数理解析研究所講究録
第 1723 巻 2011 年 101-109

複素領域での非線型偏微分方程式の解の特異点について

On the Singularities of Solutions of Nonlinear Partial Differential Equations in the Complex Domain

By

田原 秀敏 (Hidetoshi TAHARA)*

Abstract

この論文は,

$$(\partial/\partial t)^m u = F(t,x,\{(\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+|\alpha|\le m, j<m}), \quad (t,x)\in\mathbb{C}\times\mathbb{C}^n$$

という,複素領域での非線型偏微分方程式について,次の問題を論じている.「上の方程式の解で $S=\{t=0\}$ 上にのみ特異点を持つものは存在するのか?」

1. 特異点の非存在の研究は,解の解析接続によって論じられ,

2. 特異点の存在は,実際に S 上に特異点を持つ解の構成によって論じられる.

本稿は,この問題についての概説 (survey article) である.

This paper considers the following nonlinear partial differential equation

$$(\partial/\partial t)^m u = F(t,x,\{(\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+|\alpha|\le m, j<m}), \quad (t,x)\in\mathbb{C}\times\mathbb{C}^n,$$

in the complex domain. The main purpose is to examine whether or not the equation possesses solutions which admit singularities only on the hypersurface $S=\{t=0\}$. This will be done either by examining the possibility of analytic continuation of solutions or by actually constructing solutions that possess singularities only on S. This is a survey article of this problem.

§1. Introduction

Let $\mathbb{C}$ be the complex plane or the set of all complex numbers, t the variable in $\mathbb{C}_t$, and $x=(x_1,\dots,x_n)$ the variable in $\mathbb{C}^n_x=\mathbb{C}_{x_1}\times\cdots\times\mathbb{C}_{x_n}$. We use the notation: $\mathbb{N}=\{0,1,2,\dots\}$, $\mathbb{N}^*=\{1,2,\dots\}$, $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and $(\partial/\partial x)^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$.

2000 Mathematics Subject Classification(s): Primary: 35A20, Secondary: 35A10, 35B40.
Key Words: Nonlinear PDEs, holomorphic solutions, singularities.
*上智大学理工学部 (Department of Mathematics, Sophia University).

Let $m \in \mathbb{N}^*$ be fixed, set $N = \#\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m \text{ and } j < m\}$, and denote by $Z = \{Z_{j,\alpha}\}_{j+|\alpha|\leq m, j<m}$ the variable in $\mathbb{C}^N$.

Let $F(t,x,Z)$ be a function in the variables (t,x,Z) defined in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_Z^N$. In this paper we will consider the following nonlinear partial differential equation

$$\left(\frac{\partial}{\partial t}\right)^m u = F\left(t,x,\left\{\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha|\leq m \\ j<m}}\right) \tag{1.1}$$

with the unknown function $u = u(t,x)$.

For simplicity, let Ω be an open neighborhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$, and we assume:

(A) $F(t,x,Z)$ is a holomorphic function on $\Omega \times \mathbb{C}_Z^N$.

The following theorem is one of the most fundamental results in the theory of partial differential equations in the complex domain:

Theorem 1.1 (Cauchy-Kowalewski Theorem). *For any holomorphic functions $\varphi_0(x)$, $\varphi_1(x)$, $\dots, \varphi_{m-1}(x)$ in a neighborhood $x = 0$ the equation* (1.1) *has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$ satisfying*

$$\left(\frac{\partial}{\partial t}\right)^i u\Big|_{t=0} = \varphi_i(x), \quad i = 0,1,\dots,m-1.$$

By this theorem we see that the holomorphic solutions of (1.1) in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$ are completely characterized by the initial data $\varphi_0(x),\dots,\varphi_{m-1}(x)$.

But if we include into consideration the singualr solutions (that is, the solutions with some singularities) the structure of the solutions of (1.1) will become much more interesting.

In this paper we will study the following problem:

Problem 1.2. Does (1.1) admit solutions which possess singularities only on the hypersurface $S = \{t = 0\}$?

One method of arguing the non-existence of such solutions is by means of analytic continuation. We set $\Omega_+ = \{(t,x) \in \Omega; \mathrm{Re}\, t > 0\}$. If the equation (1.1) is linear, we have:

Theorem 1.3 (Zerner [13], 1971). *If the equation* (1.1) *is linear, any solution which is holomorphic on Ω_+ can be extended analytically up to some neighborhood of the origin* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$. *In other words, there does not exist a solution with singularities only on S.*

If the equation (1.1) is nonlinear, we have the following nonlinear analogue of Zerner's theorem due to Tsuno.

Theorem 1.4 (Tsuno [12], 1975). *If $u(t,x)$ is a holomorphic solution of* (1.1) *on Ω_+ and if $(\partial^i u/\partial t^i)(t,x)$ $(i = 0,1,\dots,m-1)$ are all bounded on Ω_+, then the solution can be analytically continued up to some neighborhood of the origin. In other words, there does not exist a solution which possesses singularities only on S with growth order $(\partial^i u/\partial t^i)(t,x) = O(1)$ (as $t \longrightarrow 0$) for $i = 0,1,\dots,m-1$.*

The assumption that $u(t,x)$ and all its derivatives with respect to t up to order $m-1$ are bounded in some neighborhood of the origin seemed too strong to other researchers at that time. Some might have believed that Zerner's result can be extended to the nonlinear case without any additional assumption. However, this is not possible if the equation is nonlinear, as can be seen in the following example:

Example 1.5. Let $(t,x) \in \mathbb{C}^2$. The equation

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^p \quad \text{with } p \in \mathbb{N}^*(=\{1,2,\ldots\}) \tag{1.2}$$

has a family of solutions $u(t,x) = (-1/p)^{1/p}(x+c)/t^{1/p}$ with an arbitrary $c \in \mathbb{C}$. Clearly, this has singularities only on $\{t=0\}$.

Thus, for the equation (1.2) we see the following:

(i) the singularities on $\{t=0\}$ of order $u(t,x) = O(1)$ (as $t \longrightarrow 0$) do not appear in the solution of (1.2), but

(ii) there really appear singularities on $\{t=0\}$ of order $u(t,x) = O(|t|^{-1/p})$ (as $t \longrightarrow 0$) in the solution of (1.2).

Hence, for nonlinear equations it seems better to reformulate our problem in the following form:

Problem 1.6. Let s be a real number. Does (1.1) admit solutions which possess singularities only on $S = \{t=0\}$ with growth order $O(|t|^s)$ (as $t \longrightarrow 0$)?

In view of this problem, Tsuno's result is stated in the following form:

Corollary 1.7 (Corollary to Tsuno's theorem). *If $u(t,x)$ is a holomorphic solution of (1.1) on Ω_+ and if $u(t,x) = O(|t|^{m-1})$ (as $t \longrightarrow 0$) uniformly in x in some neighborhood of $x=0$, then the solution can be analytically continued up to some neighborhood of the origin. In other words, there does not exist a solution which possesses singularities only on S with growth order $u(t,x) = O(|t|^{m-1})$ (as $t \longrightarrow 0$).*

If $s \geq m-1$ holds, by Corollary 1.7 we conclude that such singularities do not appear in the solutions of (1.1). Therefore we have only to consider the case $s < m-1$ from now. In the case $s < 0$ the solution may tend to ∞ (as $t \longrightarrow 0$); this is the reason why we suppose in (A) that $F(t,x,Z)$ is entire with respect to Z.

§ 2. Non-Existence of Singularities

Suppose the condition (A). Set $I_m = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n\,;\, j+|\alpha| \leq m, j<m\}$. We may expand the function $F(t,x,Z)$ into the following convergent power series:

$$F(t,x,Z) = \sum_{\nu \in \Delta} a_\nu(t,x) Z^\nu = \sum_{\nu \in \Delta} t^{k_\nu} b_\nu(t,x) Z^\nu,$$

where

$$Z^\nu = \prod_{(j,\alpha) \in I_m} (Z_{j,\alpha})^{\nu_{j,\alpha}},$$

$a_\nu(t,x)$ and $b_\nu(t,x)$ are all holomorphic functions on Ω, and k_ν are non-negative integers. In the summation above, the set Δ has elements of the form $\nu = (\nu_{j,\alpha})_{(j,\alpha)\in I_m}$ and is a subset of $\mathbb{N}^N$; we have omitted from Δ those multi-indices ν for which $a_\nu(t,x) \equiv 0$. Moreover, we have taken out the maximum power of t from each coefficient $a_\nu(t,x)$ so that we have $b_\nu(0,x) \not\equiv 0$ for all $\nu \in \Delta$. Using this expansion, we can now write our partial differential equation as

$$\left(\frac{\partial}{\partial t}\right)^m u = \sum_{\nu\in\Delta} t^{k_\nu} b_\nu(t,x)\left[\prod_{(j,\alpha)\in I_m}\left(\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right)^{\nu_{j,\alpha}}\right].$$

We set

$$\gamma_t(\nu) = \sum_{(j,\alpha)\in I_m} j\nu_{j,\alpha}, \quad \nu\in\mathbb{N}^N$$

which is the total number of derivatives with respect to t in the term

$$\prod_{(j,\alpha)\in I_m}\left(\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right)^{\nu_{j,\alpha}}. \tag{2.1}$$

Since the highest order of differentiation with respect to t appearing in (2.1) is at most $m-1$, we have $\gamma_t(\nu) \leq (m-1)|\nu|$.

We set

$$\Delta_2 = \{\nu\in\Delta;\, |\nu|\geq 2\}.$$

If $\Delta_2 = \varnothing$, this implies that the equation is linear and we can apply Zerner's theorem. If $\Delta_2 \neq \varnothing$, the equation is nonlinear; in this case we set

$$\sigma = \sup_{\nu\in\Delta_2}\frac{-k_\nu - m + \gamma_t(\nu)}{|\nu|-1}. \tag{2.2}$$

This was introduced by Kobayashi [5]; we call this σ as *Kobayashi index.* For a neighborhood ω of $x = 0 \in \mathbb{C}^n_x$ and a function $f(t,x)$ we define the norm $\|f(t)\|_\omega = \sup_{x\in\omega}|f(t,x)|$. Then we have the following result (originally by Kobayashi [5], and improved by Lope-Tahara [6]):

Theorem 2.1. *Suppose the conditions* (A) *and* $\Delta_2 \neq \varnothing$. *Let* σ *be the Kobayashi index given in* (2.2). *If* $u(t,x)$ *is a holomorphic solution on* Ω_+ *and if* $\|u(t)\|_\omega = o(|t|^\sigma)$ *(as* $t \longrightarrow 0$), *then* $u(t,x)$ *can be extended analytically up to some neighborhood of the origin* $(0,0)\in\mathbb{C}_t\times\mathbb{C}^n_x$.

Hence we can get the following result on the non-existence of the singularities on $S = \{t = 0\}$.

Corollary 2.2. *Suppse the conditions* (A) *and* $\Delta_2 \neq \varnothing$. *Let* σ *be the Kobayashi index given in* (2.2). *Then, there appears no singularities on S with growth order* $o(|t|^\sigma)$ *(as* $t \longrightarrow 0$) *in the solutions of* (1.1).

In the equation (1.2) the number σ may be verified to be equal to $-1/p$. Hence, by the above result we see that the singularities of order $o(|t|^{-1/p})$ do not appear in the solutions of (1.2). Note further that the singularities of the solution $u(t,x) = (-1/p)^{1/p}(x+c)/t^{1/p}$ has growth order $O(|t|^{-1/p})$ (as $t \longrightarrow 0$). Thus in the case (1.2) the number $\sigma = -1/p$ is just the critical value.

§ 3. On the Singularities with Growth Order $O(|t|^{\sigma})$

In the previous section, we have shown that there appear no singularities on $S=\{t=0\}$ with growth order $o(|t|^{\sigma})$ (as $t \longrightarrow 0$) in the solutions of (1.1). But how about the singularities with growth order $O(|t|^{\sigma})$ (as $t \longrightarrow 0$)? In this section, we will study singular solutions with growth order $O(|t|^{\sigma})$ on the hypersurface S.

Set

$$\mathcal{M}=\left\{\nu \in \Delta_2 \,;\, \sigma=\frac{-k_\nu-m+\gamma_t(\nu)}{|\nu|-1}\right\}. \tag{3.1}$$

If $\mathcal{M}=\varnothing$, we have the following result on the singularities with growth order $O(|t|^{\sigma})$ (as $t \longrightarrow 0$).

Theorem 3.1 ([5], [6]). *Suppose the condition* (A) *and* $\Delta_2 \neq \varnothing$. *If* $\mathcal{M}=\varnothing$ *and if a holomorphic solution* $u(t,x)$ *of* (1.1) *in* Ω_+ *satisfies* $\|u(t)\|_\omega = O(|t|^{\sigma})$ *(as* $t \longrightarrow 0$*), then* $u(t,x)$ *can be extended analytically up to some neighborhood of the origin.*

This implies that in the case $\mathcal{M}=\varnothing$ there appear no singularities on S with growth order $O(|t|^{\sigma})$ (as $t \longrightarrow 0$) in the solutions of (1.1).

The following equation gives an example with $\mathcal{M}=\varnothing$:

Example 3.2. Let $(t,x) \in \mathbb{C}^2$ and consider the first-order nonlinear equation

$$\frac{\partial u}{\partial t}=e^{u}\left(\frac{\partial u}{\partial x}\right).$$

In this case, it is easily checked that $\sigma=0$ and $\mathcal{M}=\varnothing$. Therefore by Theorem 3.1 we see that this equation has no singular solutions with growth order $O(1)$ (as $t \longrightarrow 0$), which is just the same result as in Tsuno's theorem.

In the case $\mathcal{M} \neq \varnothing$, the growth condition $\|u(t)\|_\omega = o(|t|^{\sigma})$ (as $t \longrightarrow 0$) may not be weakened, say by assuming that we only have $\|u(t)\|_\omega = O(|t|^{\sigma})$ (as $t \longrightarrow 0$). Let us consider the equation (1.2); in this equation, $m=1$, $k_{(1,p)}=0$, $\sigma=-1/p$ and $\mathcal{M} \neq \varnothing$. Note that the solution $u(t,x)=(-1/p)^{1/p}(x+c)t^{-1/p}$ has singulatities only on $\{t=0\}$ with just the large order of $|t|^{\sigma}$.

Since k_ν is nonnegative and $\gamma_t(\nu) \le (m-1)|\nu|$, it follows that

$$\frac{-k_\nu-m+\gamma_t(\nu)}{|\nu|-1} \le \frac{-k_\nu-m+(m-1)|\nu|}{|\nu|-1}=\frac{-k_\nu-1+(m-1)(|\nu|-1)}{|\nu|-1}<m-1 \tag{3.2}$$

and so $\sigma \le m-1$. Moreover (3.2) yields that if $\sigma=m-1$ we have $\mathcal{M}=\varnothing$. This easily leads us to the following result (which is the same as Corollary 1.7).

Corollary 3.3. *If* $u(t,x)$ *is a holomorphic solution on* Ω_+ *and if* $\|u(t)\|_\omega = O(|t|^{m-1})$ *(as* $t \longrightarrow 0$*), then* $u(t,x)$ *can be extended analytically up to some neighborhood of the origin* $(0,0) \in \mathbb{C}_t \times \mathbb{C}^n_x$. *Hence, there does not exist a solution which possesses singularities only on* S *with growth order* $u(t,x)=O(|t|^{m-1})$ *(as* $t \longrightarrow 0$*).*

In the next section we will consider the following problem:

Problem 3.4. In the case $\Delta_2 \neq \varnothing$ and $\mathcal{M} \neq \varnothing$, does (1.1) has a solution $u(t,x)$ with singularities only on $\{t=0\}$ of the growth order $O(|t|^{\sigma})$ (as $t \longrightarrow 0$)?

§ 4. Construction of a Solution with Singularities

Now, suppose the conditions (A), $\Delta_2 \neq \varnothing$ and $\mathcal{M} \neq \varnothing$; then σ is a rational number and $\sigma < m-1$. Set

$$P(x,Z) = \sum_{\nu \in \mathcal{M}} b_\nu(0,x) Z^\nu \tag{4.1}$$

which is a holomorphic function on $(\Omega \cap \{t=0\}) \times \mathbb{C}_Z^N$. Note that $|\nu| \geq 2$ holds for all $\nu \in \mathcal{M}$. By the definition of σ we have

Lemma 4.1. (1) $k_\nu + m + \sigma(|\nu|-1) - \gamma_t(\nu) \geq 0$ *holds for all* $\nu \in \Delta$.
(2) $k_\nu + m + \sigma(|\nu|-1) - \gamma_t(\nu) = 0$ *holds if and only if* $\nu \in \mathcal{M}$.

In the case $\sigma \neq 0, 1, \ldots, m-2$, one way to prove the existence of singularities of the growth order $O(|t|^\sigma)$ on S is to construct a solution $u(t,x)$ of (1.1) in the form

$$u(t,x) = t^\sigma \big(\varphi(x) + w(t,x)\big) \tag{4.2}$$

where $\varphi(x)$ is a holomorphic function in a neighborhood of $x=0$ with $\varphi(x) \not\equiv 0$, and $w(t,x)$ is a function belonging in the class $\widetilde{\mathcal{O}}_+$ which is defined by Definition 4.2 given below. Denote:

- $\mathcal{R}(\mathbb{C} \setminus \{0\})$ the universal covering space of $\mathbb{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\})\ ;\ |\arg t| < \theta\}$ a sector in $\mathcal{R}(\mathbb{C} \setminus \{0\})$,
- $S(\varepsilon(s)) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\})\ ;\ 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on $\mathbb{R}_s$,
- $D_R = \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n\ ;\ |x_i| \leq R \text{ for } i = 1, \cdots, n\}$,

Definition 4.2. A function $w(t,x)$ is said to be in the class $\widetilde{\mathcal{O}}_+$ if it satisfies the following conditions c_1) and c_2): c_1) $w(t,x)$ is a holomorphic function in the domain $S(\varepsilon(s)) \times D_R$ for some positive-valued continuous function $\varepsilon(s)$ on $\mathbb{R}_s$ and $R > 0$; c_2) there is an $a > 0$ such that for any $\theta > 0$ we have $\max_{|x| \leq R} |w(t,x)| = O(|t|^a)$ (as $t \longrightarrow 0$ in S_θ).

The construction of a solution of the form (4.2) is as follows. By substituting (4.2) into (1.1) and then by cancelling the factor $t^{\sigma - m}$ from the both sides we have

$$\begin{aligned} &[\sigma]_m \varphi + \left[t\frac{\partial}{\partial t} + \sigma\right]_m w \\ &= \sum_{\nu \in \Delta} t^{k_\nu + m + \sigma(|\nu|-1) - \gamma_t(\nu)} b_\nu(t,x) \left[\prod_{(j,\alpha) \in I_m} \left([\sigma]_j \Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi + \left[t\frac{\partial}{\partial t} + \sigma\right]_j \Big(\frac{\partial}{\partial x}\Big)^\alpha w\right)^{\nu_{j,\alpha}}\right] \end{aligned}$$

where $[\lambda]_0 = 1$ and $[\lambda]_p = \lambda(\lambda-1)\cdots(\lambda-p+1)$ for $p \geq 1$. By Lemma 4.1 we see that this equation is written in the form

$$(4.3)\qquad [\sigma]_m\varphi+\left[t\frac{\partial}{\partial t}+\sigma\right]_m w$$
$$=\sum_{\nu\in\mathcal{M}} b_\nu(t,x)\left[\prod_{(j,\alpha)\in I_m}\left([\sigma]_j\left(\frac{\partial}{\partial x}\right)^\alpha\varphi+\left[t\frac{\partial}{\partial t}+\sigma\right]_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right)^{\nu_{j,\alpha}}\right]$$
$$+\sum_{\nu\in\Delta\setminus\mathcal{M}} t^{k_\nu+m+\sigma(|\nu|-1)-\gamma_t(\nu)}b_\nu(t,x)\times$$
$$\times\left[\prod_{(j,\alpha)\in I_m}\left([\sigma]_j\left(\frac{\partial}{\partial x}\right)^\alpha\varphi+\left[t\frac{\partial}{\partial t}+\sigma\right]_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right)^{\nu_{j,\alpha}}\right].$$

Since we are now considering a function $w(t,x)\in\widetilde{\mathcal{O}}_+$, we have $w(t,x)\longrightarrow 0$ (as $t\longrightarrow 0$ uniformly in x) and so by letting $t\longrightarrow 0$ in (4.3) we obtain

$$(\mathrm{I})\qquad [\sigma]_m\varphi=P\left(x,\left\{[\sigma]_j\left(\frac{\partial}{\partial x}\right)^\alpha\varphi\right\}_{(j,\alpha)\in I_m}\right)$$

which is a partial differential equation with respect to the unknown function $\varphi(x)$. Then, subtracting the equation (I) from (4.3) we obtain

$$(\mathrm{II})\qquad \left[t\frac{\partial}{\partial t}+\sigma\right]_m w=\sum_{(j,\alpha)\in I_m}\frac{\partial P}{\partial Z_{j,\alpha}}\left(x,\left\{[\sigma]_i\varphi^{(\beta)}\right\}_{(i,\beta)\in I_m}\right)\left[t\frac{\partial}{\partial t}+\sigma\right]_j\left(\frac{\partial}{\partial x}\right)^\alpha w$$
$$+G_2\left(x,\left\{[\sigma]_j\varphi^{(\alpha)}\right\}_{(j,\alpha)\in I_m},\left\{\left[t\frac{\partial}{\partial t}+\sigma\right]_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right\}_{(j,\alpha)\in I_m}\right)$$
$$+t^{1/L}R\left(t^{1/L},x,\left\{[\sigma]_j\varphi^{(\alpha)}+\left[t\frac{\partial}{\partial t}+\sigma\right]_j\left(\frac{\partial}{\partial x}\right)^\alpha w\right\}_{(j,\alpha)\in I_m}\right),$$

where L is a positive integer such that $\sigma\in\mathbb{Z}/L$, $\varphi^{(\beta)}=(\partial/\partial x)^\beta\varphi$, $\varphi^{(\alpha)}=(\partial/\partial x)^\alpha\varphi$,

$$G_2(x,Z,W)=\sum_{|\mu|\geq 2}\frac{1}{\mu!}\left(\left(\frac{\partial}{\partial Z}\right)^\mu P\right)(x,Z)W^\mu,$$
$$R(s,x,Z)=s^{L-1}\sum_{\nu\in\mathcal{M}}c_\nu(s^L,x)Z^\nu+\sum_{\nu\in\Delta\setminus\mathcal{M}}s^{l_\nu}b_\nu(s^L,x)Z^\nu$$

with $Z=\{Z_{j,\alpha}\}_{(j,\alpha)\in I_m}$, $W=\{W_{j,\alpha}\}_{(j,\alpha)\in I_m}$, $\mu=\{\mu_{j,\alpha}\}_{(j,\alpha)\in I_m}$, $c_\nu(t,x)=(b_\nu(t,x)-b_\nu(0,x))/t$, and $l_\nu=L(k_\nu+m+\sigma(|\nu|-1)-\gamma_t(\nu))-1$.

Proposition 4.3. *If* $\sigma\neq 0,1,\ldots,m-2$, *if the equation* (I) *has a holomorphic solution* $\varphi(x)$ *with* $\varphi(x)\not\equiv 0$, *and moreover if the equation* (II) *has a solution* $w(t,x)\in\widetilde{\mathcal{O}}_+$, *then we can conclude that the original equation* (1.1) *has a solution* $u(t,x)$ *which has really singularities only on* $\{t=0\}$ *with the growth order* $|t|^\sigma$.

Thus, to construct a solution in the form (4.2), it is sufficient to consider the following problem:

Problem 4.4. When the assumption in Proposition 4.3 holds?

In the case $m = 1$, this problem is solved in [7] and [8]. In the general case, Kobayashi [5] gives a sufficient condition; but still there are many equations which do not satisfy the condition in [5] and for which the problem 4.4 is affirmative. On the present situation of the research, see §6.

§ 5. In the case $\sigma = 0, 1, \dots, m-2$

In Proposition 4.3 we have excluded the case $\sigma = 0, 1, \dots, m-2$. In the case $\sigma = 0, 1, \dots, m-2$, instead of (4.2) it will be better to consider

$$u(t,x) = t^{\sigma}(a(x)\log t + b(x) + w(t,x)) \tag{5.1}$$

where $a(x)$ and $b(x)$ are holomorphic functions in a neighborhood of $x = 0$, $a(x) \not\equiv 0$, and $w(t,x)$ is a function belonging in the class $\widetilde{\mathcal{O}}_+$.

Tahara-Yamane [10] gives a sufficient condition for the equation to have a singular solution of the form (5.1) in the case $\sigma = 0, 1, \dots, m-2$; the condition in [10] corresponds to the one in [5]. Still there are many equations which do not satisfy the condition in [10].

§ 6. Supplement on the Equation (II)

We note that by the change of variable $t^{1/L} \longrightarrow t$ the equation (II) is transformed into a holomorphic equation of the form

$$\left(t\frac{\partial}{\partial t}\right)^m w = H\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^{\alpha} w\right\}_{\substack{j+|\alpha|\le m \\ j<m}}\right) \tag{6.1}$$

where $H(t,x,Z)$ be a holomorphic function in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_Z^N$ satisfying $H(0,x,0) \equiv 0$ in a neighborhood Δ_0 of $x = 0$. Therefore, if we can prove the existence of a solution $w(t,x) \in \widetilde{\mathcal{O}}_+$ of this equation (6.1), it will help to solve the problem 4.4.

We set $I_m(+) = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n ; j + |\alpha| \le m, j < m \text{ and } |\alpha| > 0\}$; for convenience, we will divide our equation (6.1) into the following three cases:

Case 1 : $\dfrac{\partial H}{\partial Z_{j,\alpha}}(0,x,0) \equiv 0$ on Δ_0 for all $(j,\alpha) \in I_m(+)$;

Case 2 : $\dfrac{\partial H}{\partial Z_{j,\alpha}}(0,0,0) \neq 0$ for some $(j,\alpha) \in I_m(+)$;

Case 3 : the other cases.

In Case 1 the equation (6.1) is recently called a *Gérard-Tahara type* partial differential equation (or before it was called a *nonlinear Fuchsian type* partial differential equation), in Case 2 the equation (6.1) is called a *spacially nondegenerate type* partial differential equation, and in Case 3 the equation (6.1) is called a *nonlinear totally characteristic type* partial differential equation.

As to the solvability in $\widetilde{\mathcal{O}}_+$ we can see:

Case 1	Gérard-Tahara [2], Gérard-Tahara [4], Tahara-Yamazawa [11] Tahara-Yamane [10]
Case 2	Gérard-Tahara [3]
Case 3	Chen-Tahara [1], Tahara [9]

References

[1] Chen, H. and Tahara, H., On totally characteristic type non-linear partial differential equations in the complex domain, *Publ. Res. Inst. Math. Sci.* **35** (1999), 621-636

[2] Gérard, R. and Tahara, H., Solutions holomorphes et singulieres d'equations aux derivees partielles singulieres non lineaires, *Publ. Res. Inst. Math. Sci.* **29** (1993), 121-151.

[3] ———, Holomorphic and singular solutions of non-linear singular partial differential equations, II, Structure of Solutions of Differential Equations, Katata/Kyoto, 1995 (edited by Morimoto and Kawai), World Sci., 1996, pp. 135–150.

[4] ———, Singular nonlinear partial differential equations, *Aspects of Mathematics, E 28, Vieweg-Verlag*, 1996.

[5] Kobayashi, T., Singular solutions and prolongation of holomorphic solutions to nonlinear differential equations, *Publ. Res. Inst. Math. Sci.* **34** (1998), 43–63.

[6] Lope, J. E. C. and Tahara, H., On the analytic continuation of solutions to nonlinear partial differential equations, *J. Math. Pures Appl.* **81** (2002), 811–826.

[7] Tahara, H., On the singularities of solutions of nonlinear partial differential equations in the complex domain, Microlocal Analysis and Complex Fourier Analysis (edited by T. Kawai and K. Fujita), World Sci., 2002, pp 273–283.

[8] ———, On the singularities of solutions of nonlinear partial differential equations in the complex domain, II, Differential Equations and Asymptotic Theory in Mathematical Physics (edited by H. Chen and R.Wong), *Series in Analysis* **2**, World Sci., 2004, pp. 343–354.

[9] ———, Solvability of partial differential equations of nonlinear totally characteristic type with resonances, *J. Math. Soc. Japan* **55** (2003), 1095–1113.

[10] Tahara, H. and Yamane, H., Logarithmic singularities of solutions to nonlinear partial differential equations, *J. Math. Soc. Japan* **60** (2008), 603–630.

[11] Tahara, H. and Yamazawa, H., Structure of solutions of nonlinear partial differential equations of Gerard-Tahara type, *Publ. Res. Inst. Math. Sci.* **41** (2005), 339–373.

[12] Tsuno, Y., On the prolongation of local holomorphic solutions of nonlinear partial differential equations, *J. Math. Soc. Japan* **27** (1975), 454-466.

[13] Zerner, M., Domaines d'holomorphie des fonctions vérifiant une équation aux derivées partielles, *C. R. Acad. Sci. Paris Sér. I. Math.* **272** (1971), 1646–1648.