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Some applications for subordination principle

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Abstract

By considering some subordinations for a more general linear transformation, an extension of the Briot-Bouquet differential subordination relations given by S. S. Miller and P. T. Mocanu (Pure and Applied Mathematics **225**, Marcel Dekker, 2000) for certain linear transformations are discussed.

1 Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}=\{z : z\in\mathbb{C}$ and $|z|<1\}$. For a positive integer n and a complex number a, let $\mathcal{H}[a,n]$ be the class of functions $f(z)\in\mathcal{H}$ of the form

$$f(z)=a+\sum_{k=n}^{\infty}a_kz^k.$$

Also, let $\mathcal{A}_n$ denote the class of functions $f(z)\in\mathcal{H}$ of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k$$

with $\mathcal{A}_1=\mathcal{A}$. If $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some real number α with $0\leqq\alpha<1$, then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. This class is denoted by $\mathcal{S}^*(\alpha)$. Similarly, we say that $f(z)$ belongs to the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some real number α with $0\leqq\alpha<1$.

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For some real numbers A and B with $-1 \leqq B < A \leqq 1$, Janowski [1] has investigated the following linear transformation

$$p(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

which is analytic snd univalent in $\mathbb{U}$. This function $p(z)$ is called the Janowski function. Moreover, as a generalization of the Janowski functions, Kuroki and Owa [2] have discussed the Janowski functions for some complex parameters A and B which satisfy

$$A \neq B, \ |B| \leqq 1 \quad \text{and} \quad |A-B| + |A+B| \leqq 2. \tag{1.1}$$

Note that the Janowski function defined by the conditions (1.1) is analytic and univalent in $\mathbb{U}$ and has a positive real part in $\mathbb{U}$ (see [2]).

We next introduce the familiar principle of differential subordinations between analytic functions. Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{1.2}$$

if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathbb{U})$, and such that $p(z) = q\big(w(z)\big) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.2) implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.3}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.2) is equivalent to the condition (1.3).

Miller and Mocanu [4] developed the definitive result concerning the Briot-Bouquet differential subordinations as follows.

Lemma 1.1 *Let n be a positive integer, and let β and γ be complex numbers with $\beta \neq 0$. Also, let $h(z)$ be convex and univalent in $\mathbb{U}$ with $h(0) = a$, and suppose that*

$$\operatorname{Re}\big(\beta h(z) + \gamma\big) > 0 \qquad (z \in \mathbb{U}) \tag{1.4}$$

with $\operatorname{Re}(\beta a + \gamma) > 0$. *If $p(z) \in \mathcal{H}[a, n]$ with $p(z) \not\equiv a$ satisfies the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \qquad (z \in \mathbb{U}),$$

then $p(z) \prec q(z) \prec h(z) \quad (z \in \mathbb{U})$, where $q(z)$ with $q(0) = a$ is the univalent solution of the differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (z \in \mathbb{U}).$$

As applications of Lemma 1.1, Miller and Mocanu [4] derived some subordination relation for certain linear transformations.

Lemma 1.2 *Let n be a positive integer. Also, let β, γ and A be complex numbers with* $\mathrm{Re}(\beta+\gamma) > 0$, *and let B be a real number with $-1 \leqq B \leqq 0$. If β, γ, A and B satisfy either*

$$\mathrm{Re}(\beta(1+AB)+\gamma(1+B^2)) \geqq |\beta A+\overline{\beta}B+2B\mathrm{Re}\,\gamma| \qquad (-1 < B \leqq 0)$$

or

$$\beta(1+A) > 0 \quad \textit{and} \quad \mathrm{Re}(\beta(1+A)+2\gamma) \geqq 0 \qquad (B=-1),$$

then $p(z) \in \mathcal{H}[1,n]$ with $p(z) \not\equiv 1$ satisfies the following subordination relation

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Az}{1+Bz} \quad \text{implies} \quad p(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \tag{1.5}$$

for $z \in \mathbb{U}$, where $q(z)$ with $q(0)=a$ is the univalent solution of the differential equation

$$q(z)+\frac{nzq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}). \tag{1.6}$$

In the present paper, applying the theory of subordinations, we will try to determine the best conditions for complex numbers β, γ, A and B to satisfy the condition (1.4) as

$$h(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

in Lemma 1.1, and deduce an extension of Lemma 1.2.

2 Some subordinations for certain linear transformations

By using the method of a certain generalization of the Janowski functions given by Kuroki and Owa [2], we first consider a certain subordination for a more general linear transformation.

Theorem 2.1 *Let a, A, B, C and D be complex numbers with $A \neq aB$ and $C \neq aD$. If a, A, B, C and D satisfy $|B| \leqq 1$, $|D| \leqq 1$ and*

$$|A-aB|+|AD-BC| \leqq |C-aD|, \tag{2.1}$$

then

$$\frac{a+Az}{1+Bz} \prec \frac{a+Cz}{1+Dz} \qquad (z \in \mathbb{U}). \tag{2.2}$$

Proof. From $A \neq aB$ and the inequality (2.1), it is clear that

$$|C-aD|-|AD-BC| > 0. \tag{2.3}$$

If we define the function $w(z)$ by

$$w(z) = \frac{(A - aB)z}{C - aD - (AD - BC)z} \qquad (z \in \mathbb{U}), \tag{2.4}$$

then from the inequality (2.3), $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$, and that

$$\frac{a + Az}{1 + Bz} = \frac{a + Cw(z)}{1 + Dw(z)} \qquad (z \in \mathbb{U}).$$

Further, noting the inequality (2.3), a simple calculation yields

$$\left| w(z) - \frac{(A - aB)(\overline{AD - BC})}{|C - aD|^2 - |AD - BC|^2} \right| < \frac{|A - aB||C - aD|}{|C - aD|^2 - |AD - BC|^2} \qquad (z \in \mathbb{U}).$$

Since the inequality (2.1) shows that

$$\frac{|A - aB|}{|C - aD| - |AD - BC|} \leqq 1,$$

we see that $w(z)$ defined by (2.4) satisfies $|w(z)| < 1 \quad (z \in \mathbb{U})$. Therefore, from the definition of the subordinations, we conclude that the subordination (2.2) holds, which completes the proof of Theorem 2.1. □

In particular, letting

$$A = b, \; C = \overline{a}e^{i\theta} \quad \text{and} \quad D = -e^{i\theta}$$

for a complex number a with $\operatorname{Re} a > 0$ and for some θ with $0 \leqq \theta < 2\pi$ in Theorem 2.1, we find the following assertion.

Corollary 2.2 *Let a be a complex number with* $\operatorname{Re} a > 0$. *For some complex numbers a, b and B with*

$$b \neq aB, \; |B| \leqq 1 \quad \textit{and} \quad |b - aB| + |b + \overline{a}B| \leqq 2\operatorname{Re} a,$$

we have

$$\frac{a + bz}{1 + Bz} \prec \frac{a + \overline{a}e^{i\theta}z}{1 - e^{i\theta}z} \qquad (z \in \mathbb{U}),$$

where $0 \leqq \theta < 2\pi$. This subordination means the following inequality

$$\operatorname{Re}\left(\frac{a + bz}{1 + Bz}\right) > 0 \qquad (z \in \mathbb{U}).$$

Remark 2.3 Taking $a = 1$ and $b = A$ in Corollary 2.2, we find the conditions in (1.1) as the conditions for complex numbers A and B to satisfy

$$\operatorname{Re}\left(\frac{1 + Az}{1 + Bz}\right) > 0 \qquad (z \in \mathbb{U}).$$

3 The Briot-Bouquet differential subordinations for certain linear transformations

By using the discussion in the previous section, and applying Lemma 1.1, we deduce an improvement of Lemma 1.2 bellow.

Theorem 3.1 *Let n be a positive integer, and let β, γ, A and B be complex numbers with* $\mathrm{Re}(\beta+\gamma)>0$, $A \neq B$ *and* $|B| \leqq 1$. *If β, γ, A and B satisfy*

$$|\beta(A-B)|+|\beta(A-B)+2B\mathrm{Re}(\beta+\gamma)| \leqq 2\mathrm{Re}(\beta+\gamma),$$

then $p(z) \in \mathcal{H}[1,n]$ with $p(z) \not\equiv 1$ satisfies the subordination relation (1.5), *where $q(z)$ with $q(0)=1$ is the solution of the differential equation* (1.6).

Proof. If we let

$$a=\beta+\gamma,\ b=\beta A+\gamma B \quad \text{and} \quad h(z)=\frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

then, a simple check gives us that

$$b-aB=(\beta A+\gamma B)-(\beta+\gamma)B=\beta(A-B) \neq 0$$

and

$$\begin{aligned}2\mathrm{Re}\,a-(|b-aB|+|b+\bar{a}B|)&=2\mathrm{Re}\,a-(|b-aB|+|b-aB+2B\mathrm{Re}\,a|)\\&=2\mathrm{Re}(\beta+\gamma)-\left(|\beta(A-B)|+|\beta(A-B)+2B\mathrm{Re}(\beta+\gamma)|\right) \geqq 0.\end{aligned}$$

Hence by Corollary 2.2, it is easy to see that

$$\mathrm{Re}\big(\beta h(z)+\gamma\big)=\mathrm{Re}\left(\frac{\beta+\gamma+(\beta A+\gamma B)z}{1+Bz}\right)=\mathrm{Re}\left(\frac{a+bz}{1+Bz}\right)>0 \qquad (z \in \mathbb{U}).$$

Therefore, since the conditions of Lemma 1.1 are satisfied, we conclude the assertion of Theorem 3.1. □

By taking $\beta=1$, $\gamma=0$ and $n=1$ in Theorem 3.1, and letting

$$p(z)=\frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}$, we obtain the following subordination implication.

Corollary 3.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

for some complex numbers A and B which satisfy the conditions in (1.1), *then*

$$\frac{zf'(z)}{f(z)} \prec \begin{cases} \dfrac{Az}{(1+B)\left\{1-(1+Bz)^{-\frac{A}{B}}\right\}} & (A \neq 0,\ B \neq 0) \\[2ex] \dfrac{Bz}{(1+Bz)\log(1+Bz)} & (A=0,\ B \neq 0) \\[2ex] \dfrac{Aze^{Az}}{e^{Az}-1} & (A \neq 0,\ B=0) \end{cases}$$

for $z \in \mathbb{U}$.

Moreover, let us consider the case that

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad B = -1$$

in Corollary 3.2. Then, from the definition of the subordinations, we find the implication that if $f(z) \in \mathcal{K}(\alpha)$, then $f(z) \in \mathcal{S}^*(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & \left(\alpha \neq \dfrac{1}{2}\right) \\ \dfrac{1}{2\log 2} & \left(\alpha = \dfrac{1}{2}\right) \end{cases}$$

for each real number α with $0 \leqq \alpha < 1$. This relationship for convex and starlike functions was proven by MacGregor [3].

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