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On a generalization of Avhadiev and Aksentev's theorem

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1. Introduction

Let

$$F(z)=\sum_{n=0}^{\infty}a_{n}z^{n}$$

be analytic and univalent in $D=\{z \mid |z|<1\}$ and suppose that $F(D)=E$. If $f(z)$ is analytic in D, $f(0)=F(0)$ and $f(D)\subset E$, then we call that $f(z)$ is subordinate to $F(z)$ in D, and we write $f(z)\prec F(z)$ in D. In 1943, Rogosinski [3] obtained the following theorem.

Theorem A. *If $f(z)\prec F(z)$ in D and if $0<p$, then*

$$\int_{0}^{2\pi}|f(re^{i\theta})|^{p}d\theta\leq\int_{0}^{2\pi}|F(re^{i\theta})|^{p}d\theta$$

for $0<r<1$.

On the other hand, in 1973, Avhadiev and Aksentev [1] obtained a theorem which is conguously with Rogosinski's theorem.

Theorem B. *If $f(z)$ and $F(z)$, with $f(0)=F(0)$, are analytic in D and $f(z)\prec F(z)$ in D, then*

$$\int_{0}^{2\pi}|\mathrm{Re}f(re^{i\theta})|d\theta\leq\int_{0}^{2\pi}|\mathrm{Re}F(re^{i\theta})|d\theta$$

for $0<r<1$.

Applying Theorem A and B, Nunokawa, Fukui and Saitoh [2] obtained the following theorem.

Theorem C. *If $f(z)$ and $F(z)$, with $f(0) = F(0)$, are analytic in D and $f(z) \prec F(z)$ in D, then*

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^2 d\theta \le \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^2 d\theta$$

for $0 < r < 1$.

2. Main result

In this paper, we will prove the following theorem.

Theorem 1. *If $f(z)$ and $F(z)$, with $f(0) = F(0)$, are analytic in D and*

$$f(z) \prec F(z) \tag{1}$$

in D, then

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^p d\theta \le \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^p d\theta$$

where $1 < p$ *and* $0 < r < 1$.

Proof. From the hypothesis (1), we can write

$$f(z) = F(\phi(z))$$

where $\phi(z)$ is analytic in D, $|\phi(z)| < 1$ in D and $\phi(0) = 0$. Then, from the harmonic function theorey, we can write

$$\begin{aligned}
&\mathrm{Re} f(z) = \mathrm{Re} f(re^{i\theta}) \\
&= \mathrm{Re} F(re^{i\theta}) \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\mathrm{Re} F(\rho e^{i\nu}))\mathrm{Re}\frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})}\, d\nu
\end{aligned}$$

where

$$|\phi(re^{i\theta})| < |z| = |re^{i\theta}| = r < \rho < 1.$$

On the other hand, we will obtain the following easy calculation

$$\begin{aligned}
&\left|\mathrm{Re}\frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})}\right| \\
&= \frac{\rho^2 - |\phi(re^{i\theta})|^2}{\rho^2 - 2\rho|\phi(re^{i\theta})|\cos\big(\nu - \arg\ \phi(re^{i\theta})\big) + |\phi(re^{i\theta})|^2} \\
&= \mathrm{Re}\frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} > 0
\end{aligned} \tag{2}$$

and

$$\int_0^{2\pi} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})}\, d\theta = \int_{|z|=r} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} \frac{dz}{iz}$$
$$= \frac{1}{i} \int_{|z|=r} \frac{\rho e^{i\nu} + \phi(z)}{z(\rho e^{i\nu} - \phi(z))} dz = 2\pi,$$

and therefore we have

$$\int_0^{2\pi} \mathrm{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})}\, d\theta = 2\pi \tag{3}$$

where $z = re^{i\theta}$ and $0 < r < \rho < 1$. Let us put

$$R(\rho, r, \phi, \theta) = \mathrm{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})}.$$

Applying (2),(3) and Hölder's inequality, we have the following inequality

$$\int_0^{2\pi} \left|\mathrm{Re} f(re^{i\theta})\right|^p d\theta$$
$$= \int_0^{2\pi} \left|\frac{1}{2\pi} \int_0^{2\pi} \Big(\mathrm{Re} F(\rho e^{i\nu})\Big) R(\rho, r, \phi, \theta) d\nu\right|^p d\theta$$
$$\leq \int_0^{2\pi} \frac{1}{(2\pi)^p} \left|\left\{\int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p \Big(R(\rho, r, \phi, \theta)\Big)^{\frac{1}{p}p} d\nu\right\}^{\frac{1}{p}}\right.$$
$$\left. \cdot \left\{\int_0^{2\pi} \Big(R(\rho, r, \phi, \theta)\Big)^{\frac{1}{q}q} d\nu\right\}^{\frac{1}{q}}\right|^p d\theta$$
$$= \frac{1}{(2\pi)^p} \int_0^{2\pi} \left|\left\{\int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p R(\rho, r, \phi, \theta) d\nu\right\}^{\frac{1}{p}}\right.$$
$$\left. \cdot \left\{\int_0^{2\pi} R(\rho, r, \phi, \theta) d\nu\right\}^{\frac{1}{q}}\right|^p d\theta$$
$$= (2\pi)^{-p+\frac{p}{q}} \int_0^{2\pi} \left(\int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p R(\rho, r, \phi, \theta) d\nu\right) d\theta$$
$$= (2\pi)^{-p+\frac{p}{q}} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p \left(\int_0^{2\pi} R(\rho, r, \phi, \theta) d\theta\right) d\nu$$
$$= (2\pi)^{-p+\frac{p}{q}+1} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p d\nu$$
$$= \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p d\nu$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p$ and $1 < q$. Putting $r \to \rho$, it completes the proof. □

Remark 1. *In Theorem 1, we can not prove it for the case* $0 < p < 1$. *It is an open problem.*

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References

[1] F. Ahadiev and L. Aksentev, The subordination principle in sufficient conditions for univalence, Dokl. Akad. Nauk SSSR, Tom 211 (1973), No. 1, Soviet Math. Dokl. Vol. 14(1973), No. 4. 934-939.

[2] M. Nunokawa. S. Fukui and H. Saitoh, Majorization of subordinate analytic functions, Sci. Rep. Fac. Edu. Gunma Univ. Vol.24(1975), 7-10.

[3] W. Rogosinski, On the coefficients of subordinate functions, Proc.London Math. Soc. (2) 48 (1943), 48-82.

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