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Geometric properties of certain analytic functions with real coefficients

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Abstract

Let $\mathcal{T}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k$$

which are analytic in the open unit disk $\mathbb{U}=\{z:|z|<1\}$ and a_k are real numbers. For a function $f(z)\in\mathcal{T}$, some sufficient conditions for starlikeness and convexity are discussed.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:|z|<1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of the univalent functions in $\mathbb{U}$. By $\mathcal{S}^*$ and $\mathcal{K}$, we denote the subclasses of $\mathcal{A}$ whose members map $\mathbb{U}$ onto the domain which are starlike and convex.

Further, the function $f(z)\in\mathcal{A}$ is said to be starlike of order α $(\alpha<1)$ in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in\mathbb{U}). \tag{1.2}$$

Similarly, $f(z)\in\mathcal{A}$ is said to be convex of order α $(\alpha<1)$ in $\mathbb{U}$ if and only if

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>\alpha \qquad (z\in\mathbb{U}) \tag{1.3}$$

We shall denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ whose members satisfy (1.2) and (1.3), respectively.

It is known that for $0\leqq\alpha<1$, $\mathcal{S}^*(\alpha)\subset\mathcal{S}^*$, $\mathcal{K}(\alpha)\subset\mathcal{K}$ and that $\mathcal{S}^*(0)\equiv\mathcal{S}^*$, $\mathcal{K}(0)\equiv\mathcal{K}$.

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Furthermore, we define $\mathcal{T}$ the class of analytic functions with real coefficients, that is,

$$\mathcal{T} := \left\{ f(z) \in \mathcal{A} : f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ a_k \in \mathbb{R} \right\}, \tag{1.4}$$

where $\mathbb{R}$ is the set of real numbers.

According to Silverman, we introduce $\mathcal{N}$ the class of analytic functions with negative coefficients, that is,

$$\mathcal{N} := \left\{ f(z) \in \mathcal{A} : f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geqq 0 \right\}. \tag{1.5}$$

We note that

$$\mathcal{N} \subset \mathcal{T} \subset \mathcal{A}.$$

Next, we define the Hadamard product or convolution by

$$(f * g)(z) = f(z) * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \tag{1.6}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$.

With a view to introducing the Srivastava-Attiya convolution operator $\mathcal{J}_{s,b}$, we begin by recalling a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.7}$$
$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \mathrm{Re}(s) > 1 \text{ when } |z| = 1).$$

Srivastava and Attiya [3] introduced the linear operator

$$\mathcal{J}_{s,b}(f) : \mathcal{A} \longrightarrow \mathcal{A}$$

defined; in term of the Hadamard product (or convolution), by

$$\mathcal{J}_{s,b}(f)(z) := G_{s,b}(z) * f(z) \qquad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}), \tag{1.8}$$

where for convenience,

$$G_{s,b}(z) := (1+b)^s \left[\Phi(z,s,b) - b^{-s}\right] \qquad (z \in \mathbb{U}). \tag{1.9}$$

It is easy to observe from (1.1) and the definition (1.7) and (1.8) that

$$\mathcal{J}_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s a_k z^k. \tag{1.10}$$

For $f(z) \in \mathcal{A}$, we define the class $\mathcal{S}_{s,b}^*(\alpha)$ by

$$f(z) \in \mathcal{S}_{s,b}^*(\alpha) \quad \Longleftrightarrow \quad \mathrm{Re}\left(\frac{z\mathcal{J}_{s,b}'(f)(z)}{\mathcal{J}_{s,b}(f)(z)}\right) > \alpha, \tag{1.11}$$

that is, $\mathcal{J}_{s,b}(f)(z)$ is in $\mathcal{S}^*(\alpha)$ $(z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C})$.

Remark 1 For $f(z) \in \mathcal{A}$, we put

$$G(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n$$

is convex ($\mathrm{Re}(c) > -1$). So we have

$$\begin{aligned}\Phi_c(f(z)) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \quad (c > -1) \\ &= (f * G)(z) \\ &= z + \sum_{n=2}^{\infty} \frac{1+c}{n+c} a_n z^n \\ &= \mathcal{J}_{1,c}(f)(z).\end{aligned}$$

2 Preliminaries

We introduce the following lemmas for our results.

Lemma 1 [4] *Let $f(z) \in \mathcal{T}$ and $\mathrm{Re}\{f'(z)\} > 0$, then the function*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \qquad (c > -1) \tag{2.1}$$

belongs to $\mathcal{K}(-c)$ for all c $(0 \leqq -c < 1)$.

Lemma 2 [1, Carathéodory] *Let $\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{U}$ and $\mathrm{Re}\{\varphi(z)\} > 0$ $(z \in \mathbb{U})$. Then,*

$$|c_n| \leqq 2 \qquad (n = 1, 2, 3, \cdots).$$

Lemma 3 [2] *Let $f(z) \in \mathcal{T}$ and suppose that*

$$\mathrm{Re}\{f'(z) + \alpha z f''(z)\} > 0 \qquad (z \in \mathbb{U}) \tag{2.2}$$

where $\alpha \geqq 1$. Then, we have

$$1 + \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \frac{\alpha - 1}{\alpha} \qquad (z \in \mathbb{U}),$$

or $f(z)$ is convex of order $\dfrac{\alpha-1}{\alpha}$.

3 Main results

Theorem 1 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}$ *and* $0 \leqq \alpha < 1$.

$$\text{(i) } If \ \ |zf''(z) + (1-\alpha)(f'(z)-1)| \leqq 1-\alpha, \ then \ f(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U}).$$

$$\text{(ii) } If \ \ \left| f'(z) + \alpha\left(1 - \frac{f(z)}{z}\right) - 1 \right| \leqq 1-\alpha, \ then \ f(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U}).$$

Proof. Using Lemma 3, we have (i) and (ii). □

Remark 2 From Theorem 1, we have the following results given by H. Silverman [5].

$$\text{(i) } \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq 1-\alpha \quad \Longrightarrow \quad f(z) \in \mathcal{K}(\alpha).$$

$$\text{(ii) } \sum_{n=2}^{\infty} (n-\alpha)|a_n| \leqq 1-\alpha \quad \Longrightarrow \quad f(z) \in \mathcal{S}^*(\alpha).$$

Next, we prove the following theorem.

Theorem 2 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. *If*

$$\operatorname{Re}\{(1-\alpha)f'(z) + zf''(z)\} > 0 \qquad (0 \leqq \alpha < 1), \tag{3.1}$$

then $|a_n| \leqq \dfrac{2(1-\alpha)}{n(n-\alpha)}$. *The result is sharp.*

Proof. The coefficient bounds are maximized at the extreme point. Now the extreme point of (3.1) may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\alpha)x^{n-1}}{n(n-\alpha)} z^n, \quad |x| = 1 \tag{3.2}$$

and the result follows. □

Remark 3 If $f(z) \in \mathcal{T}$ and $\alpha = 0$, then $|a_n| \leqq \frac{2}{n^2}$. So, we have $\sum_{n=2}^{\infty} |a_n| \leqq \frac{\pi^2 - 6}{3} = 1.289\cdots$. Moreover, in the case of $f(z) \in \mathcal{T}$, we have $f(z) \in \mathcal{K}(\alpha)$.

Theorem 3 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. *If*

$$\operatorname{Re}\left\{ f'(z) - \alpha \frac{f(z)}{z} \right\} > 0 \qquad (0 \leqq \alpha < 1), \tag{3.3}$$

then $|a_n| \leqq \dfrac{2(1-\alpha)}{n-\alpha}$. *The result is sharp.*

Proof. The coefficient bounds are maximized at the extreme point. The extreme point of (3.3) is

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\alpha)x^{n-1}}{n-\alpha} z^n, \quad |x| = 1 \tag{3.4}$$

and the result follows. □

Remark 4 In the case of $f(z) \in \mathcal{T}$, we have $f(z) \in \mathcal{S}^*(\alpha)$.

Next, in Theorem 4 below, we present the coefficient inequalities for functions in the class $\mathcal{K}(\alpha)$.

Theorem 4 *Let $0 \leqq \alpha < 1$. If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} n(n-\alpha) \left| \left(\frac{1+b}{n+b} \right)^s \right| |a_n| \leqq 1 - \alpha, \tag{3.5}$$

then $f(z) \in \mathcal{K}(\alpha)$.

Proof. Using Silverman's result (Remark 2 (i)), we can prove this theorem. □

Letting $\alpha = 0$ in Theorem 4, we have

Corollary 1 *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} n^2 \left| \left(\frac{1+b}{n+b} \right)^s \right| |a_n| \leqq 1, \tag{3.6}$$

then $f(z)$ is convex.

Furthermore, we can have

Theorem 5 *Let $0 \leqq \alpha < 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}(\alpha)$, then*

$$|a_n| \leqq \frac{2(1-\alpha)}{n(n-1)} \left| \left(\frac{n+b}{1+b} \right)^s \right| \cdot \prod_{j=2}^{n-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{3.7}$$

Proof. We set

$$p(z) := \frac{1 + \dfrac{z\mathcal{J}''_{s,b}(f)(z)}{\mathcal{J}'_{s,b}(f)(z)}}{1-\alpha} = 1 + \sum_{n=2}^{\infty} c_n z^n.$$

Then $p(z)$ is analytic with

$$p(0) = 1 \quad \text{and} \quad \operatorname{Re}\{p(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Since

$$z\mathcal{J}''_{s,b}(f)(z) = [(1-\alpha)(p(z)-1)]\,\mathcal{J}'_{s,b}(f)(z),$$

by virtue of equation

$$\mathcal{J}_{s,b}(f)(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^s a_n z^n, \tag{3.8}$$

we have

$$n(n-1)\left(\frac{1+b}{n+b}\right)^s a_n = (1-\alpha)\left[c_{n-1} + \sum_{m=2}^{n-1} m\left(\frac{1+b}{m+b}\right)^s a_m c_{n-m}\right] \quad (n \in \mathbb{N}\setminus\{1\}). \tag{3.9}$$

By applying Lemma 2, we obtain

$$n(n-1)\left|\left(\frac{1+b}{n+b}\right)^s\right||a_n| \leqq 2(1-\alpha)\left[1 + \sum_{n=2}^{n-1} m\left|\left(\frac{1+b}{m+b}\right)^s\right||a_m|\right]. \tag{3.10}$$

We shall prove, by using the principle of mathematical induction, that the inequality (3.7) is satisfied for $n \in \mathbb{N}\setminus\{1\}$. □

Putting $\alpha = 0$ in Theorem 5, we have

Corollary 2 *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}$, *then*

$$|a_n| \leqq \left|\left(\frac{n+b}{1+b}\right)^s\right|.$$

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