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ON SPIRALLIKE FUNCTIONS AND ROBERTSON FUNCTIONS

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ABSTRACT. In this short survey paper it is shown that several crucial theorems for the theory of Löwner chains whose first coefficients are normalized by e^t can be extended for Löwner chains with complex first coefficient. As a consequence of the above consideration, several new univalence and quasiconformal extension criteria for the class of spirallike functions and Robertson functions are derived.

1. INTRODUCTION

Let $\mathbb{C}$ denotes the complex plane and $\mathbb{D}$ the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Let $f_t(z) = f(z,t) = \sum_{n=1}^{\infty} a_n(t)z^n$, $a_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0,\infty)$ and analytic in $\mathbb{D}$ for each $t \in [0,\infty)$, where $a_1(t)$ is a complex-valued function on $[0,\infty)$. Then $f_t(z)$ is said to be a *Löwner chain* if $f_t(z)$ has the following properties;

1. $f_t(z)$ is univalent in $\mathbb{D}$ for each $t \in [0,\infty)$,
2. $a_1(t)$ is locally absolutely continuous on $[0,\infty)$, $|a_1(t)|$ is strictly increasing on $t \in [0,\infty)$ and $\lim_{t\to\infty} |a_1(t)| = \infty$,
3. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for $0 \leq s < t < \infty$.

It is known that if $f_t(z)$ is a Löwner chain then $f_{t_n}(\mathbb{D}) \to f_{t_0}(\mathbb{D})$ if $t_n \to t_0 \in [0,\infty)$ and $f_{t_n}(\mathbb{D}) \to \mathbb{C}$ if $t_n \to \infty$ in the sense of kernel convergence with respect to the origin. However the converse is not true in general. Also, if f_t is a Löwner chain then a strict inclusion relationship of the expanding image domains (i.e. $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$ for $0 \leq s < t < \infty$) holds. We can adopt this as the definition of Löwner chains instead of "$|a_1(t)|$ is strictly increasing on $[0,\infty)$" in the condition 2. For the precise proofs of these arguments, the reader is referred to e.g. [4, pp.136–138] or [5, pp.94–97]. If $a_1(t) = e^t$, then we shall say that $f_t(z)$ is a *standard Löwner chain*. In this case the above condition 2 is superfluous.

Standard Löwner chains and several related theorems due to Pommerenke [9] and Becker [1] play a crucial role in the theory of univalent functions. In this short survey we show that those theorems work well without normalization of Löwner chains. As a

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consequence of this argument, several new univalence and quasiconformal extension criteria are obtained, which concerning with the typical subclasses of univalent functions, for instance, spirallike functions and Robertson functions.

2. Löwner chains with complex first coefficient

Unless otherwise noted, in this section we will denote a standard Löwner chain by $h(z,t)$ and a general Löwner chain by $f(z,t)$ for convenience.

The following necessary and sufficient condition for a standard Löwner chain has been derived by Pommerenke;

Theorem A ([9, 10]). *Let $0 < r_0 \leq 1$. Let $h(z,t) = e^t z + \sum_{n=2}^{\infty} c_n(t) z^n$ be a function defined on $\mathbb{D} \times [0,\infty)$. Then the function $h(z,t)$ is a standard Löwner chain if and only if the following two conditions are satisfied;*

(i) *The function $h(z,t)$ is analytic in $z \in \mathbb{D}_{r_0}$ for each $t \in [0,\infty)$, absolutely continuous in $t \in [0,\infty)$ for each $z \in \mathbb{D}_{r_0}$ and satisfies*

$$|h(z,t)| \leq K_0 e^t \qquad (z \in \mathbb{D}_{r_0},\ t \in [0,\infty))$$

for some positive constants K_0.

(ii) *There exists a function $p(z,t)$ analytic in $z \in \mathbb{D}$ for each $t \in [0,\infty)$ and measurable in $t \in [0,\infty)$ for each $z \in \mathbb{D}$ satisfying*

$$\mathrm{Re}\, p(z,t) > 0 \qquad (z \in \mathbb{D},\ t \in [0,\infty))$$

such that

$$\dot{h}(z,t) = z h'(z,t) p(z,t) \qquad (z \in \mathbb{D}_{r_0},\ \text{almost every } t \in [0,\infty))$$

where $\dot{h} = \partial h / \partial t$ and $h' = \partial h / \partial z$.

Theorem A can be generalized for a Löwner chain which has the complex-valued first coefficient as the following form;

Theorem 1 ([6]). *Let $0 < r_1 \leq 1$. Let $f(z,t) = \sum_{n=1}^{\infty} a_n(t) z^n$, $a_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0,\infty)$, where $a_1(t)$ is a complex-valued function on $[0,\infty)$. Then the function $f(z,t)$ is a Löwner chain if and only if the following conditions are satisfied;*

(i') *The function $f(z,t)$ is analytic in $\mathbb{D}_{r_1}$ for each $t \in [0,\infty)$, locally absolutely continuous in $[0,\infty)$ for each $z \in \mathbb{D}_{r_1}$ and satisfies $\lim_{t\to\infty} |a_1(t)| = \infty$ and*

$$|f(z,t)| \leq K_1 |a_1(t)| \qquad (z \in \mathbb{D}_{r_1},\ \text{a.e.}\ t \in [0,\infty))$$

for some positive constants K_1.

(ii') *There exists a function $p(z,t)$ analytic in $\mathbb{D}$ for each $t \in [0,\infty)$ and measurable in $[0,\infty)$ for each $z \in \mathbb{D}$ satisfying*

$$\mathrm{Re}\, p(z,t) > 0 \qquad (z \in \mathbb{D},\, t \in [0,\infty))$$

such that

$$\dot{f}(z,t) = zf'(z,t)p(z,t) \qquad (z \in \mathbb{D}_{r_1}, \text{ almost every } t \in [0,\infty)) \tag{1}$$

where $\dot{f} = \partial f/\partial t$ and $f' = \partial f/\partial z$.

Sketch of the proof of Theorem 1. Let $f(z,t) = \sum_{n=1}^{\infty} a_n(t)z^n$ be a Löwner chain, where $a_1(t) \neq 0$ is a complex-valued locally absolutely continuous function on $[0,\infty)$. Set $\lambda(t) := -\arg a_1(t)$ (here $\lambda(t)$ is absolutely continuous on $t \in [0,\infty)$). Let us define $g(z,t)$ and $h(z,t)$ as follows;

$$g(z,t) = \sum_{n=1}^{\infty} b_n(t)z^n := f(e^{i\lambda(t)}z, t) \tag{2}$$

and

$$h(z,t) := \frac{1}{|a_1(0)|} g(z, b_1^{-1}(|a_1(0)|e^t)). \tag{3}$$

We can easily see that $f(z,t)$ is a Löwner chain if and only if $h(z,t)$ is a standard Löwner chain. Also it can be proved that the reparametrization (2) and (3) preserve the other properties of the sufficient part of Theorem 1 and Theorem A with $K_1 = K_0/|a_1(0)|^2$ and $r_0 = r_1$. □

For the precise proof, see [6]. The reader can refer also [2] which contains similar arguments as above.

We shall also see that a quasiconformal extension criterion for a standard Löwner chain due to Becker [1] is extended for a Löwner chain with the complex-valued first coefficient as following. Here, a sense-preserving homeomorphism f of $G \subset \mathbb{C}$ is called *k-quasiconformal* if f_z and $f_{\bar{z}}$, the partial derivatives in z and $\bar{z}$ in the distributional sense, are locally integrable on G and satisfy $|f_{\bar{z}}| \leq k|f_z|$ almost everywhere in G, where $k \in [0,1)$.

Theorem B ([2, 3, 6]). *Suppose that $f_t(z) = f(z,t)$ is a Löwner chain for which $p(z,t)$ in (1) satisfies the condition*

$$p(z,t) \in U(k) := \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\}$$

for all $z \in \mathbb{D}$ and almost all $t \in [0,\infty)$. Then $f_t(z)$ admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and the map $\hat{f}$ defined by

$$\hat{f}(z) = \begin{cases} f(z,0), & \text{if } |z| < 1, \\ f(\frac{z}{|z|}, \log|z|), & \text{if } |z| \geq 1, \end{cases}$$

is a k-quasiconformal extension of f_0 to $\mathbb{C}$.

3. Spirallike functions

A function $f \in \mathcal{A}$ is called *γ-spirallike* and known to be univalent if f satisfies

$$\operatorname{Re}\left\{e^{-i\gamma}\frac{zf'(z)}{f(z)}\right\} > 0$$

for a real number $\gamma \in (-\pi/2, \pi/2)$ in $\mathbb{D}$. We denote such a class of functions by $\mathcal{SP}(\gamma)$. If $\gamma = 0$ then $\mathcal{SP}(0)$ is the well known class of starlike functions. By constructing a Löwner chain without normalization on the first derivative and applying Theorem 1 and Theorem B we have the following new quasiconformal extension criterion;

Theorem 2 ([6]). *Let $\gamma \in (-\pi/2, \pi/2)$ and $k \in [|\tan(\gamma/2)|, 1)$. For $f \in \mathcal{A}$, if*

$$e^{-i\gamma}\frac{zf'(z)}{f(z)} \in U(k)$$

for all $z \in \mathbb{D}$, then f has a k-quasiconformal extension to $\mathbb{C}$, where $U(k)$ is a disk defined in Theorem B.

Proof. Let c be a complex constant with $\operatorname{Re} c > 0$. If we set

$$f_t(z) = e^{ct} f(z), \tag{4}$$

then $\lim_{t\to\infty} |f_t'(0)| = \lim_{t\to\infty} |e^{ct}| = \infty$ since $\operatorname{Re} c > 0$. Therefore we obtain our theorem if we put $c = e^{i\gamma}$ and apply Theorem 1 and Theorem B to (4). □

It is known [10] that a standard Löwner chain which corresponds to γ-spirallike functions is

$$h_t(z) = e^{(1-i\tan\gamma)t} f(e^{i\tan\gamma t} z). \tag{5}$$

The standard chain (5) with Theorem A and Theorem B derive another quasiconformal extension criterion from Theorem 2;

Proposition 3 ([6]). *Let $\gamma \in (-\pi/2, \pi/2)$ and $k \in [0, 1)$. For $f \in \mathcal{A}$, if*

$$\frac{zf'(z)}{f(z)} \in U(\gamma, k)$$

for all $z \in \mathbb{D}$, then f has a k-quasiconformal extension to $\mathbb{C}$, where $U(\gamma, k)$ is the hyperbolic disk in the tilted half plane $\{z \in \mathbb{C} : \operatorname{Re} e^{-i\gamma} z > 0\}$ centered at 1 with radius arctanh k, $0 \le k < 1$, *i.e.,*

$$U(\gamma, k) = \left\{w \in \mathbb{C} : \left|w - \frac{1 + e^{2i\gamma}k^2}{1 - k^2}\right| \le \frac{2k\cos\gamma}{1 - k^2}\right\}.$$

4. Robertson functions

A function $f \in \mathcal{A}$ is said to be *γ-Robertson* ($\gamma \in (-\pi/2, \pi/2)$) if f satisfies

$$\operatorname{Re}\left\{e^{-i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

for all $z \in \mathbb{D}$. This class of functions was introduced by Robertson [11] in 1969. Let $\mathcal{R}(\gamma)$ be the set of those functions. The definition of γ-Robertson functions shows immediately that $\mathcal{R}(0)$ is precisely the class of convex functions. The class $\mathcal{R}(\lambda)$ has been investigated by various authors. The reader can be referred to e.g. [7] and the references therein.

In contrast to the case of spirallike functions, the class $\mathcal{R}(\gamma)$ is not always contained in $\mathcal{S}$ for any γ, $|\gamma| \in [0, \pi/2)$. In fact, if $0 < \cos\lambda \le 1/2$ or $\cos\lambda = 1$ then $\mathcal{R}(\lambda) \subset \mathcal{S}$ and otherwise $\mathcal{R}(\gamma) \not\subset \mathcal{S}$ ([11, 8]). By using the theory of Löwner chains it can be obtained another proof for univalence of $\mathcal{R}(\gamma)$. A Löwner chain for γ-Robertson functions is given by

$$f_t(z) = f(e^{-t}z) - e^{-2i\gamma}(e^{2t}-1)e^{-t}zf'(e^{-t}z)$$

and Theorem 1 shows that if $f \in \mathcal{R}(\gamma)$ with $\gamma = 0$ or $\gamma \in [\pi/3, \pi/2)$ then f is univalent ([7]).

Lastly, we shall introduce the following quasiconformal extension criterion which is proved by essentially making use of Löwner chains with complex-valued first coefficient;

Theorem 4 ([7]). *Let $f \in \mathcal{A}$, $k \in [0,1)$ and $\gamma \in (-\pi/2, \pi/2)$, $q > -1$ be related by*

$$0 < \cos\gamma \le \begin{cases} k/2, & \text{if } -1 < q \le 0, \\ k/(2+4q), & \text{if } \quad 0 < q. \end{cases}$$

If f satisfies

$$\operatorname{Re}\left\{e^{-i\gamma}\left(1+\frac{zf''(z)}{f'(z)}+q\frac{zf'(z)}{f(z)}\right)\right\} > 0$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$. If, in addition, $f''(0) = 0$, (4) can be replaced by

$$0 < \cos\gamma \le \begin{cases} k, & \text{if } -1 < q \le 0, \\ k/(1+2q), & \text{if } \quad 0 < q. \end{cases}$$

We note that the case $q = 0$ claims a quasiconformal extension criterion for γ-Robertson functions.

References

1. J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23–43.
2. ———, *Über die Lösungsstruktur einer Differentialgleichung in der konformen Abbildung*, J. Reine Angew. Math. **285** (1976), 66–74.

3. ______, *Conformal mappings with quasiconformal extensions*, Aspects of contemporary complex analysis, Academic Press, London, 1980, pp. 37–77.
4. John B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995.
5. I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 255, New York, 2003.
6. I. Hotta, *Löwner chains with complex leading coefficient*, Monatsh. Math., to appear.
7. I. Hotta and L.-M. Wang, *Boundedness, univalence and quasiconformal extension of Robertson functions*, preprint (available at arXiv:1003.2019).
8. J. A. Pfaltzgraff, *Univalence of the integral of $f'(z)^\lambda$*, Bull. London Math. Soc. **7** (1975), no. 3, 254–256.
9. Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159–173.
10. ______, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
11. M. S. Robertson, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J. **16** (1969), 97–101.

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