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CHARACTERIZATIONS OF DERIVATIONS ON RINGS WITH A NONTRIVIAL IDEMPOTENT

RUNLING AN, JINCHUAN HOU, AND KICHI-SUKE SAITO

1. Introduction

Let $\mathcal{A}$ be a unital ring with the unit I. Recall that an additive map δ from $\mathcal{A}$ into itself is called a derivation if $\delta(AB)=\delta(A)B+A\delta(B)$ for all $A, B\in \mathcal{A}$. As well known, derivations are very important maps both in theory and applications, and were studied intensively. The question under what conditions that an additive map becomes a derivation attracted much attention of many mathematicians. Over the past years considerable attention has been paid to the question of determining derivations through their action on the zero-product elements $A, B\in \mathcal{A}$ with $AB=0$ (see [2, 3, 4]). One popular topic is to characterize maps behaving like derivations when acting on zero-product elements, that is, a map δ : $\mathcal{A}\rightarrow \mathcal{A}$ satisfying

$$\delta(A)B+A\delta(B)=0 \quad \text{for any } A, B\in \mathcal{A} \text{ with } AB=0. \tag{1.1}$$

It was shown in [3, 4] that every additive map δ satisfying Eq.(1.1) on a unital prime ring containing a nontrivial idempotent must have the form $\delta(AB)=\delta(A)B+A\delta(B)-\delta(I)AB$ for any A, B, where $\delta(I)$ is a central element. Thus every map δ satisfying Eq.(1.1) on a unital prime ring containing a nontrivial idempotent if and only if there exists an additive derivation τ and a central element C such that $\delta(A)=\tau(A)+CA$ for all A. A similar result was obtained in [1] for maps on the triangular rings. These results reveal that every map behaves like a derivation on zero-product elements (a local structure) is in fact a derivation (a global structure) when it vanishes at I. Motivated by these results, more generally, in this paper we describe additive maps δ : $\mathcal{A}\rightarrow \mathcal{A}$ which satisfy $\delta(A)B+A\delta(B)=\delta(AB)$ for every $A, B\in \mathcal{A}$ with $AB=Z$ on a unital ring containing a nontrivial idempotent P, where $Z=0$, P or I respectively, and new characterizations of derivations are got.

Recall that a ring $\mathcal{A}$ is said to be prime if $a\mathcal{A}b=0$ implies that $a=0$ or $b=0$. To recall the notion of triangular rings, let $\mathcal{A}$ and $\mathcal{B}$ be two unital rings (or algebras) with unit I_1 and I_2 respectively, and let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, that is, $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule

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satisfying, for $A \in \mathcal{A}$, $A\mathcal{M} = \{0\} \Rightarrow A = 0$ and for $B \in \mathcal{B}$, $\mathcal{M}B = \{0\} \Rightarrow B = 0$. The ring (or algebra)

$$\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \{\begin{pmatrix} X & W \\ 0 & Y \end{pmatrix} : X \in \mathcal{A}, W \in \mathcal{M}, Y \in \mathcal{B}\}$$

under the usual matrix addition and formal matrix multiplication is called a triangular ring (or algebra) over rings (algebras) $\mathcal{A}$ and $\mathcal{B}$ (ref. [5]). It is obvious that the triangular rings are unital and contain a nontrivial idempotent $P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$, which we call it the standard idempotent.

Let $\mathcal{A}$ be a unital ring containing a nontrivial idempotent P and satisfying that $PAP\mathcal{A}(I-P) = \{0\}$ and $P\mathcal{A}(I-P)B(I-P) = \{0\}$ will imply $PAP = 0$ and $(I-P)B(I-P) = 0$, respectively. Note that, the set of above rings contains all unital prime rings with a nontrivial idempotent and all triangular rings. Let $\delta : \mathcal{A} \to \mathcal{A}$ be an additive map. In Section 2, we show that, if δ satisfies Eq.(1.1), then $\delta(I) = C$ belongs to the center of $\mathcal{A}$, and there exists an additive derivation τ such that $\delta(A) = \tau(A) + CA$ for all $A \in \mathcal{A}$. Thus this result generalizes the corresponding results in [1, 3, 4]. Particularly, a linear map on a factor von Neumann algebra satisfies Eq.(1.1) if and only if it has the form $A \mapsto TA - AT + \lambda A$, where T is an element in the algebra and λ is a scalar. In Section 3, we assume that, for every $A \in \mathcal{A}$, there is some integer n (depending on A) such that $nI - A$ is invertible. Then δ satisfies that $\delta(AB) = \delta(A)B + A\delta(B)$ for any $A, B \in \mathcal{A}$ with $AB = P$ if and only if it is a derivation. As a consequence, one sees that every additive map behaving like a derivation at nontrivial idempotent-product elements on a unital prime Banach algebra is a derivation. Section 4 is devoted to characterizing the additive maps behaving like derivations at unit-product elements. Assume that the characteristic of $\mathcal{A}$ is not 3 with $\frac{1}{2}I \in \mathcal{A}$, and, for every $A \in \mathcal{A}$, there is some integer n (depending on A) such that $nI - A$ is invertible. If δ satisfies $\delta(AB) = \delta(A)B + A\delta(B)$ for every $A, B \in \mathcal{A}$ with $AB = I$, then δ is a Jordan derivation, that is, $\delta(A^2) = \delta(A)A + A\delta(A)$ for all $A \in \mathcal{A}$. Particularly, for the cases $\mathcal{A}$ is a prime ring or a triangular ring, then δ is a derivation. As a corollary of above results, we obtain that an additive map on a factor von Neumann algebra behaving like a derivation at nonzero idempotent-product elements if and only if it is a derivation.

It is worth mentioning here that the applications of our main results to Banach and operator algebras do not require any topology. It is therefore surprising to have purely algebraic results carry over directly to analytical results with no modification.

2. Maps behaving like derivations at zero-product elements

In this section, we characterize additive maps behaving like derivations at zero-product elements on unital rings containing a nontrivial idempotent.

The following is our main result.

Theorem 2.1. *Let $\mathcal{A}$ be a unital ring with the unit I. Assume that $\mathcal{A}$ contains a nontrivial idempotent P such that $PAP\mathcal{A}(I-P) = \{0\}$ and $P\mathcal{A}(I-P)B(I-P) = \{0\}$ imply $PAP = 0$*

and $(I-P)B(I-P)=0$, *respectively. Then an additive map* $\delta : \mathcal{A} \to \mathcal{A}$ *satisfies*

$$\delta(A)B + A\delta(B) = 0 \text{ for any } A, B \in \mathcal{A} \text{ with } AB = 0 \tag{2.1}$$

if and only if there exist an additive derivation τ *and a central element* C *of* $\mathcal{A}$ *such that* $\delta(A) = \tau(A) + CA$ *for all* $A \in \mathcal{A}$.

Because a prime ring satisfies the hypotheses of Theorem 2.1 if it contains a nontrivial idempotent, the following result is immediate from Theorem 2.1, which was obtained in [3, 4].

Theorem 2.2. *Let* $\mathcal{A}$ *be a unital prime ring containing a nontrivial idempotent* P, *and let* $\delta : \mathcal{A} \to \mathcal{A}$ *be an additive map. Then* δ *satisfies Eq.(2.1) if and only if there exist an additive derivation* τ *and a central element* C *of* $\mathcal{A}$ *such that* $\delta(A) = \tau(A) + CA$ *for all* $A \in \mathcal{A}$.

As an application to operator algebra theory, recall that a von Neumann algebra $\mathcal{M}$ is a subalgebra of some $\mathcal{B}(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space H, which satisfies the double commutant property: $\mathcal{M}'' = \mathcal{M}$, where $\mathcal{M}' = \{T \mid T \in \mathcal{B}(H) \text{ and } TA = AT \ \forall A \in \mathcal{M}\}$ and $\mathcal{M}'' = \{\mathcal{M}'\}'$. $\mathcal{M}$ is called a factor if its center $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. Note that every linear derivation of a von Neumann is inner and thus continuous.

Corollary 2.3. *Let* $\mathcal{M}$ *be a factor von Neumann algebra, and let* $\delta : \mathcal{M} \to \mathcal{M}$ *be a linear map. Then* δ *satisfies Eq.(2.1) if and only if there exists an element* $T \in \mathcal{M}$ *and a complex number* λ *such that* $\delta(A) = TA - AT + \lambda A$ *for all* $A \in \mathcal{M}$.

Theorem 2.1 is also a refine of a result in [1] by omitting the assumption that $\delta(I)$ is a central element.

Theorem 2.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital rings and* $\mathcal{M}$ *be a faithful* $(\mathcal{A}, \mathcal{B})$*-bimodule. Let* $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be the triangular ring. Assume that* $\delta : \mathcal{T} \to \mathcal{T}$ *is an additive map. Then* δ *satisfies Eq.(2.1) if and only if there exist a central element* C *of* $\mathcal{T}$ *and an additive derivation* $\tau : \mathcal{T} \to \mathcal{T}$ *such that* $\delta(T) = \tau(T) + CT$ *for all* $T \in \mathcal{T}$.

Gilfeather and Larson introduced a concept of nest subalgebras of von Neumann algebras, which is a generalization of Ringrose's original concept of nest algebras. Let $\mathcal{R}$ be a von Neumann algebra acting on a complex Hilbert space H. A nest $\mathcal{N}$ in $\mathcal{R}$ is a totally ordered family of orthogonal projections in $\mathcal{R}$ which is closed in the strong operator topology, and which includes 0 and I. A nest is said to be non-trivial if it contains at least one non-trivial projection. If P is a projection, we let $P^{\perp}$ denote $I - P$. The nest subalgebra of $\mathcal{R}$ associated to a nest $\mathcal{N}$, denoted by $\mathrm{Alg}\mathcal{N}$, is the set of all elements $A \in \mathcal{R}$ satisfying $PAP = AP$ for each $P \in \mathcal{N}$. When $\mathcal{R} = \mathcal{B}(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space H, $\mathrm{Alg}\mathcal{N}$ is the usual one on the Hilbert space H.

From Theorem 2.4, we get a characterization of additive maps behaving like derivations at zero-product elements between nest subalgebras of factor von Neumann algebras.

Corollary 2.5. *Let* $\mathcal{N}$ *be a non-trivial nest in a factor von Neumann algebra* $\mathcal{R}$, *and let* $\mathrm{Alg}\mathcal{N}$ *be the associated nest algebra. Assume that* $\delta : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ *is an additive map. Then* δ *satisfies Eq.(2.1) if and only if there exist a derivation* τ *of* $\mathrm{Alg}\mathcal{N}$ *and a scalar* λ *such that* $\delta(A) = \tau(A) + \lambda A$ *for all* $A \in \mathrm{Alg}\mathcal{N}$.

To prove Theorem 2.1, we need the following lemma.

Lemma 2.5. *Assume that $\mathcal{A}$ is a ring in Theorem 2.1 and $\delta : \mathcal{A} \to \mathcal{A}$ is an additive map which satisfies Eq.(2.1). Then $\delta(P) = \delta(P)P + P\delta(P) - \delta(I)P$ and $\delta(I)P = P\delta(I)$ for every idempotent $P \in \mathcal{A}$.*

Proof For any idempotent P, as $(I-P)P = 0$, we have $\delta((I-P)P) = (\delta(I) - \delta(P))P + (I-P)\delta(P) = 0$, that is $\delta(P) = \delta(P)P + P\delta(P) - \delta(I)P$. Similarly, $P(I-P) = 0$ implies that $\delta(P) = \delta(P)P + P\delta(P) - P\delta(I)$. So the lemma follows. □

The sketch of proof of Theorem 2.1 The "if" part is obvious, we only check the "only if" part.

Let $P = P_1$ be a nontrivial idempotent in $\mathcal{A}$, and $P_2 = I - P_1$. Set $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, $i, j = 1, 2$, then $\mathcal{A} = \mathcal{A}_{11} \dot{+} \mathcal{A}_{12} \dot{+} \mathcal{A}_{21} \dot{+} \mathcal{A}_{22}$. Also, we regard $P_1 = I_1$ and $P_2 = I_2$ as the unit of $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$, respectively. Since δ is additive, for any $A_{ij} \in \mathcal{A}_{ij}$, we can write $\delta(A_{ij}) = \delta_{11}(A_{ij}) + \delta_{12}(A_{ij}) + \delta_{21}(A_{ij}) + \delta_{22}(A_{ij})$, where $\delta_{11} : \mathcal{A}_{ij} \to \mathcal{A}_{11}$, $\delta_{12} : \mathcal{A}_{ij} \to \mathcal{A}_{12}$, $\delta_{21} : \mathcal{A}_{ij} \to \mathcal{A}_{21}$ $\delta_{22} : \mathcal{A}_{ij} \to \mathcal{A}_{22}$ are additive maps, $i, j \in \{1, 2\}$.

The proofs are finished by intensive study of additive maps $\delta_{i,j}$, $i, j \in \{1, 2\}$, and we mainly check the following two claims.

Claim 1. $\delta(I)$ is a central element.

Define $\tau(A) = \delta(A) - \delta(I)A$, then τ also satisfies Eq.(2.1) and $\delta(I) = 0$. Therefore we may assume that $\delta(I) = 0$. Next we show δ is a derivation.

Claim 2. δ_{ij} satisfies the following conditions $i, j \in \{1, 2\}$.

(1) $\delta_{22}(X) = 0$, $\delta_{12}(X) = X\delta_{12}(I_1)$, $\delta_{21}(X) = \delta_{21}(I_1)X \quad \forall X \in \mathcal{A}_{11}$;

(2) $\delta_{11}(W) = 0$, $\delta_{12}(W) = -\delta_{12}(I_1)W$, $\delta_{21}(W) = -W\delta_{21}(I_1) \quad \forall W \in \mathcal{A}_{22}$;

(3) $\delta_{11}(Y) = -Y\delta_{21}(I_1)$, $\delta_{21}(Y) = 0$, $\delta_{22}(Y) = \delta_{21}(I_1)Y \quad \forall Y \in \mathcal{A}_{12}$;

(4) $\delta_{22}(Z) = Z\delta_{12}(I_1)$, $\delta_{12}(Z) = 0$, $\delta_{11}(Z) = -\delta_{12}(I_1)Z \quad \forall Z \in \mathcal{A}_{21}$;

(5) $\delta_{12}(XY) = \delta_{11}(X)Y + X\delta_{12}(Y)$, $\delta_{11}(X_1X_2) = \delta_{11}(X_1)X_2 + X_1\delta_{11}(X_2) \quad \forall X_1, X_2 \in \mathcal{A}_{11}, Y \in \mathcal{A}_{12}$;

(6) $\delta_{12}(YW) = \delta_{12}(Y)W + Y\delta_{22}(W)$, $\delta_{22}(W_1W_2) = \delta_{22}(W_1)W_2 + W_1\delta_{22}(W_2) \quad \forall W_1, W_2 \in \mathcal{A}_{22}, Y \in \mathcal{A}_{12}$;

(7) $\delta_{21}(ZX) = \delta_{21}(Z)X + Z\delta_{11}(X)$, $\delta_{21}(WZ) = \delta_{22}(W)Z + W\delta_{21}(Z) \quad \forall X \in \mathcal{A}_{11}, W \in \mathcal{A}_{22}, Z \in \mathcal{A}_{21}$;

(8) $\delta_{11}(YZ) = \delta_{12}(Y)Z + Y\delta_{21}(Z)$, $\delta_{22}(ZY) = \delta_{21}(Z)Y + Z\delta_{12}(Y) \quad \forall Y \in \mathcal{A}_{12}, Z \in \mathcal{A}_{21}$.

Now it is easy to check that δ is a derivation by claim 2.

3. Maps behaving like derivations at nontrivial idempotent-product elements

In this section we characterize the additive maps behaving like derivations at nontrivial idempotent-product elements on unital rings. The following is the main result.

Theorem 3.1. *Let $\mathcal{A}$ be a unital ring with unit I. Assume that, for every $A \in \mathcal{A}$, there exists some integer n such that $nI - A$ is invertible, and assume further that $\mathcal{A}$ contains a nontrivial idempotent P such that $PAPA(I-P) = 0$ and $PA(I-P)B(I-P) = 0$ implies respectively* *that $PAP = 0$ and $(I - P)B(I - P) = 0$. Then an additive map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\delta(AB) = \delta(A)B + A\delta(B) \text{ for any } A, B \in \mathcal{A} \text{ with } AB = P \tag{3.1}$$

if and only if δ is a derivation.

In particular, we have the following corollaries.

Corollary 3.2. *Let $\mathcal{A}$ be a unital prime ring. Assume that, for every $A \in \mathcal{A}$, there exists some integer n such that $nI - A$ is invertible. If an additive map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies Eq.(3.1) for some nontrivial idempotent $P \in \mathcal{A}$, then δ is a derivation.*

If $\mathcal{A}$ is unital (real or complex) Banach algebra, then $nI - A$ is invertible whenever $n > \|A\|$. So, the following corollaries are immediate from Corollary 3.2 without any more additional assumptions.

Corollary 3.3. *Let $\mathcal{A}$ be a unital prime Banach algebra. Then every additive map satisfies Eq.(3.1) for some nontrivial idempotent in $\mathcal{A}$ if and only if δ is a derivation.*

Corollary 3.4. *Let $\mathcal{A}$ be a factor von Neumann algebra. Then every additive map satisfies Eq.(3.1) for some nontrivial idempotent in $\mathcal{A}$ if and only if δ is a derivation.*

For triangular rings (algebras), by Theorem 3.1. we have

Theorem 3.5.*Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings with units I_1 and I_2, respectively, and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring and P be the standard idempotent of it. Assume that, for every $A \in \mathcal{A}$, there is some integer n such that $nI_1 - A$ is invertible. Then every additive map $\delta : \mathcal{T} \to \mathcal{T}$ satisfies Eq.(3.1) for the standard idempotent in $\mathcal{A}$ if and only if δ is a derivation.*

Thus by Theorem 3.5, we get

Corollary 3.6. *Let $\mathcal{N}$ be a non-trivial nest in a factor von Neumann algebra $\mathcal{R}$ and $\mathrm{Alg}\mathcal{N}$ be the associated nest algebra. Then an additive map $\delta : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ satisfying Eq.(3.1)for an idempotent element Q satisfying $PQ = Q$ and $QP = P$ for some nontrivial projection $P \in \mathcal{N}$ if and only if δ is a derivation.*

The sketch of proof of Theorem 3.1 We use the decomposition and notations in section 2. By investigation $AB = P$, we prove Claim 2 in section 2 is true for δ_{ij} $(i, j = 1, 2)$. Then it is easy to check δ is a derivation.

4. Maps behaving like derivations at unit-product elements

In this section, we discuss the additive maps behaving like derivations at unit-product elements on unital rings with a non-trivial idempotent.

The following is the main result. Note that here we assume, in addition, that the ring is of characteristic not 3 and contains the half of unit. We do not know if these assumptions may be deleted.

Theorem 4.1. *Let $\mathcal{A}$ be a unital ring with unit I and of characteristic not 3. Assume that $\mathcal{A}$ satisfies the following conditions:*

(i) $\frac{1}{2}I \in \mathcal{A}$;

(ii) *there exists a non-trivial idempotent* $P \in \mathcal{A}$ *such that, for any* $A \in \mathcal{A}$, $PAP\mathcal{A}(I-P) = \{0\}$ *and* $P\mathcal{A}(I-P)A(I-P) = \{0\}$ *imply* $PAP = 0$ *and* $(I-P)A(I-P) = 0$, *respectively;*

(iii) *for any* $A \in \mathcal{A}$, *there exists some integer* n *such that* $nI - A$ *is invertible.*

If $\delta : \mathcal{A} \to \mathcal{A}$ *is an additive map satisfying*

$$\delta(AB) = \delta(A)B + A\delta(B) \text{ for any } A, B \in \mathcal{A} \text{ with } AB = I, \tag{4.1}$$

then δ *is a Jordan derivation.*

A well known result due to Herstein [6] states that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. Since every unital ring containing $\frac{1}{2}I$ is of characteristic not 2, the following result is immediate from Theorem 4.1.

Theorem 4.2. *Let* $\mathcal{A}$ *be a unital prime ring with unit* I *and of characteristic not 3. Assume that* $\mathcal{A}$ *contains* $\frac{1}{2}I$ *and a non-trivial idempotent* P *and, for every* $A \in \mathcal{A}$, *there exists some integer* n *such that* $nI - A$ *is invertible. Then* $\delta : \mathcal{A} \to \mathcal{A}$ *is an additive map satisfying Eq.(4.1) if and only if* δ *is a derivation.*

In particular, applying above result to operator algebras, we have

Corollary 4.3. *Let* $\mathcal{A}$ *be a unital prime Banach algebra containing a non-trivial idempotent* P, *and let* $\delta : \mathcal{A} \to \mathcal{A}$ *be an additive map. Then* δ *satisfies Eq.(4.1) if and only if* δ *is a derivation.*

For triangular rings, by Theorem 4.1 we get

Corollary 4.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital rings of characteristic not 3 with units* I_1 *and* I_2, *respectively, and* $\mathcal{M}$ *be a faithful* $(\mathcal{A}, \mathcal{B})$*-bimodule. Let* $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be the triangular ring. Assume that* $\frac{1}{2}I_1 \in \mathcal{A}$ *and* $\frac{1}{2}I_2 \in \mathcal{B}$, *and further for any* $A \in \mathcal{A}$, $B \in \mathcal{B}$, *there are some integers* n_1, n_2 *such that* $n_1 I_1 - A$ *and* $n_2 I_2 - B$ *are invertible. Then every additive map* $\delta : \mathcal{T} \to \mathcal{T}$ *satisfies Eq.(4.1) if and only if* δ *is a derivation.*

From Corollary 4.4, we have

Corollary 4.5. *Let* $\mathcal{N}$ *be a non-trivial nest in a factor von Neumann algebra* $\mathcal{R}$ *and let* $\mathrm{Alg}\mathcal{N}$ *be the associated nest algebra. Then every additive map* $\delta : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ *satisfies Eq.(4.1) if and only if* δ *is a derivation.*

From Corollary 3.6 and Corollary 4.5, we obtain

Corollary 4.6. *Let* $\mathcal{N}$ *be a non-trivial nest in a factor von Neumann algebra* $\mathcal{R}$ *and let* $\mathrm{Alg}\mathcal{N}$ *be the associated nest algebra. Then every additive map* δ *satisfies* $\delta(AB) = \delta(A)B + A\delta(B)$ *for any* $A, B \in \mathcal{A}$ *with* $AB = Q$ *for some nonzero idempotent element* Q *with* $PQ = Q$ *and* $QP = P$ *for some nonzero projection* $P \in \mathcal{N}$ *if and only if* δ *is a derivation.*

From Corollary 3.4 and Corollary 4.3, we get

Corollary 4.7. *Let* $\mathcal{A}$ *be a factor von Neumann algebra. Then every additive map on* $\mathcal{A}$ *satisfies* $\delta(AB) = \delta(A)B + A\delta(B)$ *for any* $A, B \in \mathcal{A}$ *with* $AB = P$ *for some nonzero idempotent element* $P \in \mathcal{A}$ *if and only if* δ *is a derivation.*

The sketch of proof of Theorem 4.1 We use the decomposition and notations in section 2. By investigation $AB = I$, we prove (1)-(2) and (5)-(8) of Claim 2 in section 2 is true for δ_{ij}

$(i, j = 1, 2)$. We obtain the following equalities for any $Y \in \mathcal{A}_{12}$ and $Z \in \mathcal{A}_{21}$,

$$\begin{aligned} &\delta_{11}(Y) = -Y\delta_{21}(I_1), \quad \delta_{22}(Y) = \delta_{21}(I_1)Y, \quad Y\delta_{21}(Y) = \delta_{21}(Y)Y = 0, \\ &\delta_{21}(XY) = \delta_{21}(Y)X, \quad \delta_{21}(YW) = W\delta_{21}(Y), \quad \forall X \in \mathcal{A}_{11}, Y \in \mathcal{A}_{12}, W \in \mathcal{A}_{22}. \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\delta_{22}(Z) = Z\delta_{12}(I_1), \quad \delta_{11}(Z) = -\delta_{12}(I_1)Z, \quad \delta_{12}(Z)Z = Z\delta_{12}(Z) = 0, \\ &\delta_{12}(ZX) = X\delta_{12}(Z), \quad \delta_{12}(ZW) = \delta_{12}(Z)W, \quad \forall X \in \mathcal{A}_{11}, Z \in \mathcal{A}_{21}, W \in \mathcal{A}_{22}. \end{aligned} \tag{4.3}$$

Then it is to check δ is a Jordan derivation.

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REFERENCES

[1] R. An and J. Hou, Characterizations of derivations on triangular rings: additive maps derivable at idempotents, Lin. Alg. Appl., 431 (2009), 1070-1080.

[2] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Characterizing homomorphisms and derivations on C^*-algebras, Pro. Roy. Soc. Edinburrgh. Sect. A., 137 (2007) 1-7.

[3] M. Brešar, Characterizing homomorphisms, multipliers and derivations in rings with idempotents, Pro. Roy. Soc. Edinburgh. Sect. A., 137 (2007) 9-21.

[4] M. A. Chebotar, W.-F. Ke and P.-H. Lee. Maps characterized by action on zero products, Pacific. J. Math. Vol. 216 No. 2. (2004) 217-228.

[5] W. S. Cheung, Commuting maps of triangular algebras, J. London. Math. Soc., 63 (2001) 117-127.

[6] I. N. Herstein, Jordan derivation on prime rings, Proc. Amer. Math. Soc., 8 (1957), 1104-1110.

[7] J. Zhang, W. Yu, Jordan derivations of triangular algebras, Lin. Alg. Appl., 419 (2006) 251-255.

DEPARTMENT OF MATHEMATICS, TAIYUAN UNIVERSITY OF TECHNOLOGY, TAIYUAN, 030024, P. R. CHINA.
E-mail address, Runling An: `runlingan@yahoo.com.cn`

DEPARTMENT OF MATHEMATICS, TAIYUAN UNIVERSITY OF TECHNOLOGY, TAIYUAN, 030024, P. R. CHINA.
E-mail address, Jinchuan Hou: `jinchuanhou@yahoo.com.cn`

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NIIGATA UNIVERSITY, NIIGATA 950-2181, JAPAN.
E-mail address, Kichi-Suke Saito: `saito@math.sc.niigata-u.ac.jp`