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A note on lowness for Robinson theories

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Abstract

We show following two theorems. Theorem A: for thick simple existentially universal domain, the equality of Lascar strong types is definable by an existential type. Theorem B: for thick low existentially universal domain, Lascar strong types equal strong types. Theorem A is already proved by Ben-Yaacov [2].

1 Preliminaries

Definition 1.1 We say that an L-structure M is κ-existentially universal domain (e.u.domain) if

- if $\Sigma(x)$ is a partial existential type over A ($|A| < \kappa$) which is finitely satisfiable in M, then Σ is satisfiable in M, and
- for $|A|, |B| < \kappa$, and $f : A \to B$: a bijection such that $\mathrm{etp}(a) \subset \mathrm{etp}(f(a))$ for all tuples a from A, f extends to an automorphism of M.

Remark 1.1 *An e.u.domain M is an existentially closed model for the universal theory of M,* $\mathrm{Th}(M)_\forall$.

Let $\mathcal{M}$ be a κ-e.u.domain for a enoughly big cardinal κ. Put $T = \mathrm{Th}_\forall(\mathcal{M})$. $M, N, \dots$ denote existentially closed models of T, $a, b, \dots$ denote finite tuples in $\mathcal{M}$, and $A, B, \dots$ denote small subsets of $\mathcal{M}$.

Definition 1.2 Let $\Sigma(x, B)$ be an existential type over B.

1. We sat that $\Sigma(x, B)$ divides over A if there exists an existentially indiscernible sequence $(B_i : i < \omega)$ over A with $B_0 = B$ such that $\bigcup_{i<\omega} \Sigma(x, B_i)$ is not realized in $\mathcal{M}$.

2. We say that $\Sigma(x)$ forks over A if there exists a small set of dividing $(/A)$ existential formulas Ψ (with parameters) such that $\mathcal{M} \models \Sigma \to \bigvee \Psi$.

Remark 1.2

- *If $\Sigma(x)$ divides over A, then there is an existential formula $\varphi(x)$ such that $\Sigma \vdash \varphi(x)$ and $\varphi(x)$ divides over A.*
- *It is not known whether that if Σ forks over A, then there are an existential formula θ where $\Sigma \vdash \theta$ and dividing $(/A)$ existential formulas $\psi_1, \ldots, \psi_n$ such that $\mathcal{M} \models \theta \to \bigvee_{i=1}^{n} \psi_i$.*

Definition 1.3 We say that $\mathcal{M}$ is simple if for all $a \in \mathcal{M}$, $A \subset \mathcal{M}$, there exists $B \subset A$ with $|B| \leq |T| + \aleph_0$ such that $\mathrm{etp}(a/A)$ does not fork over B.

Fact 1.1 *[3] Suppose that $\mathcal{M}$ is simple. Then, Σ forks over A if and only if Σ divides over A.*

Definition 1.4

1. We say that $\mathrm{lstp}(a) = \mathrm{lstp}(b)$ if for any bounded $\emptyset$-invariant equivalence relation $E(x, y)$, $E(a, b)$ holds.
2. We say that $d(a, b) \leq 1$ if there is an existentially indiscernible sequence I such that $a, b \in I$.
3. We say that $d(a, b) \leq n$ if there exist $a_0, \ldots, a_n$ with $a_0 = a, a_n = b$ such that $d(a_i, a_{i+1}) \leq 1$ for any $i < n$.
4. We say that $d(a, b) < \omega$ if $d(a, b) \leq n$ for some $n < \omega$.

Fact 1.2 *[3]* $\mathrm{lstp}(a) = \mathrm{lstp}(b)$ *if and only if* $d(a, b) < \omega$.

Fact 1.3 *[3] If $(a_i : i < \lambda)$ is an enoughly long sequence and $A \subset \mathcal{M}$, then there is an existentially indiscernible sequence $(b_i : i < \omega)$ such that for any $n < \omega$, there are $i_0 < \cdots < i_{n-1} < \lambda$ such that* $\mathrm{etp}(b_0, \ldots, b_{n-1}/A) = \mathrm{etp}(a_{i_0}, \ldots, a_{i_{n-1}}/A)$.

Fact 1.4 *[3] Suppose that $\mathcal{M}$ is simple. Then, for all a, $A \subset B$, there exists a' such that*

- $\mathrm{lstp}(a'/A) = \mathrm{lstp}(a/A)$ *and*
- $\mathrm{etp}(a'/B)$ *does not fork over A.*

We write $a \mathop{\smile\!\!\!\!\!\!|}\, b$ to mean that $\mathrm{etp}(a/b)$ does not fork over $\emptyset$.

Fact 1.5 (Independence theorem for simple e.u.domain, [3]) *Suppose that $\mathcal{M}$ is simple and a_1, a_2, b_1, b_2 satisfy the following:*

- $\mathrm{lstp}(a_1) = \mathrm{lstp}(a_2)$,
- $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\, b_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}\, b_2$, $b_1 \mathop{\smile\!\!\!\!\!\!\mid}\, b_2$.

Then, there exists a such that

- $a \models \mathrm{etp}(a_1/b_1) \cup \mathrm{etp}(a_2/b_2)$
- $a \mathop{\smile\!\!\!\!\!\!\mid}\, b_1 b_2$.

2 Proof of Thorem A

In this section, we prove Tehorem A. For simplicity, we show over $\emptyset$.

Definition 2.1 We say that $\mathcal{M}$ is thick if "$d(x, y) \leq 1$" is definable by an existential type. If $\mathcal{M}$ is thick, then we assume that $q_1(x, y)$ defines "$d(x, y) \leq 1$".

Lemma 2.1 *Suppose that $\mathcal{M}$ is thick. Then, "$d(x, y) \leq 2$" is definable by an existential type.*

Proof: It is defined by $\{\exists z \varphi(x, z) \wedge \varphi(z, y) |\ \varphi(x, y) \in q_1(x, y)\}$.

Lemma 2.2 *Suppose that $\mathcal{M}$ is thick and simple. Then, the following are equivalent:*

1. $\mathrm{lstp}(a) = \mathrm{lstp}(b)$
2. $d(a, b) \leq 2$
3. $q_1(x, a) \cup q_1(x, b)$ *does not fork over* $\emptyset$

Proof: $(3 \to 2 \to 1)$ is trivial. $(1 \to 2)$ Let c be a tuple such that $\mathrm{lstp}(c) = \mathrm{lstp}(a) = \mathrm{lstp}(b)$ and $c \mathop{\smile\!\!\!\!\!\!\mid}\, ab$. Take a' such that $\mathrm{etp}(a'a) = \mathrm{etp}(ac)$. Then $\mathrm{lstp}(a') = \mathrm{lstp}(a)$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}\, a$. So, by independence theorem, we can get a_2 such that $a_2 \models \mathrm{etp}(a/c) \cup \mathrm{etp}(a'/a)$ and $a_2 \mathop{\smile\!\!\!\!\!\!\mid}\, ac$.

Iterating this, we can get a sequence $(a_i : i < \omega)$ such that $\mathrm{etp}(a_i a_j) = \mathrm{etp}(ac)$ for each $j < i < \omega$. By compactness and Fact 1.3, we can assume this sequence is existentially indiscernible. So, we get existentially indiscernible sequences I, J such that $a, c \in I$ and $b, c \in J$.

Theorem A *[2] Suppose that $\mathcal{M}$ is thick and simple. Then, "*$\mathrm{lstp}(x) = \mathrm{lstp}(y)$*" is definable by an existential type.*

Proof: By above lemmas.

3 Proof of Theorem B

In this section, we prove Tehorem B. Again for simplicity, we show over $\emptyset$.

Definition 3.1 We say that $\mathrm{stp}(a) = \mathrm{stp}(b)$ if for any definable (by an existential formula over $\emptyset$) finite equivalence relation $E(x, y)$, $E(a, b)$ holds.

Definition 3.2

1. Let $\varphi(x, y)$ be an existential formula. An existential formula $\psi(y_0, \ldots, y_{k-1})$ where $\mathrm{lh}(y_i) = \mathrm{lh}(y)$ for each $i < k$ is said to be a k-inconsistency witness for φ if $\mathcal{M} \models \forall y_0 \ldots y_{k-1}(\psi(y_0, \ldots, y_{k-1}) \to \neg\exists x \bigwedge_{i<k} \varphi(x, y_i))$.
2. Let $\Sigma(x)$ be an existential type and $\varphi(x, y)$ be an existential formula.
   - We say that $D(\Sigma, \varphi) \geq 0$ if Σ is satisfiable.
   - We say that $D(\Sigma, \varphi) \geq n + 1$ if there is a natural number k, a k-inconsistency witness ψ, and an existentially indiscernible sequence $(b_i : i < \omega)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi) \geq n$ for each $i < \omega$ and $\mathcal{M} \models \psi(b_{i_0}, \ldots b_{i_{k-1}})$ for all $i_0, \ldots, i_{k-1} < \omega$.
3. We say that $\mathcal{M}$ is low if
   - $\mathcal{M}$ is simple and
   - $D(x = x, \varphi) < \omega$ for any existentiall formula φ.

Lemma 3.1 *Suppose that $\mathcal{M}$ is thick and low. Then,*

1. *$\{a : \varphi(x, a)$ divides over $\emptyset\}$ is definable by an existential type.*
2. *$\{(a, b) : \varphi(x, a) \wedge \varphi(x, b)$ does not divide over $\emptyset\}$ is definable by an existential type if it is restricted to $(p \otimes p)^{\mathcal{M}} = \{(a, b) : a, b \models p, a \downarrow b\}$. So, it is definable by an existential universal formula if it is restricted to $(p \otimes p)^{\mathcal{M}}$*

Proof: (1) Note that by lowness, for any $\varphi(x, y)$ there is an existentiall formula ψ such that for all a, if $\varphi(x, a)$ divides over $\emptyset$, then φ devides by an existentially indiscernible sequence in which any k-elements satisfies ψ.

(2) For $a, b \models p$ where $a \downarrow b$, the following are equivalent:

1. $\varphi(x,a) \wedge \varphi(x,b)$ does not divide over $\emptyset$
2. there exist a^* and b^* such that
   - $\mathcal{M} \models \varphi(a^*, a)$ and $a^* \perp a$;
   - $\mathcal{M} \models \varphi(b^*, b)$ and $b^* \perp b$;
   - $\text{lstp}(a^*) = \text{lstp}(b^*)$

By Theorem A, "$\text{lstp}(a^*) = \text{lstp}(b^*)$" is expressible by an existential type. "$a^* \perp a$" is expressible by "$D(\text{etp}(a/a^*), \varphi, \psi) \geq D(p, \varphi, \psi)$" for any φ, ψ.

We sat that $E_{p(x),\varphi(x,y)}(b,c)$ if for all $a \models p$ with $a \perp bc$, $\varphi(x,a) \wedge \varphi(x,b)$ does not divide over $\emptyset$ if and only if $\varphi(x,a) \wedge \varphi(x,c)$ does not divide over $\emptyset$.

Lemma 3.2 *Suppose that $\mathcal{M}$ is thick and low. For any $a \models p$ where $\varphi(x,a)$ does not divide over $\emptyset$, $E_{p(x),\varphi(x,y)}$ is a definable (by an existential formula) finite equivalence relation on $(p^2)^{\mathcal{M}}$.*

Proof: We can check that $E_{p,\varphi}$ is a bounded equivalence relation boundedness is by "$\text{lstp}(x) = \text{lstp}(y) \Rightarrow E_{p,\varphi}(x,y)$". On the other hand, by the above lemma $\neg E_{p,\varphi}$ is definable by an existential type. So, $E_{p,\varphi}$ is a finite equivalence relation. Let $a_1, \ldots, a_n$ be representations of classes. Then $\bigcup\{\neg E(x, a_i) : i \leq n\}$ is not satisfiable. For simplicity, we assume $n = 3$. There exists an eixistential formula $\varphi(x,y)$ such that

1. $\neg E(x, a_i) \vdash \varphi(x, a_i)$ for each $i \leq 3$
2. $\mathcal{M} \models \neg\exists x \varphi(x, a_1) \wedge \varphi(x, a_2) \wedge \varphi(x, a_3)$.

Put $\psi(x,y) = \neg\varphi(x,y)$. Note that $\mathcal{M} \models \forall x(\psi(x,a_1) \leftrightarrow \varphi(x,a_2) \wedge \varphi(x,a_3))$. So, $\psi(x,a_1)$ is also existential. By a symmetric argument, $\psi(x,a_2), \psi(x,a_3)$ are all existential. Then we have

$$E(x,y) \leftrightarrow \bigwedge_{i \leq 3} (\psi(x,a_i) \leftrightarrow \psi(y,a_i)).$$

We can omit parameters a_i's because this does not depend on a choice of representations and $\psi(x,a_i)$ is existential universal.

Theorem B *Suppose that $\mathcal{M}$ is thick and low. Then,* $\text{stp} = \text{lstp}$

Proof: If $\text{stp}(a) = \text{stp}(b)$, then by the above lemma $a, b \models E_{p,\varphi}$ for any φ. Take c such that $\text{lstp}(c) = \text{lstp}(a)$ and $c \perp ab$. Then, $q_1(x,a) \cup q_1(x,c)$ does not divide by Lemma 3. Then, $q_1(x,b) \cup q_1(x,c)$ does not divide by $E_{p,\varphi}(a,b)$. Again by Lemma 3, we have $\text{lstp}(b) = \text{lstp}(c)$.

References

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