数理解析研究所講究録
第 1714 巻 2010 年 105-126

SOME APPLICATIONS OF THE COLORED ALEXANDER INVARIANT

JUN MURAKAMI

ABSTRACT. Recent study of the volume conjecture of knots reveals the relation between $\mathcal{U}_q(sl_2)$ quantum invariants of knots, e.g. the colored Jones polynomial, and the hyperbolic volume of the complements. The colored Alexander invariant is a quantum invariant coming from the non-integral highest weight representations of $\mathcal{U}_q(sl_2)$ at $q = \exp(\pi\sqrt{-1}/N)$ for an integer $N \geq 2$. In this note, we introduce some derivatives of the colored Alexander invariant and discuss the relation between them and the hyperbolic volume.

INTRODUCTION

The discovery of the Jones polynomial gave rise to various knot invariants, which are now called the quantum invariants. As one of such invariants, the colored Alexander invariant of knots and links was introduced in [1] by using the quantum R-matrix related to the non-integral highest weight representation of the quantum enveloping algebra $\mathcal{U}_q(sl_2)$ at $q = \exp(\pi\sqrt{-1}/N)$ for an integer $N \geq 2$. On the other hand, Kashaev introduced an quantum invariant by using the q-analogue of dilogarithm function, and he observed that his invariant is related to the hyperbolic volume of the knot complement. Such relation is verified only for small class of knots and links, but it is expected also for various generalized cases, and is now called the volume conjecture. In [22], it is showed that Kashaev's invariant is obtained by certain specializations of the colored Jones invariant and the colored Alexander invariant. The relation between the colored Jones invariant with generic parameter q and the hyperbolic volume is first discussed by Gukov in [13]. The relation between the colord Alexander invariant and the hyperbolic volume is discussed in [25] and [4]. In this note, we discuss the relation between some derivation of the colored Alexander invariant and the hyperbolic volume.

The $6j$-symbol is introduced for studying gravity theory by using the representation theory of the Lie algebra sl_2. The quantized version of the $6j$-symbol is introduced in [19], in which the face model of the colored Jones invariants of knots and links are constructed with the quantum $6j$-symbol instead of the quantum R-matrix. The quantum $6j$-symbol is also used to construct the Turaev-Viro invariant of three manifods in [30], which is a version of the Witten-Reshetikhin-Turaev invariant [29]. On the other hand, Kashaev constructed knot invariants in [16] from quantized dilogarithm functions and observed in [17] that certain limit of his invariants coincide with the hyperbolic volume of the knot compliment, and it turned out in [22] that the Kashaev invariant is the colored Jones invariant of spin $\frac{N-1}{2}$ at $q = \xi$, where ξ is the primitive $2n$-th root of unity $\exp(\frac{\pi\sqrt{-1}}{2N})$. In other words, the Kashaev invariant comes from the n dimensional irreducible representation of $\mathcal{U}_\xi(sl_2)$.

For the case $q = \xi$, we have other invariants related to $\mathcal{U}_\xi(sl_2)$, such as the colored Alexander invariant [1], [25], [12], the logarithmic invariant [26] , and the Hennings invariant [15]. The colored Alexander invariant relates to the central deformation of the n dimensional irreducible representation of $\mathcal{U}_\xi(sl_2)$, which is a non-integral highest weight representation. Let $\widetilde{\mathcal{U}}_\xi(sl_2)$ be the small (or restricted) quantum group which is a quotient of $\mathcal{U}_\xi(sl_2)$. The logarithmic invariant is defined by using the radical part of a non-semisimple representation of $\widetilde{\mathcal{U}}_\xi(sl_2)$. The Hennings invariant is an invariant of 3-manifolds coming from the right integral given by the finite dimensional Hopf algebra structure of $\widetilde{\mathcal{U}}_\xi(sl_2)$. The logarithmic and Hennings invariants are both related to the logarithmic conformal field theory [9], and these invariants can be expressed in terms of the colored Alexander invariant as in [26].

The main purpose of this paper is to investigate several versions of the quantum $SL(2,\mathbb{C})$ invariants which may relate to the hyperbolic volume. The author found that the colored Alexander invariant for knots has good relation with the hyperbolic volume of the knot complements, and here two subjects are discussed. One is a generalization of the Hennings invariant, and another one is a generalization of the quantum $6j$-symbols. These are constructed from the colored Alexander invariant and are expected to have a strong relation to the hyperbolic volume. For the generalized $6j$-symbol, relation to the hyperbolic volume is confirmed as in Theorem 4.4. For the generalization of the Hennings invariant, the relation to the hyperbolic volume is not checked yet, but, if we consider the case for a knot in S^3, this invariant is nothing other than Kashaev's invariant nor the logarithmic invariant, and these invariants are specializations of the colored Alexander invariant. So, for S^3 case, relation to the hyperbolic volume is already observed.

To get the $6j$-symbol from the colored Alexander invariant, we first compute the Crebush-Gordan quantum coefficients (CGQC) of the tensor product of two non-integral highest modules, and then combine them to get the corresponding quantum $6j$-symbols. Since the spin is given as a continuous parameter in the above construction, it may be natural to associate $SL(2,\mathbb{C})$, while the usual one is regarded as the $SU(2,\mathbb{C})$ quantum $6j$-symbol.

The CGQC naturally corresponds to a trivalent vertex of a colored graph, and the colored Alexander invariant is easily generalized to the invariant of colored graphs which is essentially equal to the invariant given in [12] for odd n, and we construct the face model for such invariants by using the $SL(2,\mathbb{C})$ quantum $6j$-symbols along with the method in [19]. This model is a generalization of those for the Conway function and the Alexander polynomial constructed by O. Viro [32] using the quantum supergroup $gl(1|1)$, and we generalize them concretely by using $\mathcal{U}_\xi(sl_2)$.

The relation between the Kashaev invariant and the hyperbolic volume of the knot complement is not proved yet for general case, but a nice geometric explanation is given by Yokota in [33]. This suggest that the $SU(2,\mathbb{C})$ quantum $6j$-symbol should relate to the hyperbolic volume and some geometric data of a hyperbolic tetrahedron, and this leads to the volume furmulae in [28], [31], [27]. Such idea is also applied in [5], [6] for discussing the relation between the colored Jones invariant of hyperbolic links in $S^2 \times S^1$ and the hyperbolic volumes of their complements. It is observed in [25], [4] that the colored Alexander invariant relate to the hyperbolic volume of a cone manifold whose core is the given knot. Here, we show that the $SL(2,\mathbb{C})$ quantum $6j$-symbol relates to the volume of a truncated tetrahedron as the relation between the Kashaev invariant and the volume of the complement.

This paper is organized as follows. Various versions of the volume conjecture for quantum invariants are explained in the first section. In the second section, we generalize Henning invariant coming from the right integral of the small quantum group to invariants of a knot in a 3-manifold. One of these invariants can be thought as a generalization of Kashaev's invariant. In section 3, we extend the colored Alexander invariant to an invariant of colored graphs. We introduce the Crebsch-Gordan quantum coefficient (CGQC) actually to define the invariant at the trivalent vertex. By using representation theory, CGQC is defined uniquely up to a scalar multiple, and here we give a suitable normalization so that the invariant has a good symmetry around the vertex. Combining CGQC and certain R-matrix, we define an invariant of oriented colored graphs. In the last section, we actually give the $6j$-symbol coming from the above invariant by using CGQC. We also discuss the relation between the $6j$-symbol and the hyperbolic volume.

The contents of sections 3 and 4 are obtained by a joint work with F. Costantino and the detail is given in [8].

1. Volume conjectures of knots

In 1990's, R. Kashaev [16] introduced new knot invariants $K_N(L)$ for a knot L in S^3 by using the q-analogue of the dilogarithm function for every integer $N \geq 2$. Then he conjectured the following relation between his invariant and the hyperbolic volume in [17] which is checked for three hyperbolic knot 4_1, 5_2 and 6_1.

Kashaev's Conjecture

$$\lim_{N\to\infty} \frac{2\pi \log |K_N(L)|}{N} = \mathrm{Vol}(S^3 \setminus L),$$

where $\mathrm{Vol}(S^3 \setminus L)$ is the hyperbolic volume of the complement of L.

H. Murakami and the author showed in [22] that $K_N(L)$ is equal to the colored Jones invariant corresponding to the n dimensional irreducible representation of the quantum group $\mathcal{U}_q(sl_2)$ whose parameter q is specialized to the $2n$-th root of unity $\exp(\pi\sqrt{-1}/N)$, and is also equal to the colored Alexander invariant in [25] which is introduced in [1] whose parameter is specialized to the $2n$-th root of unity, too. Moreover, Kashaev's conjecture is generalized to the following volume conjecture.

Volume Conjecture

$$\lim_{N\to\infty} \frac{2\pi \log |K_N(L)|}{N} = v_3\, V_G(S^3 \setminus L),$$

where V_G means the Gromov's simplicial volume and v_3 is the volume of the regular ideal tetrahedron, i.e. $v_3 = 1.01\ldots$.

The avobe two conjectures are for the absolute value of $K_N(L)$, and $K_N(L)$ itself is a complex number in general. For a hyperbolic link L, the following conjecture is proposed in [23].

Complexified Volume Conjecture

$$\lim_{N\to\infty} \frac{2\pi \log |K_N(L)|}{N} = \mathrm{Vol}(S^3 \setminus L) + \sqrt{-1}\ \mathrm{CS}(S^3 \setminus L),$$

where $\mathrm{CS}(S^3 \setminus L)$ is the Chern-Simons invariant of the complement of L.

On the other hand, S. Gukov considered the case that the parameter q is deformed around the $2n$-th root of unity $\exp(\pi\sqrt{-1}/N)$ in [13]. Let α be a complex number close to 1 and let $q = \exp(\alpha\,\pi\sqrt{-1}/N)$. Then the following is conjectured.

Parametrized Volume Conjecture

$$2\pi\alpha \lim_{n\to\infty} \frac{V_L^N(e^{2\pi\alpha\sqrt{-1}/N})}{N} = \mathrm{Vol}(K_\alpha) + \sqrt{-1}\ \mathrm{CS}(K_\alpha) + \text{correction terms},$$

where $V_L^N(q)$ is the colored Jones invariant corresponding to the N-dimensional irreducible representation of $\mathcal{U}_q(sl_2)$, and K_α is a deformation of the hyperbolic structure of $S^3 \setminus K$ corresponding to the parameter α. The correction term is explicitly given by H. Murakami and Y. Yokota in [24].

In the parametrized volume conjecture, the parameter q is deformed and the representation is fixed. But there is another way of deformation: the parameter q is fixed to the $2N$-th root of unity and the representation is deformed. If q is generic, then the irreducible representation of $\mathcal{U}_q(sl_2)$ is rigid, i.e. there is no way to deform it. However, if q is $2N$-th root of unity and the dimension of the irreducible representation is N, then the representation has one-parameter deformation and the knot invariant corresponding to this representation can be also deformed. The highest weight of the usual N-dimensional irreducible representation is $(N-1)/2$, and, the highest weight of the deformed representation can be any complex number λ. Such a deformed invariant is first constructed in [1] and its relation to the hyperbolic volume is discussed in [25]. This invariant is called the colored *Alexander invariant*, which is a basis of this note, and it is denoted by $\Phi_\lambda^N(L)$. For a real β, the limit of $\Phi_{N\beta}^N(L)$ is conjectured in [25] as follows.

Volume Conjecture for the colored Alexander invariant

$$2\pi \lim_{N\to\infty} \Phi_{N\beta}^N(L) = \mathrm{Vol}(L_\beta) + \sqrt{-1}\ \mathrm{CS}(L_\beta),$$

where L_β is the cone manifold with cone angle $2\pi\beta$. The R-matrix for the colored Jones invariant and the colored Alexander invariant is similar, and the resulting knot invariant is also similar. This explains the similarity of the above conjecture and Gukov's conjecture.

Another way of generalization is considered for closed 3-manifold. For a closed 3-manifold M, let $\tau_N(M)$ be the Witten-Reshetikhin-Turaev (WRT) invariant of M, which is constructed by a linear combination of the colored Jones invariants. Let us consider the limit $\lim\limits_{N\to\infty} \dfrac{2\pi \log \tau_N(M)}{N}$. Since the absolute value $|\tau_N(M)|$ is not of exponential growth with respect to N and

$$\lim_{N\to\infty} \frac{2\pi \log |\tau_N(M)|}{N} = 0.$$

However, if we apply Kashaev's way of computation to get the limit naively, in other words, if we apply the saddle point method without considering the condition which certify that the value at the saddle point gives the limit, some relation to the volume is observed in [21] for 3-manifolds obtained by surgeries of the figure eight knot as follows.

Volume Conjecture for closed 3-manifolds

$$\underset{N\to\infty}{\text{o-lim}} \frac{2\pi \log |\tau_N(M)|}{N} = \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M).$$

Here $\underset{N\to\infty}{\text{o-lim}} \dfrac{2\pi \log |\tau_N(M)|}{N}$ means the value at the saddle point and is called the *optimistic limit*. The quantum invariant $\tau_N(M)$ is expressed as a sum of several parameters and, to tell the true, the range of the parameters is not wide enough to apply the saddle point method to get the limit.

Volume formula for hyperbolic tetrahedra

There is another quantum invariant, the Turaev-Viro (TV) invariant introduced in [30], of closed 3-manifolds constructed by using the quantum $6j$ symbol introduced in [19]. This invariant is equal to $|\tau_N(M)|^2$ and so the above observation suggests that the optimistic limit of the quantum $6j$ symbol somehow relate to the volume of a hyperbolic tetrahedron. In [28] and [27], formulas for the volume of hyperbolic tetrahedron are given by applying this ideal.

Dihedral angles

In [28], we give a formula by using the dihedral angles of the tetrahedron. In contrast with the Euclidean case, congruency of hyperbolic tetrahedra is determined by their dihedral angles, and the volume is also uniquely determined by these angles. Actual formula for a general tetrahedron was first given in [3], but the formula in [28] has good conformity to the natural symmetry of a tetrahedron.

Edge lengths

Of course, congruency of hyperbolic tetrahedra is determined by their edge lengths as in the case of Euclidean tetrahedra, and the volume is uniquely determined by these lengths. The relation between a volume formula by dihedral angles and that by edge lengths are given in [20]. By using this method, the volume formula by edge lengths is given in [27].

Extended tetrahedron

The above formulas are both analytic functions with respect to the parameters except at some special values, and these formulas also work some other types of tetrahedra including spherical tetrahedra and truncated tetrahedra.

In the last cases for closed 3-manifolds and hyperbolic tetrahedra, the volume comes from the optimistic limit. The aim of my current research is to construct some quantum things whose limits *actually* correspond to the hyperbolic volume of tetrahedra and closed 3-manifolds. I haven't succeeded yet to get such things, but here I would like to propose two thisngs: one is Kashaev's invariant for knots in 3-manifold, which can be computed from the colored Alexander invariant, and another one is the quantum 6j-symbols defined by the deformed representations of $\mathcal{U}_q(sl_2)$ corresponding to the colored Alexander invariant also.

2. Kashaev's invariant for knots in 3-manifolds

In [17], Kashaev's invariant is defined by using R-matrix and is defined for knots and links in S^3. But, in [16], this invariant is defined for a knot in closed triangulated 3-manifold where the knot is given as aHamiltonian cycle of the triangulation. Here we give another way to extend Kahsaev's invariant for knots in S^3 to knots in a closed 3-manifold by using the left integral of the small quantum group $\widehat{\mathcal{U}}_\zeta(sl_2)$.

2.1. **Quantized enveloping algebra $\mathcal{U}_q(sl_2)$.** Let q be a complex parameter other than 0 and ± 1. We use the following notations.

$$\{a\} = \{q^a - q^{-a}\} \quad (a \in \mathbb{Z}), \quad \{k\}! = \prod_{j=1}^{k} \{j\} \quad (k \in \mathbb{N}), \quad [a] = \frac{\{a\}}{\{1\}},$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{j=0}^{a-b-1} \frac{\{a-j\}}{\{a-b-j\}}.$$

Definition 2.1. For a parameter $q \neq \pm 1$, let $\mathcal{U}_q(sl_2)$ be the quantized enveloping algebra of sl_2, which is the Hopf algebra generated by E, F, K and K^{-1} with relations

$$[E,F] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \; K\,E = q\,E\,K, \; K\,F = q^{-1}\,F\,K,$$
$$K\,K^{-1} = K^{-1}\,K = 1,$$

and the Hopf algebra structure given by

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$
$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F,$$
$$S(E) = -q\,E, \quad S(F) = -q^{-1}\,F, \quad S(K) = K^{-1},$$
$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1.$$

2.2. **Colored Jones invariant.** Let λ be a spin, which is a non-negative half integer. The irreducible representation of spin λ of $\mathcal{U}_q(sl_2)$ is the $2\lambda + 1$ dimensional representation V^λ given by the following construction. Let e_0^λ, e_1^λ, $\cdots$, $e_{2\lambda}^\lambda$ be the basis of V^λ and the actions of K, E, F are given by

$$E(e_j^\lambda) = [j]\,e_{j-1}^\lambda, \quad F(e_j^\lambda) = [2\lambda - j]\,e_{j+1}^\lambda, \quad (e_{-1}^\lambda = e_n^\lambda = 0)$$
$$K(e_j^\lambda) = q^{\lambda - j}\,e_j^\lambda.$$

Then, it is well known that we can construct a framed link invariant from the representation of the universal R-matrix in $\mathrm{End}(V^\lambda \otimes V^\lambda)$. The resulting invariant is called the *colored Jones invariant corresponding to the spin* λ and is denoted by $V_\lambda(L)$. Usually, $V_\lambda(L)$ is normalized by the framing so that it does not depend on the framing. In this report, we don't apply such normalization, and so $V_\lambda(L)$ depends on the framing.

Let L be a p-component framed link with components K_1, K_2, $\cdots$, K_p. Then, assigning spins λ_1, λ_2, $\cdots$, λ_p to these components respectively, we get a link invariant $V_{\lambda_1,\cdots,\lambda_p}(L)$ from the universal R-matrix, which is also called the colored Jones invariant.

2.3. **Witten-Reshetikhin-Turaev invariant.** From the colored Jones invariant, we can construct a 3-manifold invariant, which is called the *Witten-Reshetikhin-Truraev (WRT) invariant.* Let N be an odd integer greater than 1, and let q be the primitive N-th root of unity $\zeta = \exp(2\pi\sqrt{-1}/N)$. Let M be a 3-manifold obtained by the surgery along a p-component framed link $L = K_1 \cup K_2 \cup \cdots \cup K_p$, and $V^N_{\lambda_1,\lambda_2,\cdots,\lambda_p}(L)$ be the colored Jones invariant with colors $\lambda_1, \lambda_2, \cdots, \lambda_p \in \{0, 1/2, \cdots, (N-2)/2\}$ assigned to the components

$K_1, K_2, \cdots, K_p$ respectively. Then the $SO(3)$ version of the WRT invariant $\tau_N(M)$ be defined by

$$\tilde{\tau}_N(L) = \sum_{\lambda_1,\lambda_2,\cdots,\lambda_p=0}^{(N-3)/2} \left(\prod_{i=1}^{p}[2\,\lambda_i+1]\right) V^N_{\lambda_1,\lambda_2,\cdots,\lambda_p}(L),$$

$$\tau_N(L) = \frac{\tilde{\tau}_N(L)}{\tilde{\tau}_N(\bigcirc_+)^{\sigma_+(L)}\,\tilde{\tau}_N(\bigcirc_-)^{\sigma_-(L)}}.$$

where $\sigma_+(L)$ and $\sigma_-(L)$ are the numbers of the positive eigenvalues and negative eigenvalues of the linking matrix of L respectively. Then $\tau_N(L)$ does not change by the Kirby moves of types I and II, and this is a invariant of the 3-manifold M, and we denote it by $\tau_N(M)$.

Now let us consider a knot K in the 3-manifold M. Then K is given by a knot K_0 in S^3 having no intersection with the link L, and let

$$\tilde{\tau}^{\lambda}_N(L, K_0) = \sum_{\lambda_1,\lambda_2,\cdots,\lambda_p=0}^{(N-3)/2} \left(\prod_{i=1}^{p}[2\,\lambda_i+1]\right) V^N_{\lambda,\lambda_1,\lambda_2,\cdots,\lambda_p}(K_0 \cup L),$$

and

$$\tau^{\lambda}_N(M, K) = \frac{\tilde{\tau}^{\lambda}_N(L, K_0)}{\tilde{\tau}_N(\bigcirc_+)^{\sigma_+(L)}\,\tilde{\tau}_N(\bigcirc_-)^{\sigma_-(L)}},$$

Then $\tau_N(M, K)$ is an invariant of K in M and it is called the colored Jones invariant of the knot K in M.

2.4. **Left integral of $\widehat{\mathcal{U}_\zeta}(sl_2)$.** Let N be a odd integer greater than 1 and ζ be a primitive N-th root of 1. The small quantum group $\widehat{\mathcal{U}_\zeta}(sl_2)$ is a quotient of $\mathcal{U}_\zeta(sl_2)$ by the two-sided ideal generated by E^N, F^N and $K^N - 1$. Then $\widehat{\mathcal{U}_\zeta}(sl_2)$ is a finite dimensional Hopf algebra with dimension N^3, and there is a left integral λ which is a linear functional from $\widehat{\mathcal{U}_\zeta}(sl_2)$ to $\mathbb{C}$ satisfying

$$(2.1) \qquad (id \otimes \lambda)\,\Delta(x) = \lambda(x)\,id \quad : \quad \widehat{\mathcal{U}_\zeta}(sl_2) \to \widehat{\mathcal{U}_\zeta}(sl_2).$$

This functional λ is actually given by

$$\lambda(F^a\,K^b\,E^c) = \delta_{a,n-1}\,\delta_{b,1}\,\delta_{c,n-1}.$$

2.5. **Hennings invariant.** Let M be a 3-manifold determined by a framed link L, and T_L be a string link of T. The relation (2.1) of λ is similar to the second Kirby move KII and the Hennings invariant of 3-manifold is defined by using λ as follows.

$$\psi_\zeta(M) = \frac{\lambda \otimes \cdots \otimes \lambda(\Gamma_\zeta(T_L))}{\lambda(\Gamma_\zeta(T_{\bigcirc_+}))^{\sigma_+(L)}\,\lambda(\Gamma_\zeta(T_{\bigcirc_-}))^{\sigma_-(L)}}.$$

The relation between the Henings invariant and the WRT invariant is given in [2] as follows.

Theorem 2.2. *(Q.Chen-S.Kuppum-P.Srinivasan) If ζ is a complex root of unity of odd order greater than 1, then*

$$\psi_\zeta(M) = h(M)\, \tau_N(M),$$

where $h(M)$ is the order of $H_1(M)$ if it is finite and 0 otherwise.

2.6. **Center of $\widehat{\mathcal{U}}_\zeta(sl_2)$.** Let $Z(\widehat{\mathcal{U}}_\zeta(sl_2))$ be the center of $\widehat{\mathcal{U}}_\zeta(sl_2)$. It contains the Casimir element

$$C = (\zeta - \zeta^{-1})\, E\, F + \frac{\zeta\, K + \zeta^{-1}\, K^{-1}}{\zeta - \zeta^{-1}}.$$

Let $b_j = (\zeta^{2j+1} + \zeta^{-2j-1})/(\zeta - \zeta^{-1})$ and

$$\phi(x) = \prod_{i=0}^{N-1} (x - b_i).$$

Set

$$\phi_j(x) = \prod_{0 \le i \le N-1, b_i \ne b_j} (x - b_i), \quad 0 \le j \le (N-1)/2.$$

Since $b_j = b_{N-1-j}$, $\deg(\phi_j) = N-2$ for $1 \le j < (N-1)/2$ and $\deg(\phi_{(N-1)/2}) = N-1$. Let

$$P_j = \frac{1}{\phi_j(b_j)}\phi_j(C) - \frac{\phi_j'(b_j)}{\phi_j(b_j)^2}(C - b_j)\phi_j(C), \quad 0 \le j \le (N-1)/2,$$

$$N_j = \frac{1}{\phi_j(b_j)}(C - b_j)\phi_j(C), \quad 0 \le j < (N-1)/2,$$

$$\pi_j = \frac{1}{N}\sum_{i=1}^{N} \zeta^{2ij}\, K^i, \quad 0 \le N-1,$$

$$T_j = \sum_{i=j+1}^{N-1-j} \pi_i, \quad 0 \le j < (N-1)/2,$$

and

$$N_j' = T_j\, N_j, \qquad 0 \le j \le (N-1)/2.$$

Then

$$\{P_i, N_j, N_j', \quad 0 \le (N-1)/2, \quad 0 \le j < (N-1)/2\}$$

is the basis of $Z(\widehat{\mathcal{U}}_\zeta(sl_2))$ and $\dim Z(\widehat{\mathcal{U}}_\zeta(sl_2)) = (3N-1)/2$. These basis elements satisfy the following commutation relations.

$$P_i\, P_j = \delta_{ij}\, P_i, \quad P_i\, N_j = \delta_{ij}\, N_j, \quad P_i\, N_j' = \delta_{ij}\, N_j',$$

$$N_i N_j = N_i N_j' = N_i' N_j' = 0.$$

2.7. **Logarithmic invariant of a knot in 3-manifolds.** For a knot K in the 3-manifoled M, let

$$\psi_\varsigma(M, K) = \frac{id \otimes \lambda \otimes \cdots \otimes \lambda(\Gamma_\varsigma(T_{K\cup L}))}{\lambda(\Gamma_\varsigma(T_{\bigcirc_+}))^{\sigma_+(L)}\, \lambda(\Gamma_\varsigma(T_{\bigcirc_-}))^{\sigma_-(L)}}.$$

Then $\psi_\varsigma(M, K)$ is an element of $Z(\widehat{\mathcal{U}}_\varsigma(sl_2))$ and it is expressed as a linear combination of $\{P_i, N_j, N_j'\}$ as follows.

$$\psi_\varsigma(M, K) = \sum_{i=0}^{(N-1)/2} a_i(M, K)\, P_i + \sum_{j=0}^{(N-3)/2} \left(b_j(M, K)\, N_j + b_j'(M, K)\, N_j'\right).$$

The coefficients $a_i(M, K)$, $b_j(M, K)$ and $b_j'(M, K)$ are invariants of K, and $a_i(M, K)$ is a constant multiple of $h(M)\, \tau_N^i(M, K)$ if $i < (N-1)/2$. The invariants $b_j(M, K)$ and $b'_J(M, K)$ are generalizations of the logarithmic invariant of a knot in S^3 given in [26].

Let M_K be the 3-manifold obtained from M by the surgery along K, then the Hennings invariant is given by

$$\psi_\varsigma(M_K) = \lambda(\phi_\varsigma(M, K)).$$

However, it is known that

$$\lambda(P_i) = \lambda(N_j) = 0,$$

and we get

$$\psi_\varsigma(M, K) = \sum_j c_j\, b_j'(M, K),$$

where c_j is a constant which does not depend on K nor M. It means that b_j' is a constant multiple of $h(M_L)\, \tau_N^j(M, K)$.

Hence, $a_i(M, K), b_i'(M, K)$ $(0 \le i < (N-1)/2)$ are related to $\tau_N^i(M, K)$. But, $b_i(M, K)$ and $a_{(N-1)/2}(M, K)$ are not expressed by $\tau_N^i(M, K)$. If $M = S^3$, $b_i(S^3, K)$ is expressed in [26] by using the colored Alexander invariant, and $a_{(N-1)/2}(S^3, K)$ is Kashaev's invariant. So it may be natural to say that $a_{(N-1)/2}(M, K)$ is a generalization of Kashaev's invariant, and $b_i(M, K)$ is a generalization of the logarithmic invariant.

Question 2.3. *To define an invariant, Kashaev first uses triangulated 3-manifold in* [16] *and then reformulated to use the R-matrix in* [17]. *So it is a problem whether the invariant $a_{(N-1)/2}(M, K)$ coincides with Kashaev's invariant or not.*

3. The colored Alexander invariant of graphs

In this section, we generalize the colored Alexander invariant to an invariant of oriented colored graphs. Such generalization is given in [12], and here we give another construction starting from the Crebsch-Gordan quantum coefficients (CGQC).

3.1. **Highest weight representations of $\mathcal{U}_q(sl_2)$.** Let $N \in \mathbb{N}$ and let ξ be the primitive $2N$-th root of unity $\exp(\frac{\pi\sqrt{-1}}{N})$. For a complex number a, ξ^a means $\exp(\frac{\pi\sqrt{-1}a}{N})$. From now on, we assume that $q = \xi$, and adding to the notations in §2.1, let

$$\{a\} = \{\xi^a - \xi^{-a}\}$$

for any complex number a, and, if $a - b \in \mathbb{N} \cup \{0\}$, let

$$\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{j=0}^{a-b-1} \frac{\{a-j\}}{\{a-b-j\}}.$$

Lemma 3.1. *For each $a \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ there is a simple representation V^a of $U_\xi(sl_2)$ of dimension n whose basis is $\{e_0^a,\ e_1^a,\ \cdots,\ e_{N-1}^a\}$ and on which the actions of E, F and K are given by*

$$E(e_j^a) = [j]\, e_{j-1}^a,\ F(e_j^a) = [2a-j]\, e_{j+1}^a,\ K(e_j^a) = \xi^{a-j}\, e_j^a\ (e_{-1}^a = e_N^a = 0).$$

Two such representations V^a and V^b are isomorphic iff $a - b \in 2n\mathbb{Z}$. The representation $(V^a)^*$ is isomorphic to V^{N-1-a}, a duality pairing realizing this isomorphism being:

$$\cap_{a,b}(e_i^a,\, e_j^b) = \delta_{b,N-1-a}\, \delta_{i,N-1-j}\, \xi^{-(a-i)(N-1)}. \tag{3.1}$$

Similarly, an invariant vector in $V^a \otimes V^b$ is given by:

$$\cup_{a,b} = \delta_{b,N-1-a} \sum_{i=0}^{N-1} \xi^{(b-N+1+i)(N-1)}\, e_i^a \otimes e_{N-1-i}^b. \tag{3.2}$$

The basis $\{e_0^a,\, e_1^a,\, \cdots,\, e_{N-1}^a\}$ of V^a is called the *weight basis* of V^a.

3.2. **Crebsch-Gordan quantum coefficients.** Let us consider the tensor product $\pi_\alpha \otimes \pi_\beta$. By using standart argument about the weight space decomposition, we get the following decomposition of the tensor product.

Proposition 3.2. *Let V_α, V_β be highest weight representations of non-half-integer parameters α, β. If $\alpha + \beta$ is not a half-integer, then*

$$V^a \otimes V^b = \bigoplus_{a+b-c=0,1,\cdots,N-1} V^c. \tag{3.1}$$

The weight basis e_t^c of V_c is a linear combination of the tensors $e_u^a \otimes e_v^b$ of the weight basis of V^a and V^b. Referring to [7], we get the following.

Theorem 3.3 (Clebsch-Gordan decomposition). *If $a+b-c \in \{0, 1, \cdots, N-1\}$, any $\mathcal{U}_\xi(sl_2)$ module map $\iota_\gamma^{\alpha,\beta} : V^c \longrightarrow V^a \otimes V^b$ is a scalar multiple of the inclusion map $Y_c^{a,b} : V^c \to V^a \otimes V^b$ given by*

$$Y_c^{a,b}(e_t^c) = \sum_{u+v-t=a+b-c} C_{u,v,t}^{a,b,c}\, e_u^a \otimes e_v^b,$$

where

(3.2)

$$C_{u,v,t}^{a,b,c} = \sqrt{-1}^{\,c-a-b}(-1)^{(v-t)}\xi^{\frac{v(2b-v+1)-u(2a-u+1)}{2}} \begin{bmatrix} 2c \\ 2c-t \end{bmatrix}^{-1} \begin{bmatrix} 2c \\ a+b+c-N+1 \end{bmatrix}$$
$$\sum_{z+w=t} (-1)^z \xi^{\frac{(2z-t)(2c-t+1)}{2}} \begin{bmatrix} a+b-c \\ u-z \end{bmatrix} \begin{bmatrix} 2a-u+z \\ 2a-u \end{bmatrix} \begin{bmatrix} 2b-v+w \\ 2b-v \end{bmatrix}.$$

The coefficient $C_{u,v,t}^{a,b,c}$ defined above is called the *Crebsch-Gordan quantum coefficient* (CGQC). In order to get invariants of graphs, the operators $L_{a,b}^c$ and $R_{a,b}^c$ in Figure 1 must be equal, and we define $Y_{a,b}^c$ by $Y_{a,b}^c = L_{a,b}^c$. The equality of $L_{a,b}^c$ and $R_{a,b}^c$ comes from the following lemma.

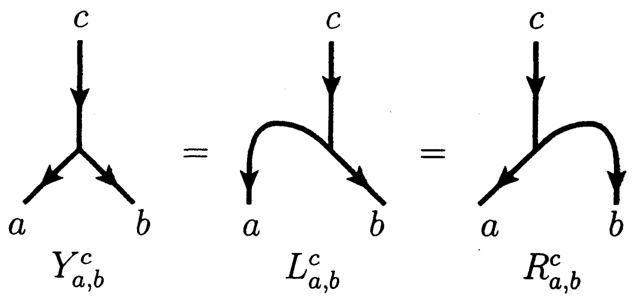


FIGURE 1. The first equality is the definition of $Y_{a,b}^c$, the second is Lemma 3.4.

Lemma 3.4. *It holds that*

$$C_{N-1-u,t,v}^{N-1-a,c,b}\, \xi^{-a(N-1)}\, \xi^{(N-1)u} = C_{t,N-1-v,u}^{c,N-1-b,a}\, \xi^{-(N-1-b)(N-1)}\, \xi^{(N-1)(N-1-v)}.$$

By using Lemma 3.4 three times, we get the following.

Proposition 3.5. *The projection* $Y^c_{a,b} : V^a \otimes V^b \to V^c$ *is given by*

$$Y^c_{a,b}(e^a_u \otimes e^b_v) = C^{N-1-b,N-1-a,N-1-c}_{N-1-v,N-1-u,N-1-t}(e^c_t). \tag{3.3}$$

3.3. **The R-matrix.** The R-matrix corresponding to the colored Alexander invariant is given in [1], and is also used in [25]. The construction of the representation of $\mathcal{U}_{\xi_n}(sl_2)$ is a little bit different from that in [25], and we define ${}^{b}_{a}R : V^a \otimes V^b \to V^b \otimes V^a$ as follows:

(3.4) ${}^{b}_{a}R(e^a_u \otimes e^b_v) =$

$$\sum_{m \geq 0} \{m\}! \xi_n^{2(a-u)(b-v)-m(a-b-u+v)-\frac{m(m+1)}{2}} \begin{bmatrix} u \\ u-m \end{bmatrix} \begin{bmatrix} 2b-v \\ 2b-v-m \end{bmatrix} e^b_{v+m} \otimes e^a_{u-m}.$$

We denote ${}^{a}_{b}R^{h,k}_{u,v}$ the coefficient of $R(e^a_u \otimes e^b_v)$ with respect to $e^b_h \otimes e^a_k$.

Proposition 3.6. *The morphism* ${}^{b}_{a}R$ *given above is the R-matrix of the non-integral representations, in other words,* ${}^{b}_{a}R$ *satisfies*

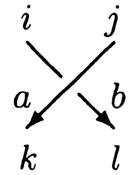
$$ {}^{b}_{a}R\,\Delta(x) = \Delta(x)\,{}^{b}_{a}R \tag{3.5}$$

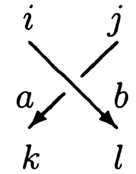
as mappings from $V^a \otimes V^b$ *to* $V^b \otimes V^a$ *for any* $x \in \mathcal{U}_{\xi_n}(sl_2)$*, and*

$$({}^{c}_{b}R \otimes \mathrm{id})\,(\mathrm{id} \otimes {}^{c}_{a}R)\,({}^{b}_{a}R \otimes \mathrm{id}) = (\mathrm{id} \otimes {}^{b}_{a}R)\,({}^{c}_{a}R \otimes \mathrm{id})\,(\mathrm{id} \otimes {}^{c}_{b}R) \tag{3.6}$$

as mappings from $V^a \otimes V^b \otimes V^c$ *to* $V^c \otimes V^b \otimes V^a$.

The R-matrix given by (3.4) is represented graphically as follows.

${}^{b}_{a}R^{ij}_{kl}$: 

, $({}^{b}_{a}R^{-1})^{ij}_{kl}$: 

.

3.4. **Invariants of trivalent planar graphs.** Let now Γ be a framed oriented connected trivalent graph in S^3 and let us fix once and for all a natural number $N \geq 2$ as well as a root $\xi = \exp(\frac{\pi\sqrt{-1}}{N})$.

Definition 3.7 (Coloring). A coloring on Γ is a map col : $\{\text{edges}\} \to \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ such that for each three uple of edges e_1, e_2, e_3 sharing a vertex v (possibly two edges coinciding) it holds:

$$f_v(e_1) + f_v(e_2) + f_v(e_3) \in \{0, 1, \ldots N-1\},$$

where $f_v(e_i)$ is $\mathrm{col}(e_i)$ if v is the end of e_i and $N - 1 - \mathrm{col}(e_i)$ otherwise.

Given an trivalent graph Γ embedded in S^3 equipped with an orientation of its edges, a framing and a coloring (such a datum will be from now on called *colored oriented graph*), we can associate to it a complex number which we shall denote $\langle\Gamma,\ \mathrm{col}\rangle_N$ by the following construction.

(1) Choose an edge e_0 of Γ and cut Γ open along e_0.
(2) Move by an isotopy Γ so to put it in a $(1,1)$-tangle diagram and so that the two open strands initially contained in e_0 are directed towards the bottom.
(3) Assigning $\cap$ operators to the maximal points, $\cup$ operators to the minimal points, R-matrices to the crossing points and the Crebsh-Gordan operators $Y_a^{b,c}$ and $Y_{a,b}^{c}$ to the trivalent vertices as in [19], we associate to the diagram D of Γ obtained in (3) an operator $\mathrm{op}(D) : V^{\mathrm{col}(e_0)} \to V^{\mathrm{col}(e_0)}$ and hence, by Schur's lemma, a scalar $\lambda(D) \in \mathbb{C}$.
(4) Define the scalar associated to D as $i(D) = \lambda(D) \begin{bmatrix} 2\,\mathrm{col}(e_0)+N \\ 2\,\mathrm{col}(e_0)+1 \end{bmatrix}^{-1}$.

Theorem 3.8. *The scalar $i(D)$ is independent on all the choices of the above construction and is therefore an invariant $<\Gamma,\mathrm{col}>_N\in\mathbb{C}$ of the colored oriented graph embedded in S^3.*

The invariance under the Reidemeister moves is proved as usual quantum invariants. The extra thing is to show the invariance under the choice of edge e_0 to cut the graph open. The factor $\begin{bmatrix} 2\,\mathrm{col}(e_0)+N \\ 2\,\mathrm{col}(e_0)+1 \end{bmatrix}^{-1}$ is added for this purpose and the following lemma assures the invariance.

Lemma 3.9. *Let $\theta(a,b,c)$ be a θ-graph embedded in the standard way in the plane, colored by $a,b,c \in \mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$ and such that the edges colored by a,b have the same source, while that colored by c has a different source. Then, letting $\theta_a(a,b,c)$ be the diagram obtained by cutting open θ along the a-colored edge it holds that*

$$\lambda(\theta_a(a,b,c)) = \begin{bmatrix} 2a+N \\ 2a+1 \end{bmatrix}, \tag{3.7}$$

and so we get $i(\theta_a(a,b,c)) = i(\theta_b(a,b,c)) = 1$.

This lemma comes from the following relation.

$$\begin{bmatrix} 2a+N \\ 2a+1 \end{bmatrix} = \sum_{t=0}^{N-1+c-b-a} C^{b,N-1-c,N-1-a}_{a+b-c+t,N-1-t,N-1}\, C^{c,N-1-b,a}_{t,N-1+c-a-b-t,0}.$$

To prove this relation, we need the following lemma.

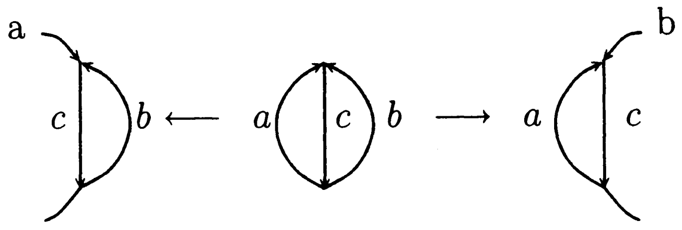


FIGURE 2. On the left $\theta_a(a,b,c)$; on the right $\theta_b(a,b,c)$.

Lemma 3.10. *For any parameter a, b and a non-negative integer c, we have*

$$\sum_{s=0}^{c} q^{\pm(a+b-c+2)s} \begin{bmatrix} a-s \\ a-c \end{bmatrix} \begin{bmatrix} b+s \\ b \end{bmatrix} = q^{\pm(b+1)c} \begin{bmatrix} a+b+1 \\ a+b-c+1 \end{bmatrix}. \tag{3.8}$$

For generic q, this relation is true for the case that a and b are non-negative integers by (51) in [18]. The both sides of (3.8) are Laurent polynomials with respect to the variable q^a, q^b, and they are equal for any positive integers a and b. Therefore, these two polynomials are equal and the both sides of (3.8) coinside for any a and b for generic q. Then we can specialize q to the $2N$-th root of unity and we get (3.8).

The formula (3.7) implies that

$$Y_c^{a,b} Y_{a,b}^c Y_c^{a,b} Y_{a,b}^c = \begin{bmatrix} 2c+N \\ 2c+1 \end{bmatrix} Y_c^{a,b} Y_{a,b}^c.$$

Hence, $\begin{bmatrix} 2c+N \\ 2c+1 \end{bmatrix}^{-1} Y_c^{a,b} Y_{a,b}^c$ is the identity on the subspace of $V^a \otimes V^b$ isomorphic to V^c. Therefore, the decomposition of $V^a \otimes V^b$ is expressed by $Y_c^{a,b}$ and $Y_{a,b}^c$ as follows.

$$\mathrm{id} = \sum_{c: a+b-c=0,1,\cdots,n-1} \begin{bmatrix} 2c+N \\ 2c+1 \end{bmatrix}^{-1} Y_c^{a,b} Y_{a,b}^c. \tag{3.9}$$

Remark 3.11. Colored graphs include colored links, and the invariant defined above is the colored Alexander invariant given in [1] and discussed in [12], [25]. Lemma 3.9 gives a new proof for the independence of the string to cut to make a (1, 1)-tangle. It is first proved in [1] by computation, and then refined in [12] for more general cases theoretically. Comparing with the proof in [12], we see that $\begin{bmatrix} 2a+N \\ 2a+1 \end{bmatrix}^{-1}$ corresponds to $d(a)$ in Definition 2.3 of [12] expressing the "virtual degree" of the representation V^a.

4. $SL(2,\mathbb{C})$ QUANTUM $6j$-SYMBOLS

4.1. **Construction.** The quantum $6j$-symbol is defined by the relation in Figure 3. The left diagram means that the composition of two inclusions $V_j \to V_{j_{12}} \otimes V_{j_3}$ and $V_{j_{12}} \to V_{j_1} \otimes V_{j_2}$ and the right diagram means that the composition of two inclusions $V_j \to V_{j_1} \otimes V_{j_{23}}$ and $V_{j_{23}} \to V_{j_2} \otimes V_{j_3}$. Let

$$\iota_l : V_j \to V_{j_{12}} \otimes V_{j_3} \to (V_{j_1} \otimes V_{j_2}) \otimes V_{j_3}$$
$$(\text{resp.} \quad \iota_r : V_j \to V_{j_1} \otimes V_{j_{23}} \to V_{j_1} \otimes (V_{j_2} \otimes V_{j_3}))$$

be the inclusion corresponding to the left (resp. right) diagram. Then Figure 3 means that

$$\iota_l(v) = \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_\xi \iota_r(v)$$

for $v \in V_j$. The quantum $6j$-symbol for non-integral highest weight representations are given as follows.

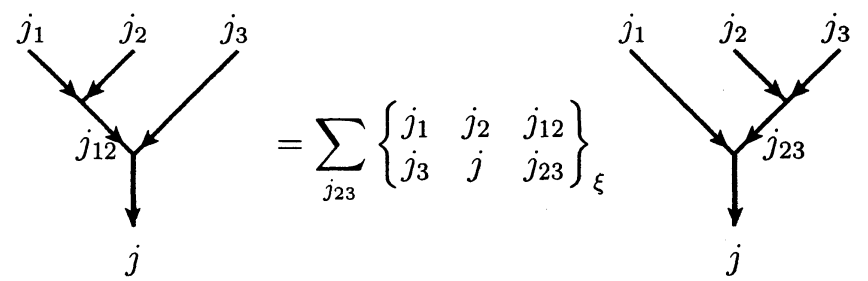


FIGURE 3. The quantum $6j$-symbol

Theorem 4.1. *For $a, b, \cdots, f$ satisfying $a+b-e, a+f-c, b+d-f, d+e-c \in \mathbb{Z}$,*

$$(4.1) \quad \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_\xi =$$
$$(-1)^{N-1+B_{afc}} \begin{bmatrix} 2f+N \\ 2f+1 \end{bmatrix}^{-1} \frac{\{B_{dec}\}!\,\{B_{abe}\}!}{\{B_{bdf}\}!\,\{B_{afc}\}!} \begin{bmatrix} 2e \\ A_{abe}+1-N \end{bmatrix} \begin{bmatrix} 2e \\ B_{ecd} \end{bmatrix}^{-1}$$
$$\sum_{z=\max(0,-B_{bdf}+B_{dec})}^{\min(B_{dec},B_{afc})} (-1)^z \begin{bmatrix} A_{afc}+1 \\ 2c+z+1 \end{bmatrix} \begin{bmatrix} B_{acf}+z \\ B_{acf} \end{bmatrix} \begin{bmatrix} B_{bfd}+B_{dec}-z \\ B_{bfd} \end{bmatrix} \begin{bmatrix} B_{dce}+z \\ B_{dfb} \end{bmatrix},$$

where

$$(4.2) \qquad A_{xyz} = x+y+z, \qquad B_{xyz} = x+y-z.$$

Remark. Such $6j$-symbol is already given in [10] for the case that n is odd.

This theorem is proved as follows. Using the value of the theta graph (3.7), we have the relation in Figure 3. This gives the following expression of the quantum $6j$-symbol.

$$(4.3)\quad \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_\xi = \left(C^{a,f,c}_{m_2,m_1,m_3}\right)^{-1} \sum_{m_4,m_5,m_6} C^{b,d,f}_{m_5,m_4,m_1}\, C^{a,b,e}_{m_2,m_5,m_6}\, C^{e,d,c}_{m_6,m_4,m_3}.$$

Let us put $m_1 = 0$, $m_3 = 0$, $m_4 = \alpha$, then $m_2 = a + f - c$, $m_5 = b + d - f - \alpha$,

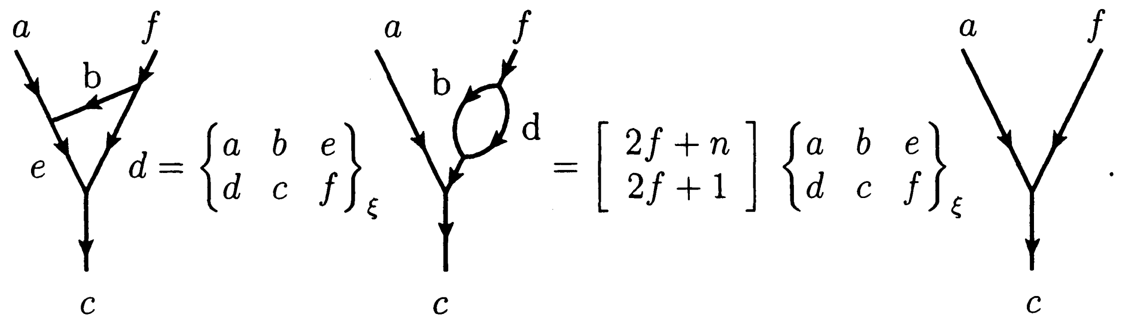


FIGURE 4. Another expression of the quantum $6j$-symbol

$m_6 = e + d - c - \alpha$. By (3.2), we have

$$\begin{bmatrix} 2f+N \\ 2f+1 \end{bmatrix} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_\xi = \begin{bmatrix} a & f & c \\ B_{afc} & 0 & 0 \end{bmatrix}_\xi^{-1} \sum_\alpha \begin{bmatrix} N-1-d & N-1-b & N-1-f \\ N-1-\alpha & N-1-B_{bdf}+\alpha & N-1 \end{bmatrix}_\xi \begin{bmatrix} a & b & e \\ B_{afc} & B_{bdf}-\alpha & B_{dec}-\alpha \end{bmatrix}_\xi \begin{bmatrix} e & d & c \\ B_{dec}-\alpha & \alpha & 0 \end{bmatrix}_\xi.$$

By using (3.8) and other simpler relations for quantum binomials, we get (4.1).

4.2. **Relations among the quantum $6j$-symbols.** From the definition of $SL(2,\mathbb{C})$ quantum $6j$-symbols, they satisfy the following relations.

Orthogonarity relation:

$$\sum_f \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_\xi \begin{Bmatrix} d & b & f \\ a & c & g \end{Bmatrix}_\xi = \delta_{eg}.$$

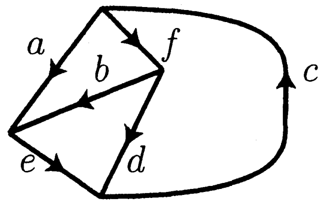


FIGURE 5. Colored oriented tetrahedral graph

Pentagon relation:

$$\sum_h \begin{Bmatrix} a & b & f \\ g & c & h \end{Bmatrix}_\xi \begin{Bmatrix} a & h & g \\ e & d & i \end{Bmatrix}_\xi \begin{Bmatrix} b & c & h \\ d & i & j \end{Bmatrix}_\xi = \begin{Bmatrix} f & c & g \\ d & e & j \end{Bmatrix}_\xi \begin{Bmatrix} a & b & f \\ j & e & i \end{Bmatrix}_\xi .$$

4.3. **Values of tetrahedra.** The value $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{tet}$ of the colored oriented graph corresponding to a tetrahedron in Figure 5 is given as follows.

Theorem 4.2. *Assume that none of the colors a, b, c, d, e, f is a half-integer and they satisfy the admissibility condition for triples (a, f, c), (a, b, e), (b, d, f) and (e, d, c), i.e.*

$$\begin{aligned} 0 \leq & a+f-c,\ a+b-e,\ b+d-f,\ e+d-c \leq N-1, \\ & a+f-c,\ a+b-e,\ b+d-f,\ e+d-c \in \mathbb{Z}. \end{aligned}$$

Then $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{tet}$ *is given as follows.*

(4.4)

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{tet} = (-1)^{N-1+B_{afc}} \frac{\{B_{dec}\}!\,\{B_{abe}\}!}{\{B_{bdf}\}!\,\{B_{afc}\}!} \begin{bmatrix} 2e \\ A_{abe}+1-N \end{bmatrix} \begin{bmatrix} 2e \\ B_{ecd}! \end{bmatrix}^{-1}$$

$$\sum_{z=\max(0,-B_{bdf}+B_{dec})}^{\min(B_{dec},B_{afc})} (-1)^z \begin{bmatrix} A_{afc}+1 \\ 2c+z+1 \end{bmatrix} \begin{bmatrix} B_{acf}+z \\ B_{acf} \end{bmatrix} \begin{bmatrix} B_{bfd}+B_{dec}-z \\ B_{bfd} \end{bmatrix} \begin{bmatrix} B_{dce}+z \\ B_{dfb} \end{bmatrix},$$

Here $A_{xyz} = x+y+z$ and $B_{xyz} = x+y-z$ as in (4.2).

Remark 4.3. From the above formula, $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{tet}$ is a Laurent polynomial for the parameters ξ^a, ξ^b, ξ^c, ξ^d, ξ^e and ξ^f.

4.4. **Relations of the values of the tetrahedra.** From the relations of the $SL(2,\mathbb{C})$ quantum $6j$-symbols, we have the following relations.

Orthogonal relation:

$$\begin{bmatrix} f+N \\ f+1 \end{bmatrix}^{-1} \begin{bmatrix} g+N \\ g+1 \end{bmatrix}^{-1} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{tet} \begin{Bmatrix} d & b & f \\ a & c & g \end{Bmatrix}_{tet} = \delta_{eg},$$

Pentagon relation:

$$\sum_h \begin{bmatrix} h+N \\ h+1 \end{bmatrix}^{-1} \begin{Bmatrix} a & b & f \\ g & c & h \end{Bmatrix}_{tet} \begin{Bmatrix} a & h & g \\ e & d & i \end{Bmatrix}_{tet} \begin{Bmatrix} b & c & h \\ d & i & j \end{Bmatrix}_{tet} = \begin{Bmatrix} f & c & g \\ d & e & j \end{Bmatrix}_{tet} \begin{Bmatrix} a & b & f \\ j & e & i \end{Bmatrix}_{tet}.$$

Symmetry: Since the change of the orientation of an edge colored by i corresponds to the change of the color i to $\bar{i} = N-1-i$, we have the following symmetry.

$$(4.5) \quad \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{tet} = \begin{Bmatrix} b & \bar{e} & \bar{a} \\ c & f & d \end{Bmatrix}_{tet} = \begin{Bmatrix} c & \bar{f} & a \\ b & e & \bar{d} \end{Bmatrix}_{tet} = \begin{Bmatrix} d & e & c \\ \bar{a} & f & b \end{Bmatrix}_{tet} = \begin{Bmatrix} e & d & c \\ \bar{f} & a & \bar{b} \end{Bmatrix}_{tet} = \begin{Bmatrix} f & \bar{b} & a \\ e & c & d \end{Bmatrix}_{tet}.$$

4.5. **Volume of a truncated tetrahedron.** Let T be the truncated tetrahedron with dihedral angles θ_a, θ_b, θ_c, θ_d, θ_e, θ_f.

Theorem 4.4. *Let T be the truncated tetrahedron with oriented labeled edges as in Fignre 5, and let $0 < \theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f < \pi$ be the dihedral angles at the edges. If θ_i, θ_j, θ_k are three dihedral angles meeting at the same vertex, then they satisfy $\theta_i + \theta_j + \theta_k < \pi$ since T is a truncated tetrahedron. Put*

$$a_N = \left\lfloor \left(1 - \frac{\theta_a}{\pi}\right) \frac{N-1}{2} \right\rfloor, \quad \cdots, \quad f_N = \left\lfloor \left(1 - \frac{\theta_f}{\pi}\right) \frac{N-1}{2} \right\rfloor,$$

and $\bar{a}_N = N-1-a_N$, $\cdots$, $\bar{f}_N = N-1-f_N$. Using these parameters, the volume of T is given as follows.

$$\mathrm{Vol}(T) = \lim_{N\to\infty} \frac{\pi}{2N} \log\left(\begin{Bmatrix} a_N & b_N & e_N \\ d_N & c_N & f_N \end{Bmatrix}_{tet} \begin{Bmatrix} \bar{a}_N & \bar{b}_N & \bar{e}_N \\ \bar{d}_N & \bar{c}_N & \bar{f}_N \end{Bmatrix}_{tet} \right).$$

Remark 4.5. $\begin{Bmatrix} a_N & b_N & e_N \\ d_N & c_N & f_N \end{Bmatrix}_{tet}$ is defined for tetrahedron with oriented edges, and it is not symmetric with respect to the natural symmetry of the tetrahedron. However, the limit of the above sum becomes symmetric.

The above theorem is proved by using the Schläfli differential equality and Ushijima's edge length formula in [31].

Question 4.6. *In [11], a 3-manifold invariant is constructed from such 6j-symbols. Explain the relation between this invariant and the hyperbolic volume of the 3-manifold.*

References

1. Y. Akutsu, T. Deguchi and T. Ohtsuki, Invariants of colored links. *J. Knot Theory Ramifications* **1** (1992), 161–184.
2. Q. Chen, S. Kuppum and P. Srinivasan, On the relation between the WRT invariant and the Hennings invariant, *Math. Proc. Camb. Phil. Soc.* **146** (2009), 151–163.
3. Y. Cho and H. Kim, On the volume formula for hyperbolic tetrahedra. *Discrete Comput. Geom.* **22** (1999), 347–366.
4. J. Cho and J. Murakami, Some limits of the colored Alexander invariant of the figure-eight knot and the volume of hyperbolic orbifolds. *J. Knot Theory Ramifications* **18** (2009), 1271–1286.
5. F. Costantino, Coloured Jones invariants of links and the volume conjecture. *J. Lond. Math. Soc.* (2) **76** (2007), 1–15.
6. F. Costantino, $6j$-symbols, hyperbolic structures and the volume conjecture. *Geom. Topol.* **11** (2007), 1831–1854.
7. F. Costantino, Integrality of Kauffman brackets of trivalent graphs. preprint, *arXiv:0908.0542.*
8. F. Costantino and J. Murakami On SL(2, C) quantum 6j-symbol and its relation to the hyperbolic volume. *arXiv:1005.4277*
9. B. L. Feigin, A.M. Gainutdinov, A. M. Semikhatov, and I. Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center,* Comm. Math. Phys. **265** (2006), no. 1, 47–93.
10. N. Geer and B. Patureau-Mirand. Polynomial 6j-symbols and states sums. preprint, *arXiv:0911.1353.*
11. N. Geer, B. Patureau-Mirand and V. Turaev Modified 6j-Symbols and 3-Manifold Invariants. preprint, *arXiv:0910.1624.*
12. N. Geer and N. Reshetikhin, On invariants of graphs related to quantum sl_2 at root of unity. *Lett. Math. Phys.* **88** *(2009), 321–331.*
13. S. Gukov, Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial. *Comm. Math. Phys.* **255** (2005), 577–627.
14. R. Hartley, The Conway potential function for links. *Comment. Math. Helv.* **58** (1983), 365–378.
15. M. Hennings, Invariants of links and 3-manifolds obtained from Hopf algebras. *J. London Math. Soc.* (2) **54** (1996), 594–624.
16. R. Kashaev, A link invariant from quantum dilogarithm, *Modern Phys. Lett. A* **10** (1995), no. 19, 1409–1418.
17. R. Kashaev, The hyperbolic volume of knots from the quantum dilogarithm. *Lett. Math. Phys.* **39** (1997), 269–275.

18. A.N. Kirillov, Clebsch-Gordan quantum coefficients. *J. Soviet Math.* **53** (1991), 264–276.
19. A.N. Kirillov and N. Yu. Reshetikhin, Representations of the algebra $\mathcal{U}_q(sl_2)$, q-orthogonal polynomials and invariants of links. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, Adv. Ser. Math. Phys. 7, World Scientific, Teaneck 1989, 285–339.
20. J. Milnor, The Schläfli differential equality, John Milnor Collected Papers Volume 1 Geometry, 281-295, Pblish or Perish, Houston, Texas.
21. H. Murakami, Optimistic calculations about the Witten-Reshetikhin-Turaev invariants of closed three-manifolds obtained from the figure-eight knot by integral Dehn surgeries. Recent progress towards the volume conjecture (Kyoto, 2000). Surikaisekikenkyusho Kokyuroku **1172** (2000), 70–79.
22. H. Murakami and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot. *Acta Math.* **186** (2001), 85–104.
23. H. Murakami, J. Murakami, M. Okamoto, T. Takata, Y. Yokota, Kashaev's conjecture and the Chern-Simons invariants of knots and links. *Experiment. Math.* **11** (2002), 427–435.
24. H. Murakami and Y. Yokota, The colored Jones polynomials of the figure-eight knot and its Dehn surgery spaces. *J. Reine Angew. Math.* **607** (2007), 47–68.
25. J. Murakami, Colored Alexander invariants and cone-manifolds. *Osaka J. Math.* **45** (2008), 541–564.
26. J. Murakami and K. Nagatomo, Logarithmic knot invariants arising from restricted quantum groups. *Internat. J. Math.* **19** (2008), 1203–1213.
27. J. Murakami and A. Ushijima, A volume formula for hyperbolic tetrahedra in terms of edge lengths. *J. Geom.* **83** (2005), 153–163.
28. J. Murakami and M. Yano, On the volume of a hyperbolic and spherical tetrahedron. *Comm. Anal. Geom.* **13** (2005), 379–400.
29. N.Yu. Reshetikhin and V. G. Tuaev, Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103** (1991), 547–597.
30. V.G. Turaev and O. Ya. Viro, State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology* **31** (1992), 865–902.
31. A. Ushijima, A volume formula for generalised hyperbolic tetrahedra. In *Non-Euclidean geometries*, Math. Appl. (N. Y.) 581, Springer, New York 2006, 249–265.
32. O. Viro, Quantum relatives of the Alexander polynomial. *St. Petersburg Math. J.* **18** (2007), 391-457.
33. Y. Yokota, On the volume conjecture for hyperbolic knots, *arXiv:math/0009165.*

Jun Murakami
Department of Mathematics
Faculty of Science and Engineering
Waseda University
3-4-1 Ohkubo, Shinjuku-ku
Tokyo 169-8555, JAPAN
e-mail: *murakami@waseda.jp*