$

on $\mathcal{D}$.

4.2 Construction of a representation of the CCR with n degrees of freedom from a representation of $\mathrm{QPS}_n(\Lambda)$

We next consider constructing a representation of the CCR with n degrees of freedom from a representation of $\mathrm{QPS}_n(\Lambda)$. A method for that is suggested by Theorem 4.2.

Let $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ be a representation of $\mathrm{QPS}_n(\Lambda)$ on a Hilbert space $\mathcal{H}$ with $\mathcal{D}$ dense in $\mathcal{H}$. Throughout this subsection, we assume the following:

(A) The parameter Λ is regular with generating matrix G given by (3.7).

Let F_1, F_2, F_3 and F_4 be as in (3.21). Then we can define $\mathbf{Q}(\Lambda) = (Q_1(\Lambda), \cdots, Q_n(\Lambda))$ and $\mathbf{P}(\Lambda) = (P_1(\Lambda), \cdots, P_n(\Lambda))$ by

$$\mathbf{Q}(\Lambda) := F_1\hat{\mathbf{Q}} + F_2\hat{\mathbf{P}}, \tag{4.7}$$

$$\mathbf{P}(\Lambda) := F_3\hat{\mathbf{Q}} + F_4\hat{\mathbf{P}}. \tag{4.8}$$

Theorem 4.3 *Assume (A). Then $\left(\mathcal{H}, \mathcal{D}, \{Q_j(\Lambda), P_j(\Lambda)\}_{j=1}^n\right)$ is a representation of the CCR with n degrees of freedom.*

The next theorem shows that every representation of $\mathrm{QPS}_n(\Lambda)$ with condition (A) comes from a representation of the CCR with n degrees of freedom:

Theorem 4.4 *Assume (A). Let $\mathbf{Q}(\Lambda)$ and $\mathbf{P}(\Lambda)$ be defined by (4.7) and (4.8) respectively. Then*

$$\hat{\mathbf{Q}} = A\mathbf{Q}(\Lambda) + B\mathbf{P}(\Lambda), \quad \hat{\mathbf{P}} = C\mathbf{Q}(\Lambda) + D\mathbf{P}(\Lambda) \tag{4.9}$$

on $\mathcal{D}$.

5 Irreducibility

For a Hilbert space $\mathcal{H}$, we denote by $\mathsf{B}(\mathcal{H})$ the set of all bounded linear operators B on $\mathcal{H}$ with $D(B) = \mathcal{H}$. Let A be a linear operator on $\mathcal{H}$. We say that A *strongly commutes* with $B \in \mathsf{B}(\mathcal{H})$ if $BA \subset AB$ (i.e., for all $\psi \in D(A)$, $B\psi \in D(A)$ and $BA\psi = AB\psi$). For a set A of linear operators on $\mathcal{H}$, we define

$$\mathsf{A}' := \{B \in \mathsf{B}(\mathcal{H}) | BA \subset AB, \forall A \in \mathsf{A}\}. \tag{5.1}$$

We call A′ the *strong commutant* of A.

We say that A is *irreducible* if $\mathsf{A}' = \{cI | c \in \mathbb{C}\}$ ($\mathbb{C}$ is the set of complex numbers).

Lemma 5.1 *Let S be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $B \in \mathsf{B}(\mathcal{H})$ such that $BS \subset SB$. Then, for all $t \in \mathbb{R}$, $Be^{itS} = e^{itS}B$.*

Theorem 5.2 *Assume (A) in Subsection 3.2. Let $\left(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n\right)$ be a representation of the CCR with n degrees of freedom. Suppose that, for each $j = 1, \cdots, n$, Q_j and P_j are essentially self-adjoint on $\mathcal{D}$ and $\{\bar{Q}_j, \bar{P}_j\}_{j=1}^n$ is irreducible. Then the representation $\left(\mathcal{H}, \mathcal{D}, \{\bar{\hat{Q}}_j, \bar{\hat{P}}_j\}_{j=1}^n\right)$ of $\mathrm{QPS}_n(\Lambda)$ given by (4.4) is irreducible.*

We can apply Theorem 5.2 to the quasi-Schrödinger representation $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$ discussed in Section 2.

Theorem 5.3 *Assume (A). Then $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ is irreducible.*

6 Weyl Representations of $\mathrm{QPS}_n(\Lambda)$

6.1 Definition and basic facts

As is well known, a Weyl representation of the CCR with n degrees of freedom on a Hilbert space $\mathcal{H}$ is defined to be a set $\{Q_j, P_j\}_{j=1}^n$ of $2n$ self-adjoint operators on $\mathcal{H}$ obeying the Weyl relations:

$$e^{itQ_j} e^{isP_k} = e^{-ist\delta_{jk}} e^{isP_k} e^{itQ_j}, \tag{6.1}$$

$$e^{itQ_j} e^{isQ_k} = e^{isQ_k} e^{itQ_j}, \tag{6.2}$$

$$e^{itP_j} e^{isP_k} = e^{isP_k} e^{itP_j}, \quad j, k = 1, \cdots, n, s, t \in \mathbb{R}. \tag{6.3}$$

Based on an analogy with Weyl representations of CCR, we introduce a concept of Weyl representation of $\mathrm{QPS}_n(\Lambda)$.

Definition 6.1 Let $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ be a set of self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ is a *Weyl representation of* $\mathrm{QPS}_n(\Lambda)$ if

$$e^{it\hat{Q}_j} e^{is\hat{P}_k} = e^{-ist\delta_{jk}} e^{is\hat{P}_k} e^{it\hat{Q}_j}, \tag{6.4}$$

$$e^{it\hat{Q}_j} e^{is\hat{Q}_k} = e^{-ist\theta_{jk}} e^{is\hat{Q}_k} e^{it\hat{Q}_j}, \tag{6.5}$$

$$e^{it\hat{P}_j} e^{is\hat{P}_k} = e^{-ist\eta_{jk}} e^{is\hat{P}_k} e^{it\hat{P}_j}, \quad j, k = 1, \cdots, n, s, t \in \mathbb{R}. \tag{6.6}$$

We call these relations the *deformed Weyl relations* with parameter Λ.

For a linear operator A on a Hilbert space, we denote its spectrum by $\sigma(A)$.

Proposition 6.2 *Let* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ *be a Weyl representation of* $\mathrm{QPS}_n(\Lambda)$. *Then it is a self-adjoint representation of* $\mathrm{QPS}_n(\Lambda)$. *Moreover, for each* $j = 1, \cdots, n$, $\hat{Q}_j$ *and* $\hat{P}_j$ *are purely absolutely continuous with*

$$\sigma(\hat{Q}_j) = \mathbb{R}, \quad \sigma(\hat{P}_j) = \mathbb{R}, \quad j = 1, \cdots, n. \tag{6.7}$$

Remark 6.3 The converse of Proposition 6.2 does not hold. Indeed, there exists a self-adjoint representation of $\mathrm{QPS}_n(\Lambda)$ which is not a Weyl one [4].

Proposition 6.4 *The set* $\{e^{it\hat{Q}_j}, e^{it\hat{P}_j} | t \in \mathbb{R}, j = 1, \cdots, n\}$ *is irreducible if and only if so is* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$.

7 Uniqueness Theorems on Weyl Representations of $\mathrm{QPS}_n(\Lambda)$

For each regular parameter Λ, every Weyl representation of $\mathrm{QPS}_n(\Lambda)$ on a *separable* Hilbert space is unitarily equivalent to a direct sum of a quasi-Schrödinger representation $\{\bar{q}_j, \bar{p}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$:

Theorem 7.1 *Assume (A). Let* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ *be a Weyl representation of* $\mathrm{QPS}_n(\Lambda)$ *on a separable Hilbert space* $\mathcal{H}$. *Then there exist closed subspaces* $\mathcal{H}_\ell$ *such that the following (i)-(iii) hold:*

(i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$ *(*N *is a positive integer or* ∞*).*

(ii) For each $j = 1, \cdots, n$, $\hat{Q}_j$ *and* $\hat{P}_j$ *are reduced by each* $\mathcal{H}_\ell, \ell = 1, \cdots, N$. *We denote by* $\hat{Q}_j^{(\ell)}$ *(resp.* $\hat{P}_j^{(\ell)}$*) the reduced part of* $\hat{Q}_j$ *(resp.* $\hat{P}_j$*) to* $\mathcal{H}_\ell$.

(iii) For each ℓ, *there exists a unitary operator* $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n)$ *such that*

$$U_\ell \hat{Q}_j^{(\ell)} U_\ell^{-1} = \bar{q}_j, \quad U_\ell \hat{P}_j^{(\ell)} U_\ell^{-1} = \bar{p}_j, \quad j = 1, \cdots, n, \tag{7.1}$$

where $\{\bar{q}_j, \bar{p}_j\}_{j=1}^n$ *is the quasi-Schrödinger representation of* $\mathrm{QPS}_n(\Lambda)$ *defined by (3.11).*

Theorem 7.1 tells us that, under the assumption there, every Weyl representation $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$ is unitarily equivalent to a direct sum of the quasi-Schrödinger representation $\{\bar{q}_j, \bar{p}_j\}_{j=1}^n$, because the operator

$$U := \oplus_{\ell=1}^N U_\ell : \mathcal{H} \to \oplus^N L^2(\mathbb{R}^n),$$

is unitary and

$$U\hat{Q}_j U^{-1} = \oplus^N \bar{q}_j, \quad U\hat{P}_j U^{-1} = \oplus^N \bar{p}_j.$$

Remark 7.2 There exist self-adjoint representations of $\mathrm{QPS}_n(\Lambda)$ which are not uinitarily equivalent to $\{\bar{q}_j, \bar{p}_j\}_{j=1}^n$ [4].

Theorem 7.1 and the irreducibility of the representation $\{\bar{q}_j, \bar{p}_j\}_{j=1}^n$ immediately lead us to the following fact:

Corollary 7.3 *Assume (A). Let* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ *be an irreducible Weyl representation of* $\mathrm{QPS}_n(\Lambda)$ *on a separable Hilbert space* $\mathcal{H}$. *Then there exists a unitary operator* $W : \mathcal{H} \to L^2(\mathbb{R}^n)$ *such that*

$$W\hat{Q}_jW^{-1} = \bar{q}_j, \quad W\hat{P}_jW^{-1} = \bar{p}_j, \quad j = 1, \cdots, n.$$

Applying this corollary to the case where $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ is a quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$, we obtain the following result:

Corollary 7.4 *Let* Λ *be regular. Let* G *and* G' *be two generating matrices of* Λ*:* G *is given by (3.7) and*

$$G' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},$$

where A', B', C' *and* D' *are* $n \times n$ *real matrices. Let* $\{\bar{q}'_j, \bar{p}'_j\}_{j=1}^n$ *be the quasi-Schrödinegr representation of* $\mathrm{QPS}_n(\Lambda)$ *with generating matrix* G'*:*

$$\hat{\mathbf{q}}' := A'\mathbf{q} + B'\mathbf{p}, \quad \hat{\mathbf{p}}' = C'\mathbf{q} + D'\mathbf{p}.$$

Then there exists a unitary operator $V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that*

$$V\bar{q}'_jV^{-1} = \bar{q}_j, \quad V\bar{p}'_jV^{-1} = \bar{p}_j, \quad j = 1, \cdots, n. \tag{7.2}$$

Corollary 7.4 shows that, for each regular parameter Λ, quasi-Schrödinger representations of $\mathrm{QPS}_n(\Lambda)$ are unique up to unitary equivalences.

Acknowledgement

This work is supported by the Grant-In-Aid No.21540206 for Scientific Research from Japan Society for the Promotion of Science (JSPS).

References

[1] A. Arai, Generalized weak Weyl relation and decay in quantum dynamics, *Rev. Math. Phys.* **17** (2005), 1-39.

[2] A. Arai, Spectrum of time operators, *Lett. Math. Phys.* **80** (2007), 211–221.

[3] A. Arai, On the uniqueness of weak Weyl representations of the canonical commutation relation, *Lett. Math. Phys.* **85** (2008), 15–25. Erratum: *Lett. Math. Phys.* **89** (2009), 287.

[4] A. Arai, Representations of a quantum phase space with general degrees of freedom, mp_arc 09-122.

[5] A. Arai, Strong time operators in algebraic quantum mechanics and quantum field theory, *RIMS Kôkyûroku Bessatsu* **B16** (2010), 1–13.

[6] A. Arai and Y. Matsuzawa, Construction of a Weyl representation from a weak Weyl representation of the canonical commutation relation, *Lett. Math. Phys.* **83** (2008), 201–211.

[7] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, Field theory on noncommutative spacetimes: Quasiplanar Wick products, *Phys. Rev. Lett. D* **71** (2005), 025022(1–12).

[8] S. Doplicher, K. Fredenhagen and J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields, *Commun. Math. Phys.* **172** (1995), 187–220.

[9] V. Gayral, J. M. Gracia-Bondia and F. Ruiz Ruiz, Position-dependent noncommutative products: classical construction and field theory, *Nucl. Phys.* **727**[PM](2005), 513–536.

[10] H. Grosse and M. Wohlgenannt, Induced gauge theory on a noncommutative space, *Eur. Phys. J.* C **52** (2007), 435–450.

[11] Y. Habara, A new approach to scalar field theory on noncommutative space, *Prog. Theor. Phys.* **107** (2002), 211–230.

[12] F. Hiroshima, S. Kuribayashi and Y. Matsuzawa, Strong time operators associated with generalized Hamiltonians, *Lett. Math. Phys.* **87** (2009), 115–123.

[13] L. Jonke and S. Meljanac, Representations of non-commutative quantum mechanics and symmetries, *Eur. Phys. J* C **29** (2003), 433–439.

[14] K. Li and J. Wang, The topological AC effect on non-commutative phase space, *Eur. Phys. J.* C **50** (2007), 1007–1011.

[15] Y.-G. Miao, H. J. W. Müller-Kirsten and D. K. Park, Chiral bosons in noncommutative spacetime, *J. High Energy Phys.* **08** (2003), 038.

[16] L. R. Riberio, E. Passos, C. Furtado and J. R. Nascimento, Landau analog levels for dipoles in non-commutative space and phase space, *Eur. Phys. J.* C **56** (2008), 597–606.

[17] J.-Z. Zhang, Consistent deformed bosonic algebra in noncommutative quantum mechanics, *Int. J. Mod. Phys.* A **23**(2008), 1393–1403.