数理解析研究所講究録
第 1727 巻 2011 年 22-34

N-Fractional Calculus Operator Method to Some Second Order homogeneous Euler's Equation

Tsuyako Miyakoda and Katsuyuki Nishimoto

Abstract

In this article, the solutions to homogeneous second order Euler's equation

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot az + \varphi \cdot b = 0, \quad (z \neq 0)$$

where

$$\varphi_0 = \varphi = \varphi(z), \quad \varphi_\alpha = \frac{d^\alpha \varphi}{dz^\alpha}(for\ \alpha > 0)$$

are discussed by means of N- fractional calculus operator.
We have the following fractional differintegrated forms as particular solutions;

$$(i)\varphi = (z^{-(2\gamma+a)})_{-(1+\gamma)} \equiv \varphi_{[1](a,b)} \quad (denote)$$

$$(\gamma = \frac{1}{2}(1 - a + \sqrt{p}),\ p \neq 0)$$

$$(ii)\varphi = (z^{-(2\delta+a)})_{-(1+\delta)} \equiv \varphi_{[2](a,b)}$$

$$(\delta = \frac{1}{2}(1 - a - \sqrt{p}),\ p \neq 0)$$

and

$$(iii)\varphi = (z^{-1})_{\frac{1}{2}(a-3)} \equiv \varphi_{[3](a,b)} \quad (p = 0)$$

where $p = (a-1)^2 - 4b$.

1 Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)
Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, where C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+. (Here D contains the points over the curve C). And, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{\nu+1}} \quad (\nu \notin Z^-), \end{aligned} \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \quad (m \in Z^+), \tag{2}$$

where

$$-\pi \le arg(\zeta - z) \le \pi \ for\ C_-, \quad 0 \le arg(\zeta - z) \le 2\pi \quad for\ C_+,$$

$$\zeta \ne z, \quad z \in C, \quad \nu \in R, \quad \Gamma;\ Gamma\ function,$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

Notice that (1) is reduced to Goursat's integral for $\nu = n (\in \mathbf{Z}^+)$ and is reduced to the famous Cauchy's integral for $\nu = 0$. that is, (1) is an extension of Cauchy's integral and of Goursat's integral, consequently, Cauchy and Goursat's integrals are special cases of (1).

(II) On the fractional calculus operator N^ν [3]
Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu f(z) = \Big(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{\nu+1}}\Big) \quad (\nu \notin Z^-), \quad (Refer\ to [1]) \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$(N^\beta \circ N^\alpha) f = (N^\beta N^\alpha) f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| \leq \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$) " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{_- N^\nu\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(log(z-c))_\alpha = -e^{i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z - c \neq 0$ in (i), and $z - c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad (u = u(z), v = v(z))$$

2 Solutions to some homogeneous Euler's equation

Theorem 1. Let $\varphi \in F = \{\varphi : 0 \neq |\varphi_\nu| < \infty,\ \nu \in R\}$ then the second order homogeneous Euler's equation

$$L(\varphi; z; a, b) := \varphi_2 \cdot z^2 + \varphi_1 \cdot az + \varphi \cdot b = 0 \ (z \neq 0) \tag{1}$$

$$(\varphi_\alpha = d^\alpha \varphi / dz^\alpha \quad for \ \ \alpha > 0,\ \varphi_0 = \varphi = \varphi(z))$$

has particular solutions of the forms in fractional differintegrate form as follows;

(i)

$$\varphi = (z^{-(2\gamma+a)})_{-(1+\gamma)} \equiv \varphi_{[1](a,b)} \ \ (denote) \tag{2}$$

$$(\gamma = \frac{1}{2}\{-(a-1) + \sqrt{(a-1)^2 - 4b}\},\ (a-1)^2 - 4b \neq 0)$$

(ii)

$$\varphi = (z^{-(2\delta+a)})_{-(1+\delta)} \equiv \varphi_{[2](a,b)} \tag{3}$$

$$(\delta = \frac{1}{2}\{-(a-1) - \sqrt{(a-1)^2 - 4b}\},\ (a-1)^2 - 4b \neq 0)$$

and

(iii)

$$\varphi = (z^{-1})_{\frac{1}{2}(a-3)} \equiv \varphi_{[3](a,b)} \tag{4}$$

$$((a-1)^2 - 4b = 0).$$

Proof

We operate the N-fractional calculus operator of order α ($\alpha \notin Z^-$) N^α to the both sides of equation (1), then

$$(\varphi_2 \cdot z^2)_\alpha + (\varphi_1 \cdot az)_\alpha + (\varphi \cdot b)_\alpha = 0. \tag{5}$$

From the index low we notice

$$N^\alpha \varphi_m = (\varphi_m)_\alpha = \varphi_{m+\alpha} \quad (m = 2, 1) \tag{6}$$

And by Lemma (iv) we have

$$\begin{aligned} N^{\alpha}(\varphi_2 \cdot z^2) &= (\varphi_2 \cdot z^2)_{\alpha} \\ &= \sum_{k=0}^{2} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} (\varphi_2)_{\alpha-k} (z^2)_k \qquad (7) \\ &= \varphi_{2+\alpha} \cdot z^2 + \varphi_{1+\alpha} \cdot 2\alpha z + \varphi_{\alpha} \cdot \alpha(\alpha-1), \qquad (8) \end{aligned}$$

$$\begin{aligned} N^{\alpha}(\varphi_1 \cdot az) &= (\varphi_1 \cdot az)_{\alpha} = \sum_{k=0}^{1} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} (\varphi_1)_{\alpha-k} \cdot (az)_k \qquad (9) \\ &= \varphi_{1+\alpha} \cdot az + \varphi_{\alpha} \cdot a\alpha \qquad (10) \end{aligned}$$

and

$$N^{\alpha}(\varphi \cdot b) = \varphi_{\alpha} \cdot b. \qquad (11)$$

Therefore we obtain

$$\varphi_{2+\alpha} \cdot z^2 + \varphi_{1+\alpha} \cdot (2\alpha + a)z + \varphi_{\alpha} \cdot \{\alpha^2 + \alpha(a-1) + b\} = 0. \qquad (12)$$

We choose α such that

$$\alpha^2 + \alpha(a-1) + b = 0, \qquad (13)$$

that is

$$\alpha = \frac{1}{2}\{-(a-1) + \sqrt{p}\} \equiv \gamma \quad , \quad \alpha = \frac{1}{2}\{-(a-1) - \sqrt{p}\} \equiv \delta \qquad (14)$$

where $p = (a-1)^2 - 4b$.
When $\alpha = \gamma$ and $p \neq 0$, we have

$$\varphi_{2+\gamma} \cdot z^2 + \varphi_{1+\gamma} \cdot z(2\gamma + a) = 0 \qquad (15)$$

from (12) by applying (14). Setting

$$\psi = \psi(z) = \varphi_{1+\gamma}, \qquad (\varphi = \psi_{-(1+\gamma)}) \qquad (16)$$

and we obtain

$$\psi_1 \cdot z^2 + \psi \cdot z(2\gamma + a) = 0. \qquad (17)$$

Then a particular solution to this equation is given by

$$\psi(z) = z^{-(2\gamma+a)} \qquad (18)$$

Therefore we obtain

$$\varphi(z) = (z^{-(2\gamma+a)})_{-(1+\gamma)} = \varphi_{[1](a,b)}. \tag{19}$$

Inversely the function shown by (19) satisfy the equation (1) cleary. Since

$$\varphi_{1+\nu} = (z^{-(2\gamma+a)})_{-(1+\gamma)+1+\gamma} = z^{-(2\gamma+a)}, \tag{20}$$

and

$$\varphi_{2+\nu} = (z^{-(2\gamma+a)})_{-(1+\gamma)+2+\gamma} = (z^{-(2\gamma+a)})_1, \tag{21}$$

we have

$$LHS\ of\ (15) = (z^{-(2\gamma+a)})_1 \cdot z^2 + (z^{-(2\gamma+a)}) \cdot z(2\gamma + a) = 0. \tag{22}$$

Therefore the function (2) satisfies the equation (1).
When $\alpha = \delta$ and $p \neq 0$, we have

$$\varphi_{2+\delta} \cdot z^2 + \varphi_{1+\delta} \cdot z(2\delta + a) = 0 \tag{23}$$

instead of (15). Therefore in the same way we obtain

$$\varphi(z) = (z^{-(2\delta+a)})_{-(1+\delta)} = \varphi_{[2](a,b)}. \tag{24}$$

When $p = 0$, the case is $\gamma = \delta = \frac{1}{2}(1 - a)$, we have

$$\varphi(z) = (z^{-1})_{\frac{1}{2}(a-3)} = \varphi_{[3](a,b)}. \tag{25}$$

Notice that in our N-fractional calculus operator (NFCO)-method, the original homogeneous linear second order ordinary differential equation (1) is reduced to a variable separable form one.

3 Familiar forms of Solutions

In this section we show the translated forms (familiar forms) of the solutions obtained in §2.
Theorem 2. The solutions shown in Theorem 1 are writen like as the following familiar forms;

(i)

$$\varphi_{[1](a,b)} = -e^{i\pi\gamma}\frac{\Gamma(\gamma+a-1)}{\Gamma(2\gamma+a)}z^{-\gamma-a+1} \tag{1}$$
$$(|\frac{\Gamma(\gamma+a-1)}{\Gamma(2\gamma+a)}| < \infty, \quad \gamma = \frac{1}{2}(1-a-\sqrt{p}), \ p \neq 0).$$

(ii)

$$\varphi_{[2](\nu,b)} = -e^{i\pi\delta}\frac{\Gamma(\delta+a-1)}{\Gamma(2\delta+a)}z^{-\delta-a+1} \tag{2}$$
$$(|\frac{\Gamma(\delta+a-1)}{\Gamma(2\delta+a)}| < \infty, \quad \gamma = \frac{1}{2}(1-a+\sqrt{p}), \ p \neq 0)$$

(iii)

$$\varphi_{[3](a,b)} = \begin{cases} e^{i\pi\frac{1}{2}(3-a)}\Gamma(\frac{a}{2}-\frac{1}{2})z^{\frac{1}{2}-\frac{a}{2}} & \text{for } \frac{1}{2}(a-3) \notin Z^-, p=0 \\ (\log z)_{-\frac{1}{2}(a-3)+1} & \text{for } \frac{1}{2}(a-3) \in Z^-, p=0 \end{cases} \tag{3}$$

where $p = (a-1)^2 - 4b$.

4 A Special case

When $a = 1$ and $b = -\nu^2$, we have the following corollary from Theorem 1.
Corollary 1. Let $\varphi \in F = \{\varphi : 0 \neq |\varphi_\nu| < \infty, \ \nu \in R\}$ then the second order homogeneous Euler's equation

$$L(\varphi; z; 1, -\nu^2) = 0 \quad (z \neq 0) \tag{1}$$

has particular solutions of the forms in fractional differintegrated form as follows;

(i)

$$\varphi = (z^{-(2\nu+1)})_{-(1+\nu)} = \varphi_{[1](1,-\nu^2)} \tag{2}$$

(ii)

$$\varphi = (z^{2\nu-1})_{\nu-1} = \varphi_{[2](1,-\nu^2)} \tag{3}$$

and

(iii)

$$\varphi = (z^{-1})_{-1} = \varphi_{[3](1,-\nu^2)} \quad (when\ \nu = 0) \tag{4}$$

5 Some Illustrative Example

[I] Let $a = 1$ and $b = -1$. The equation is

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot z - \varphi = 0 \quad (z \neq 0) \tag{1}$$

and the solutions are

$$\varphi = \varphi_{[1](1,-1)} = (z^{-3})_{-2} = \frac{1}{\Gamma(3)} z^{-1} \tag{2}$$

and

$$\varphi = \varphi_{[2](1,-1)} = (z)_0 = z. \tag{3}$$

The function given by (2) and (3) satisfy the equation (1) clearly.
[II] Let $a = 5$ and $b = 1$. The equation is

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot 5z + \varphi = 0 \quad (z \neq 0) \tag{4}$$

and the solutions are

$$\begin{aligned} \varphi = \varphi_{[1](5,1)} &= (z^{-1-2\sqrt{3}})_{1-\sqrt{3}} \\ &= K z^{-2-\sqrt{3}} \quad (K = e^{i\pi(\sqrt{3}-1)} \frac{1}{\Gamma(2+\sqrt{3})} \Gamma(1+2\sqrt{3})) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \varphi = \varphi_{[2](5,1)} &= (z^{-1+2\sqrt{3}})_{1+\sqrt{3}} \\ &= K' z^{-2+\sqrt{3}} \quad (K' = e^{-i\pi(1+\sqrt{3})} \frac{1}{\Gamma(2-\sqrt{3})} \Gamma(1-2\sqrt{3})) \end{aligned} \tag{6}$$

The function given by (5) and (6) satisfy the equation (4) clearly.
Indeed in the case of (5), we have

$$\varphi_1 = K(-2-\sqrt{3}) z^{-3-\sqrt{3}} \tag{7}$$

and

$$\varphi_2 = K(2+\sqrt{3})(3+\sqrt{3}) z^{-4-\sqrt{3}}. \tag{8}$$

Therefore we obtain

$$LHS\ of\ (4) = K z^{-2-\sqrt{3}} \{(2+\sqrt{3}(3+\sqrt{3}) + 5(-2-\sqrt{3}) + 1\} = 0. \tag{9}$$

6 Commentary

(I) Usually the Euler's equation is solved by the change og independent variable

$$z = e^t. \tag{1}$$

Since

$$\varphi_1 = \frac{d\varphi}{dz} = \frac{d\varphi}{dt} \cdot \frac{dt}{dz} = \frac{d\varphi}{dt} e^{-t}, \tag{2}$$

$$\varphi_2 = \frac{d^2\varphi}{dz^2} = \frac{d}{dz}\left(\frac{d\varphi}{dz}\right) = \left(\frac{d^2\varphi}{dt^2} - \frac{d\varphi}{dt}\right)e^{-2t}, \tag{3}$$

therefore, for example, for §5, (1) we have

$$\frac{d^2\varphi}{dt^2} - \varphi = 0. \tag{4}$$

Particular solutions to this equation are given by

$$\varphi = e^t = z \tag{5}$$

and

$$\varphi = e^{-t} = z^{-1}. \tag{6}$$

(II) Equation §2 (1)

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot az + \varphi \cdot b = 0 \tag{7}$$

canbe solved with setting

$$\varphi = z^\lambda \phi \quad (\phi = \phi(z)) \tag{8}$$

too.

(III) Nonhomogeneous equations shall be discussed in a next paper.

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimnoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^{ν} (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field Produced from The Set of N- Fractional Calculus Operator, J. Frac Calc. Vol. 24, Nov. (2003),29 - 36.

[6] K. Nishimoto ; An application of fractional calculus to the nonhomogeneous Gauss equations, J. Coll. Engng, Nihon Univ., **B**,-28 (1987), 1 - 8.

[7] K. Nishimoto and S. L. Kalla ; Application of Fractional Calculus to Ordinary Differential Equation of Fuchs Type, Rev. Tec. Ing. Univ. Zulia, Vol. 12, No 1 (1989).

[8] K. Nishimoto ; Application of Fractional Calculus to Gauss Type Partial Differential Equations, J. Coll. Engng, Nihon Univ., **B**,-30 (1989), 81 - 87.

[9] Shih- Tong Tu, S.-J. Jaw and Shy -Der Lin ; An application of fractional calculus to Chebychev 's equation, Chung Yuan J., Vol. XIX (1990), 1 - 4.

[10] K. Nishimoto, H. M. Srivastava and Shih -Tong Tu ; Application of Fractional Calculus in Solving Certain Classes of Fuchsian Differential Equations, J. Coll. Engng, Nihon Univ., **B**,-32 (1991), 119 - 126.

[11] K. Nishimoto ; A Generalization of Gauss' Equation by Fractional Calculus Method, J. Coll. Engng, Nihon Univ., **B**,-32 (1991), 79 - 87.

[12] Shy -Der Lin, Shih -Tong Tu and K. Nishimoto ; A generalization of Legendre's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 35 - 43.

[13] N. S. Sohi, L. P. Singh and K. Nishimoto ; A generalization of Jacobi's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 45 - 51.

[14] K. Nishimoto ; Solutions of Gauss equation in fractional calculus, J. Frac. Calc. Vo1. 3, May(1993), 29 - 37.

[15] K. Nishimoto ; Solutions of homogeneous Gauss equations, which have a logarithmic function, in fractional calculus, J. Frac. Calc. Vol. 5, May (1994), 11 - 25.

[16] K. Nishimoto ; Application of N-transformation and N-fractional calculus method to nonhomogeneous Bessel equations (I), J. Frac. Calc. Vol. 8, Nov. (1995), 25 - 30.

[17] K. Nishimoto ; Operator N^{ν} method to nonhomogeneous Gauss and Bessel equations , J. Frac. Calc. Vol. 9, May (1996), 1 - 15.

[18] K. Nishimoto and Susana S. de Romero ; N-fractional calculus operator N^{ν} method to nonhomogeneous and homogeneous Whittaher equations (I), J. Frac. Calc. Vol. 9, May (1996), 17 - 22.

[19] K. Nishimoto and Judith A. de Duran ; N-fractional calculus operator N^{ν} method to nonhomogeneous Fukuhara equations (I), J. Frac. Calc. Vol. 9, May (1996), 23 - 31.

[20] K. Nishimoto ; N-fractional calculus operator N^{ν} method to nonhomogeneous Gauss equation, J. Frac. Calc. Vol. 10, Nov. (1996), 33 - 39.

[21] K. Nishimoto ; Kummer's twenty-four functions and N-fractional calculus, Nonlinear Analysis, Theory, Method & Applications, Vol. 30, No.2 (1997), 1271 - 1282.

[22] Shih -Tong Tu, Ding -Kuo Chyan and Wen -Chieh Luo ; Some solutions to the nonhomogeneous Jacobi equations Via fractional calculus operator N^{ν} methods, J. Frac. Calc. Vol. 12, Nov.(1997), 51 - 60.

[23] Shih -Tong Tu, Ding -Kuo Chyan and Erh -Tsung Chin ; Solutions of Gegenbauer and Chebysheff equations via operator N^{ν} methods, J. Frac. Calc. Vol. 12, Nov.(1997), 61 - 69.

[24] K. Nishimoto ; N-method to Hermite equations, J. Frac. Calc. Vol. 13, May (1998), 21 - 27.

[25] K. Nishimoto ; N-method to Weber equations, J. Frac. Calc. Vol. 14, Nov. (1998), 9 - 21.

[26] K. Nishimoto ; N-method to generalized Laguerre equations, J. Frac. Calc. Vol. 14, Nov. (1998), 1 - 8.

[27] Shy -Der Lin, Jaw -Chian Shyu, K. Nishimoto and H. M. Srivastava ; Explicit Solutions of Some General Families of Ordinary and Partial Differential Equations Associated with the Bessel Equation by Means of Fractional Calculus, J. Frac. Calc. Vol. 25, May (2004), 33 - 45.

[28] K. Nishimoto ; Solutions to Some Extended Hermite's Equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 29, May (2006), 45-56.

[29] Tsuyako Miyakoda ; Solutions to An Extended Hermite's Equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 30, Nov. (2006), 23 - 32.

[30] K. Nishimoto ; N-Fractional Calculus Operator Method to Chebyshev's Equations , J. Frac. Calc. Vol. 33, May (2008), 71- 90.

[31] K. Nishimoto ; N-fractional Calculus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May (2005), 83 - 88.

[32] K. Nishimoto ; N-fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006), 35 - 44.

[33] David Dummit and Richard M. Foote ; Abstract Algebra, Prentice Hall (1991).

[34] K. B. Oldham and J. Spanier ; The Fractional Calculus, Academic Press (1971).

[35] A.C. McBride ; Fractional Calculus and Integral Transforms of Generalized Functions, Research Notes, Vol. 31, (1979), Pitman.

[36] S.G. Samko, A.A. Kilbas and O. I. Marichev ; Fractional Integrals and Derivatives, and Some Their Applications (1987), Nauka, USSR.

[37] K. S. Miller and B. Ross ; An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, (1993).

[38] V. Kiryakova ; Generalized fractional calculus and applications, Pitman Research Notes, No.301, (1994), Longman.

[39] A. Carpinteri and F. Mainard (Ed.); Fractals and Fractional Calculus in Continuum Mechanics, (1997), Springer, Wien, New York.

[40] Igor Podlubny ; Fractional Differential Equations (1999), Academic Press.

[41] R. Hilfer (Ed.) ; Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapore, New Jersey, London, Hong Kong.

Tsuyako Miyakoda
Osaka Institute of Technology,
Nakamiya 565 - 0871, Osaka, JAPAN
Fax : + 81 - 66 - 879 - 7871

Katsuyuki Nishimoto
Institute for Applied Mathematics,
Descartes Press Co.
2- 13- 10 Kaguike, Koriyama, 963 - 8833, JAPAN
Fax : + 81 - 24 - 922 - 7596