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The Fekete-Szegö problem for p-valently Janowski starlike and convex functions

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Abstract

For p-valently Janowski starlike and convex functions defined by applying subordination for the generalized Janowski function, the sharp upper bounds of a functional $|a_{p+2}-\mu a_{p+1}^{2}|$ related to the Fekete-Szegö problem are given.

1 Introduction

Let $\mathcal{A}_p$ denote the family of functions $f(z)$ normalized by

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n} \qquad (p=1,2,3,\cdots) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Furtheremore, let $\mathcal{W}$ be the class of functions $w(z)$ of the form

$$w(z)=\sum_{k=1}^{\infty}w_{k}z^{k} \tag{1.2}$$

which are analytic and satisfy $|w(z)|<1$ in $\mathbb{U}$. Then, a function $w(z)\in\mathcal{W}$ is called the Schwarz function. If $f(z)\in\mathcal{A}_p$ satisfies the following condition

$$\mathrm{Re}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right]>0 \qquad (z\in\mathbb{U})$$

for some complex number b $(b\neq 0)$, then $f(z)$ is said to be p-valently starlike function of complex order b. We denote by $\mathcal{S}_b^*(p)$ the subclass of $\mathcal{A}_p$ consisting of all functions $f(z)$ which are p-valently starlike functions of complex order b. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}_b(p)$ of p-valently convex functions of complex order b in $\mathbb{U}$ if $f(z)\in\mathcal{A}_p$ satisfies the following inequality

$$\mathrm{Re}\left[1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}-(p-1)\right)\right]>0 \qquad (z\in\mathbb{U})$$

for some complex number b $(b\neq 0)$.

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Next, let $F(z)=\dfrac{zf'(z)}{f(z)}=u+iv$ and $b=\rho e^{i\varphi}$ $(\rho>0,\ 0\leqq\varphi<2\pi)$. Then, the condition of the definition of $\mathcal{S}_b^*(p)$ is equivalent to

$$\text{(1.3)}\qquad \operatorname{Re}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right]=1+\frac{\cos\varphi}{\rho}(u-p)+\frac{\sin\varphi}{\rho}v>0.$$

We denote by $d(l_1,p)$ the distance between the boundary line $l_1:(\cos\varphi)u+(\sin\varphi)v+\rho-p\cos\varphi=0$ of the half plane satisfying the condition (1.3) and the point $F(0)=p$. A simple computation gives us that

$$d(l_1,p)=\frac{|\cos\varphi\times p+\sin\varphi\times 0+\rho-p\cos\varphi|}{\sqrt{\cos^2\varphi+\sin^2\varphi}}=\rho,$$

that is, that $d(l_1,p)$ is always equal to $|b|=\rho$ regardless of φ. Thus, if we consider the circle C_1 with center at p and radius ρ, then we can know the definition of $\mathcal{S}_b^*(p)$ means that $F(\mathbb{U})$ is covered by the half plane separated by a tangent line of C_1 and containing C_1. For $p=1$, the same things are discussed by Hayami and Owa [3].

Then, we introduce the following function

$$\text{(1.4)}\qquad p(z)=\frac{1+Az}{1+Bz}\qquad(-1\leqq B<A\leqq 1)$$

which has been investigated by Janowski [4]. Therefore, the function $p(z)$ given by (1.4) is said to be the Janowski function. Furthermore, as a generalization of the Janowski function, Kuroki, Owa and Srivastava [6] have investigated the Janowski function for some complex parameters A and B which satisfy one of the following conditions

$$\text{(1.5)}\qquad \begin{cases}\text{(i) } A\neq B,\ |B|<1,\ |A|\leqq 1 \quad\text{and}\quad \operatorname{Re}(1-A\overline{B})\geqq|A-B|\\[2ex] \text{(ii) } A\neq B,\ |B|=1,\ |A|\leqq 1\quad\text{and}\quad 1-A\overline{B}>0.\end{cases}$$

Here, we note that the Janowski function generalized by the conditions (1.5) is analytic and univalent in $\mathbb{U}$, and satisfies $\operatorname{Re}(p(z))>0$ $(z\in\mathbb{U})$. Moreover, Kuroki and Owa [5] discussed the fact that the condition $|A|\leqq 1$ can be omitted from among the conditions in (1.5)–(i) as the conditions for A and B to satisfy $\operatorname{Re}(p(z))>0$. In the present paper, we consider the more general Janowski function $p(z)$ as follows:

$$\text{(1.6)}\qquad p(z)=\frac{p+Az}{1+Bz}\qquad(p=1,2,3,\cdots)$$

for some complex parameter A and some real parameter B $(A\neq pB,\ -1\leqq B\leqq 0)$. Then, we don't need to discuss the other cases because for the function

$$\text{(1.7)}\qquad q(z)=\frac{p+A_1z}{1+B_1z}\qquad(A_1,B_1\in\mathbb{C},\ A_1\neq pB_1,\ |B_1|\leqq 1),$$

letting $B_1=|B_1|e^{i\theta}$ and replacing z by $-e^{-i\theta}z$ in (1.7), we see that

$$p(z)=q(-e^{-i\theta}z)=\frac{p-A_1e^{-i\theta}z}{1-|B_1|z}\equiv\frac{p+Az}{1+Bz}\qquad(A=-A_1e^{-i\theta},\ B=-|B_1|)$$

maps $\mathbb{U}$ onto the same circular domain as $q(\mathbb{U})$.

Remark 1.1 For the case $B = -1$ in (1.6), we know that $p(z)$ maps $\mathbb{U}$ onto the following half plane

$$\mathrm{Re}\,(p+\overline{A})\,p(z) > \frac{p^2-|A|^2}{2}$$

and for the case $-1 < B \leqq 0$ in (1.6), $p(z)$ maps $\mathbb{U}$ onto the circular domain

$$\left|p(z) - \frac{p+AB}{1-B^2}\right| < \frac{|A+pB|}{1-B^2}.$$

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then we say that the function $p(z)$ is subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

if there exists a function $w(z) \in \mathcal{W}$ such that $p(z) = q(w(z))$ $(z \in \mathbb{U})$. In particular, if $q(z)$ is univalent in $\mathbb{U}$, then $p(z) \prec q(z)$ if and only if

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}).$$

We next define the subclasses of $\mathcal{A}_p$ by applying the subordination as follows:

$$\mathcal{S}_p^*(A,B) = \left\{ f(z) \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec \frac{p+Az}{1+Bz} \quad (z \in \mathbb{U}) \right\}$$

and

$$\mathcal{K}_p(A,B) = \left\{ f(z) \in \mathcal{A}_p : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{p+Az}{1+Bz} \quad (z \in \mathbb{U}) \right\}$$

where $A \neq pB$, $-1 \leqq B \leqq 0$. We immediately know that

(1.8) $$f(z) \in \mathcal{K}_p(A,B) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(A,B).$$

Then, we have the next theorem.

Theorem 1.2 *If $f(z) \in \mathcal{S}_p^*(A,B)$ $(-1 < B \leqq 0)$, then $f(z) \in \mathcal{S}_b^*(p)$ where*

$$b = \frac{\left|B(-pB + \mathrm{Re}(A))\cos\varphi + B\mathrm{Im}(A)\sin\varphi + |A-pB|\right|}{1-B^2} e^{i\varphi} \quad (0 \leqq \varphi < 2\pi).$$

Espesially, $f(z) \in \mathcal{S}_p^*(A,-1)$ if and only if $f(z) \in \mathcal{S}_b^*(p)$ where $b = \dfrac{p+A}{2}$.

Proof. Supposing that $\dfrac{zf'(z)}{f(z)} \prec \dfrac{p+Az}{1-z}$, it follows from Remark 1.1 that

$$\mathrm{Re}\left[(p+\overline{A})\frac{zf'(z)}{f(z)}\right] > \frac{p^2-|A|^2}{2}$$

that is, that

$$\mathrm{Re}\left[2(p+\overline{A})\frac{zf'(z)}{f(z)}\right] > \mathrm{Re}\left[2p(p+\overline{A})\right] - |p+A|^2.$$

This means that

$$\mathrm{Re}\left[\frac{2(p+\overline{A})}{|p+A|^2}\left(\frac{zf'(z)}{f(z)}-p\right)\right] > -1$$

which implies that

$$\mathrm{Re}\left[\frac{1}{\frac{1}{2}(p+A)}\left(\frac{zf'(z)}{f(z)}-p\right)\right] > -1.$$

Therefore, $f(z) \in \mathcal{S}_b^*$ where $b = \dfrac{p+A}{2}$. The converse is also completed.

Next, for the case $-1 < B \leqq 0$, by the definition of the class $\mathcal{S}_b^*(A,B)$, if a tangent line l_2 of the circle C_2 containing the point p is parallel to the straight line $L : (\cos\theta)u + (\sin\theta)v = 0 \quad (-\pi \leqq {}^\exists\theta < \pi)$, and the image $F(\mathbb{U})$ by $F(z) = \dfrac{zf'(z)}{f(z)}$ is covered by the circle C_2, then there exists a non-zero complex number b with $\arg(b) = \theta + \pi$ and $|b| = d(l_2, p)$ such that $f(z) \in \mathcal{S}_b^*(p)$, where $d(l_2, p)$ is the distance between the tangent line l_2 and the point p. Now, for the function $p(z) = \dfrac{p+Az}{1+Bz} \quad (A \neq pB,\ -1 < B \leqq 0)$, the image $p(\mathbb{U})$ is equivalent to

$$C_2 = \left\{\omega \in \mathbb{C} : \left|\omega - \frac{p-AB}{1-B^2}\right| < \frac{|A-pB|}{1-B^2}\right\}$$

and the point ξ on $\partial C_2 = \left\{\omega \in \mathbb{C} : \left|\omega - \dfrac{p-AB}{1-B^2}\right| = \dfrac{|A-pB|}{1-B^2}\right\}$ can be written by

$$\xi := \xi(\theta) = \frac{|A-pB|}{1-B^2}e^{i\theta} + \frac{p-AB}{1-B^2} \quad (-\pi \leqq {}^\exists\theta < \pi).$$

Further, the tangent line l_2 of the circle C_2 through each point $\xi(\theta)$ is parallel to the straight line $L : (\cos\theta)u + (\sin\theta)v = 0$. Namely, l_2 can be represented by

$$l_2 : \ (\cos\theta)\left(u - \frac{|A-pB|\cos\theta + p - B\mathrm{Re}(A)}{1-B^2}\right) + (\sin\theta)\left(v - \frac{|A-pB|\sin\theta - B\mathrm{Im}(A)}{1-B^2}\right) = 0$$

which implies that

$$l_2 : \ (\cos\theta)u + (\sin\theta)v - \frac{|A-pB| + \{p - B\mathrm{Re}(A)\}\cos\theta - B\mathrm{Im}(A)\sin\theta}{1-B^2} = 0.$$

Then, we see that the distance $d(l_2, p)$ between the point p and the above tangent line l_2 of the circle C_2 is

$$\left|\cos\theta \times p + \sin\theta \times 0 - \frac{|A-pB| + \{p - B\mathrm{Re}(A)\}\cos\theta - B\mathrm{Im}(A)\sin\theta}{1-B^2}\right|$$

$$= \frac{\left|-B(-pB + \mathrm{Re}(A))\cos\theta - B\mathrm{Im}(A)\sin\theta + |A-pB|\right|}{1-B^2}.$$

Therefore, if the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{p+Az}{1+Bz} \quad (A \neq pB,\ -1 < B \leqq 0)$$

holds true, then $f(z) \in \mathcal{S}_b^*$ where

$$b = \frac{\left|-B(-pB + \mathrm{Re}(A))\cos\theta - B\mathrm{Im}(A)\sin\theta + |A-pB|\right|}{1-B^2}e^{i(\theta+\pi)}.$$

Finally, setting $\varphi = \theta + \pi \ (0 \leqq \varphi < 2\pi)$, the proof of the theorem is completed. □

Noonan and Thomas [8], [9] have stated the q–th Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \qquad (n, q \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

This determinant is discussed by several authors with $q = 2$. For example, we can know that the functional $|H_2(1)| = |a_3 - a_2^2|$ is known as the Fekete-Szegö problem and they consider the further generalized functional $|a_3 - \mu a_2^2|$ where $a_1 = 1$ and μ is some real number (see, [1]). The purpose of this investigation is to find the sharp upper bounds of the functional $\left|a_{p+2} - \mu a_{p+1}^2\right|$ for functions $f(z) \in \mathcal{S}_p^*(A, B)$ or $\mathcal{K}_p(A, B)$.

2 Preliminary results

We need some lemmas to establish our results. Applying the Schwarz lemma or subordination principle.

Lemma 2.1 *If a function $w(z) \in \mathcal{W}$, then*

$$|w_1| \leqq 1.$$

Equality is attained for $w(z) = e^{i\theta} z$ for any $\theta \in \mathbb{R}$.

The following lemma is obtained by applying the Schwarz-Pick lemma (see, for example, [7]).

Lemma 2.2 *For any functions $w(z) \in \mathcal{W}$, the inequality*

$$|w_2| \leqq 1 - |w_1|^2$$

holds true. Namely, this gives us the following representation

$$w_2 = \left(1 - |w_1|^2\right) \zeta$$

for some ζ ($|\zeta| \leqq 1$).

3 p-valently Janowski starlike functions

Our first main result is contained in

Theorem 3.1 *If $f(z) \in \mathcal{S}_p^*(A, B)$, then* $\left|a_{p+2} - \mu a_{p+1}^2\right| \leqq$

$$\begin{cases} \dfrac{|(A - pB) \{(1 - 2\mu)A - ((p+1) - 2p\mu) B\}|}{2} & (|(1 - 2\mu)A - ((p+1) - 2p\mu) B| \geqq 1) \\[2ex] \dfrac{|A - pB|}{2} & (|(1 - 2\mu)A - ((p+1) - 2p\mu) B| \leqq 1) \end{cases}$$

with equality for

$$f(z)=\begin{cases}\dfrac{z^p}{(1+Bz)^{\frac{pB-A}{B}}} \quad \text{or} \quad z^pe^{Az}\ (B=0) & (|(1-2\mu)A-((p+1)-2p\mu)B| \geqq 1)\\[2ex] \dfrac{z^p}{(1+Bz^2)^{\frac{pB-A}{2B}}} \quad \text{or} \quad z^pe^{\frac{A}{2}z^2}\ (B=0) & (|(1-2\mu)A-((p+1)-2p\mu)B| \leqq 1).\end{cases}$$

Proof. Let $f(z) \in \mathcal{S}_p^*(A,B)$. Then, there exists the function $w(z) \in \mathcal{W}$ such that

$$\frac{zf'(z)}{f(z)}=\frac{p+Aw(z)}{1+Bw(z)}$$

which means that

$$(n-p)a_n=\sum_{k=p}^{n-1}(A-kB)a_kw_{n-k} \qquad (n \geqq p+1)$$

where $a_p=1$. Thus, by the help of the relation in Lemma 2.2, we see that

$$\left|a_{p+2}-\mu a_{p+1}^2\right| = \left|\frac{1}{2}(A-pB)\left\{w_2+(A-(p+1)B)\,w_1^2\right\}-\mu(A-pB)^2w_1^2\right|$$

$$=\frac{|A-pB|}{2}\left|(1-w_1^2)\zeta+\{(A-(p+1)B)-2\mu(A-pB)\}\,w_1^2\right|.$$

Then, by Lemma 2.1, supposing that $0 \leqq w_1 \leqq 1$ without loss of generality, and applying the triangle inequality, it follows that

$$\left|(1-w_1^2)\zeta+\{(A-(p+1)B)-2\mu(A-pB)\}\,w_1^2\right| \leqq 1+\{|(A-(p+1)B)-2\mu(A-pB)|-1\}\,w_1^2$$

$$\leqq \begin{cases}|(A-(p+1)B)-2\mu(A-pB)| & (|(A-(p+1)B)-2\mu(A-pB)| \geqq 1;\ w_1=1)\\[1ex] 1 & (|(A-(p+1)B)-2\mu(A-pB)| \leqq 1;\ w_1=0).\end{cases}$$

□

Especially, taking $\mu=\dfrac{p+1}{2p}$ in Theorem3.1, we obtain

Corollary 3.2 *If $f(z) \in \mathcal{S}_p^*(A,B)$, then*

$$\left|a_{p+2}-\frac{p+1}{2p}a_{p+1}^2\right| \leqq \begin{cases}\dfrac{|A(A-pB)|}{2p} & (|A| \geqq p)\\[2ex] \dfrac{|A-pB|}{2} & (|A| \leqq p)\end{cases}$$

with equality for

$$f(z)=\begin{cases}\dfrac{z^p}{(1+Bz)^{\frac{pB-A}{B}}} \quad \text{or} \quad z^pe^{Az}\ (B=0) & (|A| \geqq p)\\[2ex] \dfrac{z^p}{(1+Bz^2)^{\frac{pB-A}{2B}}} \quad \text{or} \quad z^pe^{\frac{A}{2}z^2}\ (B=0) & (|A| \leqq p).\end{cases}$$

Furthermore, putting $A = p - 2\alpha$ and $B = -1$ for some α $(0 \leqq \alpha < p)$ in Theorem 3.1, we arrive at the following result by Hayami and Owa [2, Theorem 3].

Corollary 3.3 *If $f(z) \in \mathcal{S}_p^*(\alpha)$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \begin{cases} (p-\alpha)\left\{(2(p-\alpha)+1) - 4(p-\alpha)\mu\right\} & \left(\mu \leqq \dfrac{1}{2}\right) \\ p-\alpha & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p-\alpha+1}{2(p-\alpha)}\right) \\ (p-\alpha)\left\{4(p-\alpha)\mu - (2(p-\alpha)+1)\right\} & \left(\mu \geqq \dfrac{p-\alpha+1}{2(p-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{z}{(1-z)^{2(p-\alpha)}} & \left(\mu \leqq \dfrac{1}{2} \text{ or } \mu \geqq \dfrac{p-\alpha+1}{2(p-\alpha)}\right) \\ \dfrac{z}{(1-z^2)^{p-\alpha}} & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p-\alpha+1}{2(p-\alpha)}\right). \end{cases}$$

4 p-valently Janowski convex functions

Similarly, we consider the functional $|a_{p+2} - \mu a_{p+1}^2|$ for p-valently Janowski convex functions.

Theorem 4.1 *If $f(z) \in \mathcal{K}_p(A,B)$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \begin{cases} \dfrac{p\left|(A-pB)\left\{((p+1)^2 - 2p(p+2)\mu)A - ((p+1)^3 - 2p^2(p+2)\mu)B\right\}\right|}{2(p+1)^2(p+2)} \\ \qquad (|((p+1)^2 - 2p(p+2)\mu)A - ((p+1)^3 - 2p^2(p+2)\mu)B| \geqq (p+1)^2) \\ \dfrac{p|A-pB|}{2(p+2)} \\ \qquad (|((p+1)^2 - 2p(p+2)\mu)A - ((p+1)^3 - 2p^2(p+2)\mu)B| \leqq (p+1)^2) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z^p{}_2F_1\left(p, p-\frac{A}{B}; p+1; -Bz\right) \quad \text{or} \quad z^p{}_1F_1\left(p, p+1; Az\right) \quad (B=0) \\ \qquad (|((p+1)^2 - 2p(p+2)\mu)A - ((p+1)^3 - 2p^2(p+2)\mu)B| \geqq (p+1)^2) \\ z^p{}_2F_1\left(\frac{p}{2}, \frac{pB-A}{2B}; 1+\frac{p}{2}; -Bz^2\right) \quad \text{or} \quad z^p{}_1F_1\left(\frac{p}{2}, 1+\frac{p}{2}; \frac{A}{2}z^2\right) \quad (B=0) \\ \qquad (|((p+1)^2 - 2p(p+2)\mu)A - ((p+1)^3 - 2p^2(p+2)\mu)B| \leqq (p+1)^2) \end{cases}$$

where ${}_2F_1(a,b;c;z)$ represents the ordinary hypergeometric function and ${}_1F_1(a,b;z)$ represents the confluent hypergeometric function.

Proof. By the help of the relation (1.8) and Theorem 3.1, if $f(z) \in \mathcal{K}_p(A,B)$, then

$$\left|\frac{p+2}{p}a_{p+2} - \mu\frac{(p+1)^2}{p^2}a_{p+1}^2\right| = \frac{p+2}{p}\left|a_{p+2} - \frac{(p+1)^2}{p(p+2)}\mu a_{p+1}^2\right| \leqq C(\mu)$$

where $C(\mu)$ is one of the values in Theorem 3.1. Then, dividing the both sides by $\dfrac{p+2}{p}$ and replacing $\dfrac{(p+1)^2}{p(p+2)}\mu$ by μ, we obtain the theorem. □

Now, letting $\mu = \dfrac{(p+1)^3}{2p^2(p+2)}$ in Theorem 4.1, we have

Corollary 4.2 *If $f(z) \in \mathcal{K}_p(A,B)$, then*

$$\left|a_{p+2} - \frac{(p+1)^3}{2p^2(p+2)}a_{p+1}^2\right| \leqq \begin{cases} \dfrac{|A(A-pB)|}{2(p+2)} & (|A| \geqq p) \\[2ex] \dfrac{p|A-pB|}{2(p+2)} & (|A| \leqq p) \end{cases}$$

wiht equality for

$$f(z) = \begin{cases} z^p{}_2F_1\left(p, p-\frac{A}{B}; p+1; -Bz\right) \quad or \quad z^p{}_1F_1\left(p, p+1; Az\right) \quad (B=0) & (|A| \geqq p) \\[2ex] z^p{}_2F_1\left(\frac{p}{2}, \frac{pB-A}{2B}; 1+\frac{p}{2}; -Bz^2\right) \quad or \quad z^p{}_1F_1\left(\frac{p}{2}, 1+\frac{p}{2}; \frac{A}{2}z^2\right) \quad (B=0) & (|A| \leqq p) \end{cases}$$

where ${}_2F_1(a,b;c;z)$ represents the ordinary hypergeometric function and ${}_1F_1(a,b;z)$ represents the confluent hypergeometric function.

Moreover, we suppose that $A = p - 2\alpha$ and $B = -1$ for some α $(0 \leqq \alpha < p)$. Then, we arrive at the result by the Hayami and Owa [2, Theorem 4].

Corollary 4.3 *If $f(z) \in \mathcal{K}_p(\alpha)$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \begin{cases} \dfrac{p(p-\alpha)\left\{(p+1)^2(2(p-\alpha)+1) - 4p(p+2)(p-\alpha)\mu\right\}}{(p+1)^2(p+2)} & \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)}\right) \\[2ex] \dfrac{p(p-\alpha)}{p+2} & \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \\[2ex] \dfrac{p(p-\alpha)\left\{4p(p+2)(p-\alpha)\mu - (p+1)^2(2(p-\alpha)+1)\right\}}{(p+1)^2(p+2)} & \left(\mu \geqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z^p{}_2F_1\left(p, 2(p-\alpha); p+1; z\right) & \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)} \ or \ \mu \geqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \\[2ex] z^p{}_2F_1\left(\frac{p}{2}, p-\alpha; 1+\frac{p}{2}; z^2\right) & \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \end{cases}$$

where ${}_2F_1(a,b;c;z)$ represents the ordinary hypergeometric function.

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