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ON THE GRADED RING OF SIEGEL MODULAR FORMS OF DEGREE TWO WITH RESPECT TO A NON-SPLIT SYMPLECTIC GROUP

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1. INTRODUCTION

The purpose of this article is to report my talk at the conference "Automorphic forms, automorphic representations and related topics" on January 2010. We give explicitly the graded ring of Siegel modular forms of degree two with respect to a certain discrete subgroup of a non-split symplectic group. (Theorem 1.1 below). In this section, we give an introduction for our main result and the way to prove it.

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D with the canonical involution $^{-}$. We define the group $U(2;B)$ as the unitary group with respect to the quaternion hermitian space of rank two, i.e.

$$U(2;B):=\left\{g\in GL(2;B)\ \middle|\ {}^{t}\overline{g}\begin{pmatrix}0&1\\1&0\end{pmatrix}g=\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\},$$

where ${}^{t}\overline{g}=\begin{pmatrix}\overline{a}&\overline{c}\\\overline{b}&\overline{d}\end{pmatrix}$ for $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}$. We can regard $U(2;B)$ as a subgroup of $Sp(2;\mathbb{R})$ by fixing an isomorphism $U(2;B)\otimes_{\mathbb{Q}}\mathbb{R}\simeq Sp(2;\mathbb{R})$. If $D\neq 1$, then $U(2;B)$ is a non-split $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$. Let $\mathfrak{O}$ be the maximal order of B, which is unique up to conjugation. If we take a positive divisor D_1 of D and put $D_2:=D/D_1$, then there is the unique maximal two-sided ideal $\mathfrak{A}$ of $\mathfrak{O}$ such that $\mathfrak{A}\otimes_{\mathbb{Z}}\mathbb{Z}_p=\mathfrak{O}_p$ if $p\mid D_1$ or $p\nmid D$, and $\mathfrak{A}\otimes_{\mathbb{Z}}\mathbb{Z}_p=\pi\mathfrak{O}_p$ if $p\mid D_2$, where π is a prime element of $\mathfrak{O}_p$. We treat a discrete subgroup of $Sp(2;\mathbb{R})$ defined by

$$\Gamma(D_1,D_2):=U(2;B)\cap\begin{pmatrix}\mathfrak{O}&\mathfrak{A}^{-1}\\\mathfrak{A}&\mathfrak{O}\end{pmatrix}.$$

We are interested in studying Siegel modular forms with respect to $\Gamma(D_1,D_2)$. We denote by $M_k(\Gamma)$ the space of Siegel modular forms of weight k with respect to $\Gamma=\Gamma(D_1,D_2)$. The main theorem of this paper is the following:

Theorem 1.1. *The graded ring of Siegel modular forms with resptect to* $\Gamma(1,6)$ *is given explicitly by*

$$\bigoplus_{k=0}^{\infty}M_k(\Gamma(1,6))=\mathbb{C}[E_2,E_4,\chi_{5a},E_6]\oplus\chi_{5b}\mathbb{C}[E_2,E_4,\chi_{5a},E_6]$$
$$\oplus\chi_{15}\mathbb{C}[E_2,E_4,\chi_{5a},E_6]\oplus\chi_{5b}\chi_{15}\mathbb{C}[E_2,E_4,\chi_{5a},E_6],$$

where we denote by E_k *(*$k=2,4,6$*) the Eisenstein series which are defined in* [Hir99], *and denote by* χ_{5a}, χ_{5b} *and* χ_{15} *the Siegel cusp forms of weight* 5, 5 *and* 15 *respectively, which are defined in Proposition 1.3 and 1.4 below. The four modular forms* E_2, E_4, χ_{5a}

and E_6 are agebraically independent over $\mathbb{C}$, and ${\chi_{5b}}^2$ and ${\chi_{15}}^2$ can be written by E_2, E_4, χ_{5a} and E_6. Fourier coefficients of these forms are computable and given in Appendix.

Explicit constructions of the graded ring of Siegel modular forms of split case have been studied by many authors, for example, Igusa[Igu62], Ibukiyama[Ibu91], Freitag and Salvati Manni[FS04], Gunji[Gun04] and Aoki and Ibukiyama[AI05], but, as far as the author knows, no results were known for the case of non-split $\mathbb{Q}$-forms of $Sp(2;\mathbb{R})$. We are short of available methods in the case of non-split $\mathbb{Q}$-forms because they have only point cusps. Hirai [Hir99] determined the spaces of low weights for $\Gamma(6,1)$ by using his explicit formula of Fourier coefficients of the Eisenstein series (cf. Proposition 2.2), Oda lifting (cf. [Oda77],[Sug84]) and Hashimoto's explicit dimension formula (cf. [Has84]), but he did not obtain the graded ring.

We summarize the way to prove our main theorem, Theorem 1.1. The dimension formula which we obtained in our previous work (see subsection 2.2) plays a crucial role in our work. The first step to prove Theorem 1.1 is to determine the spaces of weight $k \leq 4$. Note that the formula can not be applied for the spaces of weight $k \leq 4$. We will prove Proposition 1.2 in section 3.

Proposition 1.2.

$$M_1(\Gamma(1,6)) = \{0\}, \qquad M_2(\Gamma(1,6)) = \mathbb{C}E_2,$$
$$M_3(\Gamma(1,6)) = \{0\}, \qquad M_4(\Gamma(1,6)) = \mathbb{C}{E_2}^2 \oplus \mathbb{C}E_4.$$

The second step to prove Theorem 1.1 is to construct χ_{5a}, χ_{5b} and χ_{15}. Generally speaking, it is difficult to construct modular forms of odd weight. As for χ_{5a} and χ_{5b}, we will prove Proposition 1.3 in section 4 by detailed calculation of Fourier coefficients of the space of weight 10

Proposition 1.3. *The Siegel cusp forms χ_{5a} and χ_{5b} of weight 5 exist and are determined uniquely up to sign by the following relations:*

$$\begin{aligned}{\chi_{5a}}^2 = &\tfrac{31513745731}{416023384089600}E_{10} - \tfrac{126433528597}{311423218947072}{E_2}^5 + \tfrac{11304517601}{14285468759040}{E_2}^3E_4 \\ &- \tfrac{41742579637}{1557116094735360}{E_2}^2E_6 - \tfrac{38947571}{120147846816}E_2{E_4}^2 - \tfrac{1000259890201}{9083177219289600}E_4E_6, \\ {\chi_{5b}}^2 = &\tfrac{31513745731}{416023384089600}E_{10} + \tfrac{266799861}{1281577032704}{E_2}^5 - \tfrac{261925781}{1587274306560}{E_2}^3E_4 \\ &- \tfrac{1914649869}{6407885163520}{E_2}^2E_6 + \tfrac{935053847}{51903869824512}E_2{E_4}^2 + \tfrac{551346719209}{3406191457233600}E_4E_6.\end{aligned}$$

As for χ_{15}, we will prove Proposition 1.4 in section 5. We denote by $\{E_2, E_4, \chi_{5a}, E_6\}_*$ the Siegel cusp form of weight 20 obtained from E_2, E_4, χ_{5a} and E_6 by the differential operator which is reviewed in subsection 2.5.

Proposition 1.4. *The Siegel cusp form $\{E_2, E_4, \chi_{5a}, E_6\}_*$ is divisible by χ_{5b}, so we can define $\chi_{15} := \{E_2, E_4, \chi_{5a}, E_6\}_*/\chi_{5b}$.*

Fianlly, we will prove Theorem 1.1 in section 6. We can obtain the generating function of $\dim_{\mathbb{C}} M_k(\Gamma(1,6))$ by using the dimension formula and Proposition 1.2. It is crucial for the final step to prove the equality.

2. Preliminaries

2.1. **Siegel modular forms.** We review Sigel modular forms to fix notation. Let $Sp(2;\mathbb{R})$ be the real symplectic group of degree two, i.e.

$$Sp(2;\mathbb{R}) = \left\{ g \in GL(4,\mathbb{R}) \;\middle|\; g\begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} {}^t g = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}.$$

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two, i.e.

$$\mathfrak{H}_2 = \{ Z \in M(2;\mathbb{C}) \mid {}^t Z = Z,\ \mathrm{Im}(Z) \text{ is positive definite } \}.$$

The group $Sp(2;\mathbb{R})$ acts on $\mathfrak{H}_2$ by

$$\gamma\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\mathbb{R})$ and $Z \in \mathfrak{H}_2$. Let Γ be a discrete subgroup of $Sp(2;\mathbb{R})$ such that $\mathrm{vol}(\Gamma\backslash\mathfrak{H}_2) < \infty$. We say that a holomorphic function $F(Z)$ on $\mathfrak{H}_2$ is a Sigel modular form of weight k of Γ if it satisfies

$$f(\gamma\langle Z\rangle) = \det(CZ+D)^k f(Z), \quad \text{for } \forall\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \forall Z \in \mathfrak{H}_2.$$

If a Siegel modular form $F(Z)$ satisfies

$$\det(\mathrm{Im}(Z)^{1/2})|f(Z)| \text{ is bounded on } \mathfrak{H}_2,$$

then we say that $F(Z)$ is a Siegel cusp form. We denote by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the spaces of all Siegel modular forms (resp. cusp forms) of weight k of Γ. It is known that $M_k(\Gamma)$ and $S_k(\Gamma)$ are finite dimensional vector spaces over $\mathbb{C}$.

2.2. **Dimension formula.** Let B be an indefinite quaternion algebra over $\mathbb{Q}$. We fix an isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M(2;\mathbb{R})$ and we identify B with a subalgebra of $M(2;\mathbb{R})$. We define $U(2;B)$ and $\Gamma(D_1,D_2)$ as in section 1. It is known that $U(2;B)\otimes_{\mathbb{Q}}\mathbb{R}$ is isomorphic to $Sp(2;\mathbb{R})$ by

$$\phi : U(2;B)\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} Sp(2;\mathbb{R})$$

$$\phi(g) = \begin{pmatrix} a_1 & a_2 & b_2 & -b_1 \\ a_3 & a_4 & b_4 & -b_3 \\ c_3 & c_4 & d_4 & -d_3 \\ -c_1 & -c_2 & -d_2 & d_1 \end{pmatrix}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2;B)\otimes_{\mathbb{Q}}\mathbb{R}$$

where $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ $\in B\otimes_{\mathbb{Q}}\mathbb{R}$, and we can identify $\Gamma(D_1,D_2)$ with a discrete subgroup of $Sp(2;\mathbb{R})$ such that $\mathrm{vol}(\Gamma(D_1,D_2)\backslash\mathfrak{H}_2) < \infty$.

In our previous paper [Kit], we obtained an explicit formula for dimensions of the spaces $S_k(\Gamma(D_1,D_2))$ of weight $k \geq 5$ for general (D_1,D_2), including the vector-valued case. If we apply this formula to $S_k(\Gamma(1,2p))$ for an odd prime number p, then we have

$$\begin{aligned}\dim_{\mathbb{C}} S_k(\Gamma(1,2p)) &= \frac{(k-2)(k-1)(2k-3)}{2^7\cdot 3^2\cdot 5}\cdot(p^2-1)+\frac{1}{2^3\cdot 3}\cdot(p-1)\\ &+\frac{(-1)^k(8+(\frac{-1}{p}))+(2k-3)(8-(\frac{-1}{p}))}{2^7\cdot 3}(p-(\tfrac{-1}{p}))\\ &+\frac{[0,-1,1;3]_k}{2^2\cdot 3^2}\cdot\Big(4+\tfrac{1}{2}(\tfrac{-3}{p})(1-5(\tfrac{-3}{p}))\Big)(p-(\tfrac{-3}{p}))\\ &+\frac{2k-3}{2^2\cdot 3^2}\cdot\Big(5-\tfrac{1}{2}(\tfrac{-3}{p})(1+7(\tfrac{-3}{p}))\Big)(p-(\tfrac{-3}{p}))\\ &-\frac{1}{2^3}(1-(\tfrac{-1}{p}))-\frac{1}{3}(1-(\tfrac{-3}{p}))\\ &+\frac{2\cdot[1,0,0,-1,0;5]_k}{5}\cdot(1-(\tfrac{p}{5}))\\ &+\frac{[1,0,0,-1;4]_k}{2^2}\cdot\begin{cases}0 & \cdots\ \text{if } p\equiv 1,7 \bmod 8\\ 1 & \cdots\ \text{if } p\equiv 3,5 \bmod 8\end{cases}\\ &+\frac{1}{6}\cdot\begin{cases}(-1)^k/2 & \cdots\ \text{if } p=3\\ 0 & \cdots\ \text{if } p\equiv 1,11 \bmod 12\\ [0,1,-1;3]_k & \cdots\ \text{if } p\equiv 5 \bmod 12\\ (-1)^k & \cdots\ \text{if } p\equiv 7 \bmod 12,\end{cases}\end{aligned}$$

where $(\frac{*}{*})$ is the Legendre symbol and $[a_0,\dots,a_{m-1};m]_k$ is the function on k which takes the value a_i if $k\equiv i \bmod m$. From this formula, we have $\dim_{\mathbb{C}} S_k(\Gamma(1,6))$ as follows. Our formula is not valid for $k\le 4$. In the following table, we formally substitute $k\le 4$ in the formula.

k	0	1	2	3	4	5	6	7	8	9	10	$\cdots$	15	$\cdots$	20	$\cdots$	25	$\cdots$
dim	0	−1	0	−1	1	2	2	2	3	4	6	$\cdots$	13	$\cdots$	27	$\cdots$	47	$\cdots$

2.3. **Fourier expansion.** Let $\mathfrak{A}$ be a maximal two-sided ideal of $\mathfrak{O}$. Since the class number of $\mathfrak{O}$ is one, we can write $\mathfrak{A}=\mathfrak{O}\pi=\pi\mathfrak{O}$ for some $\pi\in\mathfrak{O}$ such that $|N\pi|=D_1$ where $\mathfrak{A}$ corresponds to (D_1,D_2) as in section 1. We define a three-dimensional $\mathbb{Q}$ vector space $B^0:=\{x\in B \mid \mathrm{Tr}(x)=0\}$ and define a lattice A and its dual lattice by

$$A:=B^0\cap\mathfrak{A}^{-1},\quad A^*:=\{y\in B^0 \mid \mathrm{Tr}(xy)\in\mathbb{Z} \text{ for any } x\in A\}.$$

Arakawa proved the following proposition in his master thesis [Ara75, Proposition 10] by the same way as, for example, Maaß[Maa71, §13].

Proposition 2.1 ([Ara75],[Hir99]). *Let $\Gamma(D_1,D_2)$ be the discrete subgroup of $Sp(2;\mathbb{R})$ defined in section 1 and k be a positive integer. Then $f(Z)\in M_k(\Gamma(D_1,D_2))$ has the following Fourier expansion*

$$f(Z)=C_f(0)+\sum_{\substack{\eta\in A^*\\ \eta J>0}} C_f(\eta)\mathbf{e}[\mathrm{Tr}(\eta ZJ)],\quad (\mathbf{e}[z]:=e^{2\pi i z})$$

where $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and $\eta J>0$ means that ηJ is positive definite when we regard η as an element of $M(2;\mathbb{R})$. In particular, $f(Z)\in S_k(\Gamma(D_1,D_2))$ is equivalent to $C_f(0)=0$.

2.4. **Eisenstein series.** By applying the method of Shimura [Shi83], Hirai [Hir99] studied the Eisenstein series E_k ($k \geq 2$: even) on $\Gamma(D_1, D_2)$ and obtained an explicit formula of Fourier coefficients of it. (Proposition 2.2 below). We define

$$A^*_{\text{prim}} := \{\eta \in A^* \mid n^{-1}\eta \notin A^* \text{ for any integer } n\}.$$

For $\eta \in A^*$, we denote by d_η and χ_η the discriminant and the Dirichlet character of $\mathbb{Q}(\eta)/\mathbb{Q}$ and denote by B_m (resp. B_{m,χ_η}) the m-th Bernoulli (resp. the generalized Bernoulli) number. We define positive integers a_η and f_η by

$$a_\eta^{-1}\eta \in A^*_{\text{prim}}, \quad (2a_\eta^{-1}\eta)^2 = d_\eta {f_\eta}^2.$$

We put $a_{\eta,p} = \operatorname{ord}_p(a_\eta)$, $f_{\eta,p} = \operatorname{ord}_p(f_\eta)$. Then the following proposition holds.

Proposition 2.2 ([Hir99] Theorem 3.10). *Let k be an even positive integer. Then the Eisenstein series E_k has the following Fourier expansion.*

$$E_k(Z) = 1 + \sum_{\substack{\eta \in A^* \\ \eta J > 0}} C(\eta)\mathbf{e}[\operatorname{Tr}(\eta Z)],$$

where

$$C(\eta) = \frac{4kB_{k-1,\chi_\eta}}{B_k B_{2k-2}} \prod_{p|D_1} \frac{(1-\chi_\eta(p)p^{k-1})(1-\chi_\eta(p)p^{k-2})}{p^{2k-2}-1} \prod_{p|D_2} \frac{1}{p^{k-1}-1} \prod_p F_p(\eta, k),$$

$$F_p(\eta,k) = \begin{cases} \displaystyle\sum_{t=0}^{a_{\eta,p}} p^{(2k-3)t} + (1+\chi_\eta(p)) \sum_{t=0}^{a_{\eta,p}-1} p^{(2k-3)t+k-1} & \cdots \text{ if } p|D_1, \\ \displaystyle\sum_{t=0}^{a_{\eta,p}} p^{(2k-3)t} - \chi_\eta(p) \sum_{t=0}^{a_{\eta,p}-1} p^{(2k-3)t+k-2} & \cdots \text{ if } p|D_2, \\ \displaystyle\sum_{t=0}^{a_{\eta,p}} \left\{ \sum_{l=0}^{a_{\eta,p}+f_{\eta,p}-t} p^{(2k-3)l+(k-1)t} - \chi_\eta(p) \sum_{l=0}^{a_{\eta,p}+f_{\eta,p}-t-1} p^{(2k-3)l+(k-1)t+k-2} \right\} & \\ & \cdots \text{ if } p \nmid D. \end{cases}$$

We see from Proposition 2.1 that

$$M_k(\Gamma(D_1,D_2)) = S_k(\Gamma(D_1,D_2)) \qquad \text{if } k \text{ is odd, and}$$
$$M_k(\Gamma(D_1,D_2)) = S_k(\Gamma(D_1,D_2)) \oplus \mathbb{C}E_k \qquad \text{if } k \text{ is even.}$$

2.5. **Rankin-Cohen type differential operators.** We quote the following proposition from Aoki and Ibukiyama[AI05].[1] For $Z \in H_2$, we write the (i,j) component of Z by z_{ij}. For Siegel modular forms $f_i \in M_{k_i}(\Gamma)$ of weight k_i ($1 \leq i \leq 4$), we define a new function $\{f_1, f_2, f_3, f_4\}_*$ by

$$\{f_1,f_2,f_3,f_4\}_* = \begin{vmatrix} k_1 f_1 & k_2 f_2 & k_3 f_3 & k_4 f_4 \\ \frac{\partial f_1}{\partial z_{11}} & \frac{\partial f_2}{\partial z_{11}} & \frac{\partial f_3}{\partial z_{11}} & \frac{\partial f_4}{\partial z_{11}} \\ \frac{\partial f_1}{\partial z_{12}} & \frac{\partial f_2}{\partial z_{12}} & \frac{\partial f_3}{\partial z_{12}} & \frac{\partial f_4}{\partial z_{12}} \\ \frac{\partial f_1}{\partial z_{22}} & \frac{\partial f_2}{\partial z_{22}} & \frac{\partial f_3}{\partial z_{22}} & \frac{\partial f_4}{\partial z_{22}} \end{vmatrix}.$$

[1]In [AI05], Proposition 2.3 has been proved in the case of general degree n.

Proposition 2.3 (Aoki and Ibukiyama [AI05]). *(i) The above function* $\{f_1, f_2, f_3, f_4\}_*$ *is a Siegel cusp form of weight* $k_1 + k_2 + k_3 + k_4 + 3$.
(ii) f_1, f_2, f_3, f_4 *are algebraiclly independent if and only if* $\{f_1, f_2, f_3, f_4\}_* \neq 0$.

3. Proof of Proposition 1.2

In this section, we will prove Proposition 1.2, that is, we will determine the spaces of weight $k \leq 4$. Note that the dimension formula is not valid for weight $k \leq 4$. (See subsection 2.2).

We prepare to calculate Fourier coefficients. If we put

$$B := \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab, \qquad a^2 = 6, b^2 = 5, ab = -ba,$$

then B is an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant 6, which is unique up to isomorphism. Let $\mathfrak{O}$ be the maximal order of B, which is unique up to conjugacy. It is known by Ibukiyama [Ibu72],[Ibu82] that $\mathfrak{O}$ can be taken as

$$\mathfrak{O} = \mathbb{Z} + \mathbb{Z}\frac{1+b}{2} + \mathbb{Z}\frac{a(1+b)}{2} + \mathbb{Z}\frac{(1+a)b}{5}.$$

If we put $\mathfrak{A} = a\mathfrak{O}$, then $\mathfrak{A}$ is the unique maximal two-sided ideal corresponding to $(1, 6)$. By a straightforward calculation, we obtain

$$A^* = \mathbb{Z}\frac{5a+b+ab}{10} + \mathbb{Z}\frac{b}{2} + \mathbb{Z}a.$$

For $\eta = x(5a + b + ab)/60 + yb/12 + za/6 \in A^*$, we denote it by $\eta = [x, y, z]$ and we can see from a direct calculation that the condition $\eta J > 0$ is equivalent to

$$\begin{cases} x > 0, \quad \text{and} \\ m_\eta := -(5x^2 + 5y^2 + 24z^2 - 2xy + 24zx) > 0. \end{cases}$$

We have the following modular forms which are obtained as products of Eisenstein series E_k's:

$$\text{weight } 2 : E_2, \qquad \text{weight } 4 : {E_2}^2, E_4,$$
$$\text{weight } 6 : {E_2}^3, E_2E_4, E_6, \qquad \text{weight } 8 : {E_2}^4, {E_2}^2E_4, E_2E_6, {E_4}^2, E_8.$$

For the sake of simplicity of Fourier coefficients, we use the following φ_k instead of E_k $(k = 2, 4, 6, 8)$:

$$\varphi_2 = E_2, \qquad \varphi_4 = -\tfrac{13}{288} \cdot (E_4 - {\varphi_2}^2),$$
$$\varphi_6 = -\tfrac{341}{113184} \cdot (E_6 - {\varphi_2}^3) - \tfrac{109}{262} \cdot \varphi_2\varphi_4, \qquad \varphi_8 = 138811E_8,$$

then we have Fourier coefficients of them as in the following tables.

m_η	η	φ_2	${\varphi_2}^2$	φ_4	${\varphi_2}^3$	$\varphi_2\varphi_4$	φ_6
0	[0, 0, 0]	1	1	0	1	0	0
3	[2, 1, −1]	48	96	1	144	1	0
4	[2, 0, −1]	72	144	−1	216	−1	1
12	[4, 2, −2]	192	2688	−2	7488	46	−6
16	[4, 0, −2]	216	5616	6	16200	−66	42
19	[4, 1, −2]	144	7200	−5	21168	19	−16
24	[5, 1, −2]	288	15552	12	45792	−36	−60
27	[6, 3, −3]	192	18816	36	166464	132	96
36	[6, 0, −3]	360	41040	−45	495288	363	21
40	[6, 2, −3]	288	52416	−4	654048	−580	100

m_η	η	${\varphi_2}^4$	${\varphi_2}^2\varphi_4$	$\varphi_2\varphi_6$	${\varphi_4}^2$	φ_8
0	[0, 0, 0]	1	0	0	0	138811
3	[2, 1, −1]	192	1	0	0	13440
4	[2, 0, −1]	288	−1	1	0	87840
12	[4, 2, −2]	14592	94	−6	1	110974080
16	[4, 0, −2]	31968	−138	114	1	719673120
19	[4, 1, −2]	42048	43	32	−2	2181836160
24	[5, 1, −2]	91008	−84	84	4	10043268480
27	[6, 3, −3]	553728	2532	−192	−4	21427714560
36	[6, 0, −3]	1736352	−4413	3261	−8	140109455520
40	[6, 2, −3]	2302848	3452	−764	−4	277771616640
43	[6, 1, −3]	3318336	−1613	992	18	441018218880
48	[8, 4, −4]	11137152	16012	3396	82	909100536960
51	[7, 2, −4]	7825536	3318	−4992	40	1337603408640
52	[7, 1, −4]	9062784	−5116	−2180	−56	1528671231360
64	[8, 0, −4]	49322592	12476	9556	90	5895562286880

From these tables and the results of the dimension formula, we can see the following:

$$M_2(\Gamma(1,6)) \supseteq \mathbb{C}E_2, \qquad M_4(\Gamma(1,6)) \supseteq \mathbb{C}{E_2}^2 \oplus \mathbb{C}E_4,$$
$$M_6(\Gamma(1,6)) = \mathbb{C}{E_2}^3 \oplus \mathbb{C}E_2E_4 \oplus \mathbb{C}E_6,$$
$$M_8(\Gamma(1,6)) = \mathbb{C}{E_2}^4 \oplus \mathbb{C}{E_2}^2E_4 \oplus \mathbb{C}E_2E_6 \oplus \mathbb{C}{E_4}^2.$$
$$\left(E_8 = \tfrac{48860325}{18184241}{E_2}^4 - \tfrac{107719950}{18184241}{E_2}^2E_4 + \tfrac{26257000}{18184241}E_2E_6 + \tfrac{387686}{138811}{E_4}^2\right)$$

We can prove Proposition 1.2 by using the spaces of weight 6 and 8. We prove the following lemma.

Lemma 3.1. *If $k(\neq 6)$ is a positive divisor of 6, then there are no non-zero cusp forms of weight k.*

Proof. We assume that there is a non-zero cusp form f of weight k. Then the Fourier coefficients of $f^2 \in S_{2k}(\Gamma(1,6))$ are:

$$\begin{aligned} C_{f^2}(0,0,0) &= C_f(0,0,0) \cdot C_f(0,0,0) = 0, \\ C_{f^2}(2,1,-1) &= 2 \cdot C_f(0,0,0) \cdot C_f(2,1,-1) = 0, \\ C_{f^2}(2,0,-1) &= 2 \cdot C_f(0,0,0) \cdot C_f(2,0,-1) = 0, \end{aligned}$$

so the Fourier coefficients of $f^{6/k} \in S_6(\Gamma(1,6))$ are also

$$C_{f^{6/k}}(0,0,0) = C_{f^{6/k}}(2,1,-1) = C_{f^{6/k}}(2,0,-1) = 0.$$

Hence we have $f^{6/k} = 0$ because of the table of Fourier coefficients of the space of weight 6 on page 7, but this contradicts the assumption that f is not zero. □

Proof of Proposition 1.2. Noting that modular forms of odd weight are necessarily cusp forms, we see that $M_1(\Gamma(1,6)) = M_3(\Gamma(1,6)) = \{0\}$ by Lemma 3.1. Also we see that $M_2(\Gamma(1,6)) = \mathbb{C}E_2$ by Lemma 3.1 because if there is a non-zero element f of $M_2(\Gamma(1,6))$ which is linearly independent of E_2, then we can assume that f is a cusp form by adjusting it by E_2.

Next, we prove $M_4(\Gamma(1,6)) = \mathbb{C}{E_2}^2 \oplus \mathbb{C}E_4$. We assume that there is a non-zero element $f \in M_4(\Gamma(1,6))$ which is linearly independent of ${E_2}^2$ and E_4. Then we can assume that $C_f(0,0,0) = C_f(2,1,-1) = 0$ by adjusting them by ${E_2}^2$ and E_4 (cf. the table on page 7). Then the Fourier coefficients of $f^2 \in S_8(\Gamma(1,6))$ are

$$\begin{aligned} C_{f^2}(0,0,0) &= C_f(0,0,0) \cdot C_f(0,0,0) = 0, \\ C_{f^2}(2,1,-1) &= 2 \cdot C_f(0,0,0) \cdot C_f(2,1,-1) = 0, \\ C_{f^2}(2,0,-1) &= 2 \cdot C_f(0,0,0) \cdot C_f(2,0,-1) = 0, \\ C_{f^2}(4,2,-2) &= 2 \cdot C_f(0,0,0) \cdot C_f(4,2,-2) + C_f(2,1,-1)^2 = 0. \end{aligned}$$

Hence we have $f^2 = 0$, and therefore $f = 0$. This contradicts the assumption.

□

4. Proof of Proposition 1.3

In this section, we will prove Propositin 1.3, that is, we will determine the spaces of weight 5 and 10. By the dimension formulla, we have $\dim_{\mathbb{C}} M_5(\Gamma(1,6)) = 2$ and $\dim_{\mathbb{C}} M_{10}(\Gamma(1,6)) = 7$. We can obtain a 6-dimensional subspace V of $M_{10}(\Gamma(1,6))$ by products of Eisenstein series E_k's:

$$V = \mathbb{C}{E_2}^5 \oplus \mathbb{C}{E_2}^3 E_4 \oplus \mathbb{C}{E_2}^2 E_6 \oplus \mathbb{C}E_2{E_4}^2 \oplus \mathbb{C}E_4E_6 \oplus \mathbb{C}E_{10}.$$

We define φ_2, φ_4 and φ_6 as in section 3 and define φ_{10} by

$$\begin{aligned} \varphi_{10} = \tfrac{31513745731}{416023384089600} \cdot (E_{10} - {\varphi_2}^5) + \tfrac{52522796831}{2889051278400} \cdot {\varphi_2}^3\varphi_4 \\ + \tfrac{21884309761}{481508546400} \cdot {\varphi_2}^2\varphi_6 - \tfrac{829232949}{1671904675} \cdot \varphi_2{\varphi_4}^2 + \tfrac{318067693}{1671904675} \cdot \varphi_4\varphi_6. \end{aligned}$$

Then we have Fourier coefficients of them as in the following table.

m_η	η	$\varphi_2{}^5$	$\varphi_2{}^3\varphi_4$	$\varphi_2{}^2\varphi_6$	$\varphi_2\varphi_4{}^2$	$\varphi_4\varphi_6$	φ_{10}
0	[0, 0, 0]	1	0	0	0	0	0
3	[2, 1, −1]	240	1	0	0	0	0
4	[2, 0, −1]	360	−1	1	0	0	0
12	[4, 2, −2]	24000	142	−6	1	0	0
16	[4, 0, −2]	52920	−210	186	1	−1	0
19	[4, 1, −2]	69840	67	80	−2	1	1
24	[5, 1, −2]	151200	−132	228	4	−2	−4
27	[6, 3, −3]	1291200	7236	−480	44	−6	−6
36	[6, 0, −3]	4137480	−14373	11685	64	−36	−24
40	[6, 2, −3]	5496480	12092	676	−28	−14	−12
43	[6, 1, −3]	8018640	−5021	9008	−78	55	23
48	[8, 4, −4]	40679520	155356	−6540	82	102	96
51	[7, 2, −4]	19124640	10182	3648	−8	−44	20
52	[7, 1, −4]	22154400	−17188	15652	112	4	−8
64	[8, 0, −4]	189615960	−326884	275764	−654	118	320

Lemma 4.1. *For a non-zero element $f \in M_5(\Gamma(1,6))$, there is a non-zero element $\chi_f \in V$ such that χ_f is divisible by f (i.e. the function χ_f/f is holomorphic).*

Proof. We can take some $g \in M_5(\Gamma(1,6))$ such that $M_5(\Gamma(1,6)) = \mathbb{C}f \oplus \mathbb{C}g$. We have either $f^2 \in V$ or $f^2 \notin V$. If $f^2 \in V$, Lemma 4.1 holds for $\chi_f = f^2$. Hereafter we assume $f^2 \notin V$. Then we have $M_{10}(\Gamma(1,6)) = V \oplus \mathbb{C}f^2$. We have either $fg \in V$ or $fg \notin V$. If $fg \in V$, then Lemma 4.1 holds for $\chi_f = fg$. If $fg \notin V$, we can write $fg = x + r \cdot f^2$ for some $x \in V$ and some $r \in \mathbb{C}^\times$. Hence we have $V \ni x = fg - r \cdot f^2 = f(g - r \cdot f)$ and $x \neq 0$. We see that Lemma 4.1 holds for $\chi_f = x$. □

Lemma 4.2. *We can find a basis χ_{5a}, χ_{5b} of $M_5(\Gamma(1,6))$ which satisfy the following conditions:*

$$C_{\chi_{5a}}(0,0,0) = 0, \qquad C_{\chi_{5a}}(2,1,-1) = 0, \qquad C_{\chi_{5a}}(2,0,-1) = 1,$$
$$C_{\chi_{5b}}(0,0,0) = 0, \qquad C_{\chi_{5b}}(2,1,-1) = 1, \qquad C_{\chi_{5b}}(2,0,-1) = 0.$$

Proof. Let f, g be a basis of $M_5(\Gamma(1,6))$. We see from Lemma 4.1 that we can take r, $s \in \mathbb{C}$ so that $f(rf + sg) \in V - \{0\}$. We put Fourier coefficients of them as

$$C_f(2,1,-1) = \alpha, \qquad C_f(2,0,-1) = \beta,$$
$$C_g(2,1,-1) = \gamma, \qquad C_g(2,0,-1) = \delta$$

We assume $\alpha = \gamma = 0$. Then Fourier coefficients of $h := f(rf + sg)$ are as follows:

$$\begin{aligned}
C_h(0,0,0) &= C_f(0,0,0) \cdot C_{f'}(0,0,0) = 0, \\
C_h(2,1,-1) &= C_f(0,0,0) \cdot C_{f'}(2,1,-1) + C_f(2,1,-1) \cdot C_{f'}(0,0,0) = 0, \\
C_h(2,0,-1) &= C_f(0,0,0) \cdot C_{f'}(2,0,-1) + C_f(2,0,-1) \cdot C_{f'}(0,0,0) = 0, \\
C_h(4,2,-2) &= C_f(0,0,0) \cdot C_{f'}(4,2,-2) + C_f(4,2,-2) \cdot C_{f'}(0,0,0) \\
&\quad + C_f(2,1,-1) \cdot C_{f'}(2,1,-1) = 0, \\
C_h(4,0,-2) &= C_f(0,0,0) \cdot C_{f'}(4,0,-2) + C_f(4,0,-2) \cdot C_{f'}(0,0,0) \\
&\quad + C_f(2,0,-1) \cdot C_{f'}(2,0,-1) = 0, \\
C_h(4,1,-2) &= C_f(0,0,0) \cdot C_{f'}(4,1,-2) + C_f(4,1,-2) \cdot C_{f'}(0,0,0) \\
&\quad + C_f(2,0,-1) \cdot C_{f'}(2,1,-1) + C_f(2,1,-1) \cdot C_{f'}(2,0,-1) = 0,
\end{aligned}$$

where $f' := rf + sg$. Hence we have $h = 0$ because of the table of Fourier coefficients of the space of weight 10. This contradicts the above. Hereafter we assume that either α or γ is non-zero. We can assume that $\alpha = 0$ and $\gamma = 1$. If $\beta = 0$, then the Fourier coefficients of h satisfy the same condition as above. So we have $\beta \neq 0$. We can assume $\beta = 1$ and $\delta = 0$. □

Proof of Proposition 1.3. We take a basis χ_{5a} and χ_{5b} which satisfy the condition of Lemme 4.2. Then we can verify that Fourier coefficients are as follows:

$$\begin{array}{llll}
C_{\chi_{5a}}(0,0,0) = 0, & C_{\chi_{5a}}(2,1,-1) = 0, & C_{\chi_{5a}}(2,0,-1) = 1, & \\
C_{\chi_{5a}}(3,0,-2) = 0, & C_{\chi_{5a}}(3,0,-1) = 0, & C_{\chi_{5a}}(3,1,-2) = -1, & C_{\chi_{5a}}(3,1,-1) = -1, \\
C_{\chi_{5b}}(0,0,0) = 0, & C_{\chi_{5b}}(2,1,-1) = 1, & C_{\chi_{5b}}(2,0,-1) = 0. & \\
C_{\chi_{5b}}(3,0,-2) = -1, & C_{\chi_{5b}}(3,0,-1) = -1, & C_{\chi_{5b}}(3,1,-2) = 0, & C_{\chi_{5b}}(3,1,-1) = 0.
\end{array}$$

By Lemma 4.1, we have $f := \chi_{5a}(\alpha\chi_{5a} + \beta\chi_{5b}) \in V$ for some $\alpha, \beta \in \mathbb{C}$. Fourier coefficients of f are

$$C_f(0,0,0) = C_f(2,1,-1) = C_f(2,0,-1) = C_f(4,2,-2) = 0,$$
$$C_f(4,0,-2) = \alpha, \quad C_f(4,1,-2) = \beta$$

by the same calculation as in the proof of Lemma 4.2. We can see from the table of Fourier coefficients of the space of weight 10 that $f = -\alpha\varphi_4\varphi_6 + (\alpha+\beta)\varphi_{10}$ and $C_f(5,1,-2) = -2\alpha - 4\beta$. On the other hand, we have

$$\begin{aligned}
C_f(5,1,-2) &= C_{\chi_{5a}}(0,0,0) \cdot C_{f'}(5,1,-2) + C_{\chi_{5a}}(5,1,-2) \cdot C_{f'}(0,0,0) \\
&\quad + C_{\chi_{5a}}(2,0,-1) \cdot C_{f'}(3,1,-1) + C_{\chi_{5a}}(3,1,-1) \cdot C_{f'}(2,0,-1) \\
&\quad + C_{\chi_{5a}}(2,1,-1) \cdot C_{f'}(3,0,-1) + C_{\chi_{5a}}(3,0,-1) \cdot C_{f'}(2,1,-1) \\
&= -2\alpha,
\end{aligned}$$

where $f' = \alpha\chi_{5a} + \beta\chi_{5b}$. Hence we have $\beta = 0$, and therefore we can assume $f = {\chi_{5a}}^2$ and

$$\begin{aligned}
f &= \varphi_{10} - \varphi_4\varphi_6 \\
&= \tfrac{31513745731}{416023384089600} E_{10} - \tfrac{126433528597}{311423218947072} {E_2}^5 + \tfrac{11304517601}{14285468759040} {E_2}^3 E_4 \\
&\quad - \tfrac{41742579637}{1557116094735360} {E_2}^2 E_6 - \tfrac{38947571}{120147846816} E_2 {E_4}^2 - \tfrac{1000259890201}{9083177219289600} E_4 E_6.
\end{aligned}$$

If $\chi_{5a}\chi_{5b} \in V$, then we have $\chi_{5a}(\chi_{5a}+\chi_{5b}) \in V$ and this contradicts the above argument. Hence $\chi_{5a}\chi_{5b} \notin V$ and $M_{10}(\Gamma(1,6)) = V \oplus \mathbb{C}\chi_{5a}\chi_{5b}$. We put ${\chi_{5b}}^2 = v + r\chi_{5a}\chi_{5b}$ for some $v \in V$ and $r \in \mathbb{C}$. Then $v = \chi_{5b}(\chi_{5b} - r\chi_{5a})$ and

$$C_f(0,0,0) = C_f(2,1,-1) = C_f(2,0,-1) = C_f(4,0,-2) = 0,$$

$$C_f(4,2,-2) = 1, \;\; C_f(4,1,-2) = -r$$

by the same calculation as above. Hence we have $v = \varphi_2{\varphi_4}^2 + \varphi_4\varphi_6 + (-r+1)\varphi_{10}$ and $C_v(5,1,-2) = -4r-2$. On the other hand, we have

$$\begin{aligned} C_v(5,1,-2) &= C_{\chi_{5b}}(0,0,0)\cdot C_{v'}(5,1,-2) + C_{\chi_{5b}}(5,1,-2)\cdot C_{v'}(0,0,0) \\ &\quad + C_{\chi_{5b}}(2,0,-1)\cdot C_{v'}(3,1,-1) + C_{\chi_{5b}}(3,1,-1)\cdot C_{v'}(2,0,-1) \\ &\quad + C_{\chi_{5b}}(2,1,-1)\cdot C_{v'}(3,0,-1) + C_{\chi_{5b}}(3,0,-1)\cdot C_{v'}(2,1,-1) \\ &= -2, \end{aligned}$$

where $v' = \chi_{5b} + r\chi_{5a}$. Hence we have $r = 0$, and therefore $v = {\chi_{5b}}^2$ and

$$\begin{aligned} {\chi_{5b}}^2 &= \varphi_2{\varphi_4}^2 + \varphi_4\varphi_6 + \varphi_{10} \\ &= \tfrac{31513745731}{416023384089600}E_{10} + \tfrac{266799861}{1281577032704}{E_2}^5 - \tfrac{261925781}{1587274306560}{E_2}^3E_4 \\ &\quad - \tfrac{1914649869}{6407885163520}{E_2}^2E_6 + \tfrac{935053847}{519038698245 12}E_2{E_4}^2 + \tfrac{551346719209}{3406191457233600}E_4E_6. \end{aligned}$$

□

5. Proof of Proposition 1.4

In this section, we will prove Proposition 1.4, that is, we will determine the spaces of weight 15 and 20. By the result of the dimension formula, we have $\dim_{\mathbb{C}} M_{20}(\Gamma(1,6)) = 28$. We can verify that the subspace V of $M_{20}(\Gamma(1,6))$ spanned by all products of E_2, E_4, χ_{5a}, χ_{5b} and E_6 is of dimension 26. If we put $\delta_{20a} := \{E_2, E_4, \chi_{5a}, E_6\}_*$ and $\delta_{20b} := \{E_2, E_4, \chi_{5b}, E_6\}_*$, then we can verify that the complementary space of V in $M_{20}(\Gamma(1,6))$ is spanned by δ_{20a} and δ_{20b} by calculating Fourier coefficients of them.

By Proposition 1.3, we see that E_2, E_4, E_6 and ${\chi_{5a}}^2 - {\chi_{5b}}^2$ are algebraically dependent over $\mathbb{C}$, so we have $\{E_2, E_4, E_6, {\chi_{5a}}^2 - {\chi_{5b}}^2\}_* = 0$. By an elementary property of the differential calculus, we have

$$\begin{array}{c} \{E_2, E_4, {\chi_{5a}}^2, E_6\} = 2\cdot\chi_{5a}\cdot\{E_2, E_4, \chi_{5a}, E_6\}_* \\ \| \\ \{E_2, E_4, {\chi_{5b}}^2, E_6\} = 2\cdot\chi_{5b}\cdot\{E_2, E_4, \chi_{5b}, E_6\}_*. \end{array}$$

Hence we see that there is a cusp form χ_{15} such that $\{E_2, E_4, \chi_{5a}, E_6\}_* = \chi_{5b}\chi_{15}$ and $\{E_2, E_4, \chi_{5b}, E_6\}_* = \chi_{5a}\chi_{15}$.

By the result of the dimension formula, we have $\dim_{\mathbb{C}} M_{15}(\Gamma(1,6)) = 13$. We see that the subspace U of $M_{15}(\Gamma(1,6))$ spanned by all products of E_2, E_4, χ_{5a}, χ_{5b} and E_6 is of dimension 12. If $\chi_{15} \in U$, then we see that $\delta_{20a} = \chi_{5b}\chi_{15} \in V$, but this is not the case. Hence we see that $M_{15}(\Gamma(1,6)) = U \oplus \mathbb{C}\chi_{15}$.

□

6. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. First, we calculate the generating function of $\dim_{\mathbb{C}} M_k(\Gamma(1,6))$. From the dimension formula in subsection 2.2 and Proposition 1.2, we see that

$$\sum_{k=0}^{\infty} \dim_{\mathbb{C}} M_k(\Gamma(1,6))t^k = 1 + t^2 + 2t^4 + \sum_{k=5}^{\infty} \dim_{\mathbb{C}} S_k(\Gamma(1,6))t^k + \sum_{k=3}^{\infty} t^{2k}$$
$$= \frac{(1+t^5)(1+t^{15})}{(1-t^2)(1-t^4)(1-t^5)(1-t^6)}.$$

By the results of the previous sections, we have obtained

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma(1,6)) \supseteq \mathbb{C}[E_2, E_4, \chi_{5a}, \chi_{5b}, E_6, \chi_{15}]. \tag{1}$$

We do not mean that six modular forms in the right side of (1) are algebraically independent over $\mathbb{C}$. We need to determine the precise structure of the right side of (1).

Lemma 6.1.
(i) E_2, E_4, χ_{5a} and E_6 are algebraically independent over $\mathbb{C}$.
(ii) ${\chi_{5b}}^2$, ${\chi_{15}}^2 \in \mathbb{C}[E_2, E_4, \chi_{5a}, E_6]$.
(iii) 1 and χ_{5b} are linearly independent over $\mathbb{C}[E_2, E_4, \chi_{5a}, E_6]$.
(iv) 1 and χ_{15} are linearly independent over $\mathbb{C}[E_2, E_4, \chi_{5a}, \chi_{5b}, E_6]$.

Proof. (i) This is followed from Proposition 2.3 because $\{E_2, E_4, \chi_{5a}, E_6\}_* = \chi_{5b}\chi_{15} \neq 0$.
(ii) This is proved by comparison of Fourier coefficients. In fact, we give the expression of ${\chi_{5a}}^2$ and ${\chi_{5b}}^2$ by E_2, E_4, χ_{5a} and E_6 in Appendix.
(iii) If $\alpha + \beta\chi_{5b} = 0$ for $\alpha, \beta \in \mathbb{C}[E_2, E_4, \chi_{5a}, E_6]$, then we have $\alpha^2 = \beta^2{\chi_{5b}}^2$. We see from (i) that α^2 and β^2 can be regarded as the squares of polynonials with four variables E_2, E_4, χ_{5a} and E_6, while ${\chi_{5b}}^2$ is not so. Hence we have $\alpha = \beta = 0$.
(iv) If $f + \chi_{5b}g = \chi_{15}(h + \chi_{5b}j)$ for $f, g, h, j \in \mathbb{C}[E_2, E_4, \chi_{5a}, E_6]$, then we have

$$2(fg - {\chi_{15}}^2 hj)\chi_{5b} = -f^2 - {\chi_{5b}}^2 g^2 + {\chi_{15}}^2 h^2 + {\chi_{5b}}^2 {\chi_{15}}^2 j^2.$$

We see from (ii) and (iii) that

$$fg = {\chi_{15}}^2 hj, \tag{2}$$

$$f^2 + {\chi_{5b}}^2 g^2 = {\chi_{15}}^2 (h^2 + {\chi_{5b}}^2 j^2). \tag{3}$$

We can see that ${\chi_{15}}^2$ is irreducible as a polynomial with 4 variables E_2, E_4, χ_{5a} and E_6. We see from (2) that either f or g is divisible by ${\chi_{15}}^2$. We see from (3) that both f and g are divisible by ${\chi_{15}}^2$. By dividing (2) and (3) by ${\chi_{15}}^2$, we obtain equations of the same shape as (2) and (3). We can repeat this infinitely, so f, g, h and j must be 0. □

We see from Lemma 6.1 that

$$\mathbb{C}[E_2, E_4, \chi_{5a}, \chi_{5b}, E_6, \chi_{15}] = \mathbb{C}[E_2, E_4, \chi_{5a}, \chi_{5b}, E_6] \oplus \chi_{15}\mathbb{C}[E_2, E_4, \chi_{5a}, \chi_{5b}, E_6]$$
$$= \mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6]$$
$$\oplus\, \mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6].$$

Hence the generating function of $\dim_{\mathbb{C}} M_k(\Gamma(1,6))$ is the same as that of dimensions of right side of (1). We have completed the proof of Theorem 1.1.

7. Appendix

We give a table of Fourier coefficients of the generators of $\bigoplus_{k=0}^{\infty} M_k(\Gamma(1,6))$ in Theorem 1.1.

η	E_2	E_4	E_6	χ_{5a}	χ_{5b}	χ_{15}
(0,0,0)	1	1	1	0	0	0
(2,1,-1)	48	960/13	2016/341	0	1	0
(2,0,-1)	72	2160/13	7560/341	1	0	0
(4,2,-2)	192	35520/13	1066464/341	0	16	0
(4,0,-2)	216	71280/13	3878280/341	6	0	0
(4,1,-2)	144	95040/13	8134560/341	−16	−27	0
(5,1,-2)	288	198720/13	24101280/341	0	0	1
(6,3,-3)	192	234240/13	39682944/341	0	12	0
(6,0,-3)	360	546480/13	149423400/341	81	0	0
(6,2,-3)	288	682560/13	239023008/341	40	0	0
(6,1,-3)	144	717120/13	10348128/11	16	135	0
(8,4,-4)	480	1141440/13	546063840/341	0	256	0
(7,2,-4)	288	1157760/13	694612800/341	0	54	112
(7,1,-4)	288	1304640/13	778117536/341	−68	0	162
(8,0,-4)	504	2283120/13	1985686920/341	−92	0	0
(8,3,-4)	144	2168640/13	2360177568/341	128	−189	0
(8,1,-4)	336	3024960/13	3938762016/341	0	85	0
(8,2,-4)	576	3516480/13	4303182240/341	−224	−432	0
(9,3,-5)	576	4544640/13	6765837120/341	0	0	−3564
(9,1,-5)	288	371520	8301345696/341	−112	0	−5103
(9,2,-5)	288	4752000/13	9366960960/341	112	162	−1296
(10,2,-6)	864	6557760/13	12363956640/341	0	0	14976
(10,0,-5)	720	6968160/13	14784532560/341	890	0	0
(12,6,-6)	768	8666880/13	20992277376/341	0	192	0

We can write ${\chi_{5b}}^2$ and ${\chi_{15}}^2$ as polynomials of 4 variables E_2, E_4, χ_{5a} and E_6 as follows. These are followed from the comparison of Fourier coefficients.

${\chi_{5b}}^2 = (5005/8149248) * {E_2}^5 - (15587/16298496) * {E_2}^3 E_4 - (4433/16298496) * {E_2}^2 E_6 + (1859/5432832) * E_2 {E_4}^2 + (4433/16298496) * E_4 E_6 + {\chi_{5a}}^2,$

${\chi_{15}}^2 = (7193626131746618585/222607917767232721152) * {E_2}^{15} - (307986483294442487/1426973831841235392) * {E_2}^{13} E_4 + (1416328854305111/54400761917701056) * {E_2}^{12} E_6 + (4087366592607641/6860451114621324) * {E_2}^{11} {E_4}^2 - (192607575137275/1394891331223104) * {E_2}^{10} E_4 E_6 + (50704311727294/69507316593) * {E_2}^{10} {\chi_{5a}}^2 - (52003816542174887/59873027909422464) * {E_2}^9 {E_4}^3 + (2912260461769/319066052303232) * {E_2}^9 {E_6}^2 + (1922370985523/6706208323188) * {E_2}^8 {E_4}^2 E_6 - (20825649443174/5346716661) * {E_2}^8 E_4 {\chi_{5a}}^2 + (102989732952024139/146356290445254912) * {E_2}^7 {E_4}^4 - (96923094941/2727060276096) * {E_2}^7 E_4 {E_6}^2 + (27583081580/203833773) * {E_2}^7 E_6 {\chi_{5a}}^2 - (92968372638167/321897999513024) * {E_2}^6 {E_4}^3 E_6 + (65$

$651791909/36815313727296) * E_2{}^6E_6{}^3 + (3387092572918/411285897) * E_2{}^6E_4{}^2\chi_{5a}{}^2 - (7304217732454747/24392715074209152) * E_2{}^5E_4{}^5 + (30622846693/629321602176) * E_2{}^5E_4{}^2E_6{}^2 - (256204744/505791) * E_2{}^5E_4E_6\chi_{5a}{}^2 - (10936889634816/19651489) * E_2{}^5\chi_{5a}{}^4 + (14944942065833/107299333171008) * E_2{}^4E_4{}^4E_6 - (27494911499/6135885621216) * E_2{}^4E_4E_6{}^3 - (1176607216174/137095299) * E_2{}^4E_4{}^3\chi_{5a}{}^2 + (10349644/597753) * E_2{}^4E_6{}^2\chi_{5a}{}^2 + (36987323269/710702030016) * E_2{}^3E_4{}^6 - (49717185583/1887964806528) * E_2{}^3E_4{}^3E_6{}^2 + (1709446981/8862945897312) * E_2{}^3E_6{}^4 + (773604236/1206117) * E_2{}^3E_4{}^2E_6\chi_{5a}{}^2 + (2503569715200/1511653) * E_2{}^3E_4\chi_{5a}{}^4 - (26102557/1042085088) * E_2{}^2E_4{}^5E_6 + (2820958987/943982403264) * E_2{}^2E_4{}^2E_6{}^3 + (509138188/116281) * E_2{}^2E_4{}^4\chi_{5a}{}^2 - (2420960/45981) * E_2{}^2E_4E_6{}^2\chi_{5a}{}^2 - (31993344000/57629) * E_2{}^2E_6\chi_{5a}{}^4 + (18421/4583952) * E_2E_4{}^4E_6{}^2 - (159653813/681765069024) * E_2E_4E_6{}^4 - (843440/3069) * E_2E_4{}^3E_6\chi_{5a}{}^2 - (136400/66417) * E_2E_6{}^3\chi_{5a}{}^2 - (137631744000/116281) * E_2E_4{}^2\chi_{5a}{}^4 - (4433/20627784) * E_4{}^3E_6{}^3 + (39651821/4431472948656) * E_6{}^5 - (301621736/348843) * E_4{}^5\chi_{5a}{}^2 + (1100/27) * E_4{}^2E_6{}^2\chi_{5a}{}^2 + (3018240000/4433) * E_4E_6\chi_{5a}{}^4 + (40993977139200000/19651489) * \chi_{5a}{}^6.$

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