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Loewner matrices of matrix convex and monotone functions
(joint work with F. Hiai)

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Some results in [3, 6] were reported. Here we collect results from them. For the detail, please see the papers.

1 Characterisations by Bhatia-Sano

In this section, we consider a C^1 function f from the interval $(0,\infty)$ into itself, with $f(0)=\lim_{t\to 0^+} f(t)=0$. Given any n distinct points $p_1,\ldots,p_n$ in $(0,\infty)$, let $L_f(p_1,\ldots,p_n)$ be the $n\times n$ matrix defined as

$$L_f(p_1,\ldots,p_n)=\left[\frac{f(p_i)-f(p_j)}{p_i-p_j}\right]. \qquad (1.1)$$

When $i=j$ the quotient in (1.1) is interpreted as $f'(p_i)$. Such a matrix is called a *Loewner matrix* associated with f.

For the function $f(t)=t^r$ where $r>0$, we use the symbol L_r for a Loewner matrix associated with this function. Thus

$$L_r=\left[\frac{p_i^r-p_j^r}{p_i-p_j}\right]. \qquad (1.2)$$

The function f is said to be *operator monotone* on $[0,\infty)$ if for two positive semidefinite matrices A and B (of any size n) the inequality $A\geqq B$ implies $f(A)\geqq f(B)$. Here, as usual, $A\geqq B$ means that $A-B$ is positive semidefinite (p.s.d. for short).

Karl Löwner (later Charles Loewner) in [9] showed that f is operator monotone if and only if for all n, and all $p_1,\ldots,p_n$, the Loewner matrices $L_f(p_1,\ldots,p_n)$ are p.s.d. and that the function $f(t)=t^r$ is operator monotone if and only if $0<r\leqq 1$. Consequently, if $0<r\leqq 1$, then the matrix (1.2) is p.s.d., and therefore all its eigenvalues are non-negative.

Recall the notion of operator convexity: Assume that f is a C^2 function from $(0,\infty)$ into itself, $f(0)=0$ and $f'(0)=0$. We say that f is *operator convex* if

$$f((1-t)A+tB)\leqq(1-t)f(A)+tf(B),\quad 0\leqq t\leqq 1,$$

for all p.s.d. matrices A and B (of any size n).

Let H^n be the subspace of $\mathbb{C}^n$ consisting of all $x = (x_1, \dots, x_n)$ for which $\sum_{i=1}^{n} x_i = 0$. An $n \times n$ Hermitian matrix A is said to be *conditionally positive definite* (c.p.d. for short) or *almost positive* if

$$\langle x, Ax \rangle \geqq 0 \quad \text{for all} \quad x \in H^n,$$

and *conditionally negative definite* (c.n.d. for short) if $-A$ is c.p.d. We refer the reader to [1, 4, 8] for properties of these matrices.

We proved:

Theorem 1.1. Let f be an operator convex function. Then all Loewner matrices associated with f are conditionally negative definite.

Theorem 1.2. Let $f(t) = tg(t)$ where g is an operator convex function. Then all Loewner matrices associated with f are conditionally positive definite.

Theorem 1.3. Let L_r be the $n \times n$ Loewner matrix (1.2) associated with distinct points $p_1, \dots, p_n$. Then
(i) L_r is conditionally negative definite for $1 \leqq r \leqq 2$, and conditionally positive definite for $2 \leqq r \leqq 3$.
(ii) L_r is nonsingular for $1 < r < 2$ and for $2 < r < 3$.
(iii) As a consequence, for $1 < r < 2$ the matrix L_r has one positive and $n-1$ negative eigenvalues, and for $2 < r < 3$ it has one negative and $n-1$ positive eigenvalues.

Here is the converse of Theorems 1.1 and 1.2:

Theorem 1.4. Let f be a C^2 function from $(0, \infty)$ into itself with $f(0) = f'(0) = 0$. Suppose all Loewner matrices L_f are conditionally negative definite. Then f is operator convex.

Theorem 1.5. Let f be a C^3 function from $(0, \infty)$ into itself with $f(0) = f'(0) = f''(0) = 0$. Suppose all Loewner matrices L_f are conditionally positive definite. Then there exists an operator convex function g such that $f(t) = tg(t)$.

Remark. Theorems 1.1, 1.2, 1.4 and 1.5 together say the following. Let f be a C^3 function from $(0, \infty)$ into itself with $f(0) = 0$. Let $g(t) = tf(t), h(t) = t^2 f(t)$. Then the following three conditions are equivalent.

(i) All Loewner matrices L_f are p.s.d.
(ii) All Loewner matrices L_g are c.n.d.
(iii) All Loewner matrices L_h are c.p.d.

2 Generalisations by Hiai-Sano

We already review characterizations in [3] for operator convexity of nonnegative functions on $[0,\infty)$ in terms of the conditional negative or positive definiteness of the Loewner matrices. Uchiyama [10] extended, by a rather different method, results in such a way that the assumption $f \geq 0$ is removed and the boundary condition $f(0) = f'(0) = 0$ is relaxed. Note that the conditional positive definiteness of the Loewner matrices and the matrix/operator monotony were related in [7] and [4, Chapter XV] for a real function on a general open interval.

We proved:

Theorem 2.1. *Let f be a real C^1 function on $(0,\infty)$. For each $n \in \mathbb{N}$ consider the following conditions:*

$(\mathrm{a})_n$ *f is n-convex on $(0,\infty)$;*

$(\mathrm{b})_n$ $\liminf_{t\to\infty} f(t)/t > -\infty$ *and* $L_f(t_1,\dots,t_n)$ *is c.n.d. for all* $t_1,\dots,t_n \in (0,\infty)$*;*

$(\mathrm{c})_n$ $\limsup_{t\searrow 0} tf(t) \geq 0$ *and* $L_{tf(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $t_1,\dots,t_n \in (0,\infty)$.

Then for every $n \in \mathbb{N}$ the following implications hold:

$$(\mathrm{a})_{2n+1} \Longrightarrow (\mathrm{b})_n, \quad (\mathrm{b})_{4n+1} \Longrightarrow (\mathrm{a})_n, \quad (\mathrm{a})_{n+1} \Longrightarrow (\mathrm{c})_n, \quad (\mathrm{c})_{2n+1} \Longrightarrow (\mathrm{a})_n.$$

Corollary 2.2. *Let f be a real C^1 function on $(0,\infty)$. Then the following conditions are equivalent:*

(a) *f is operator convex on $(0,\infty)$;*

(b) $\liminf_{t\to\infty} f(t)/t > -\infty$ *and* $L_f(t_1,\dots,t_n)$ *is c.n.d. for all* $n \in \mathbb{N}$ *and all* $t_1,\dots,t_n \in (0,\infty)$*;*

(c) $\limsup_{t\searrow 0} tf(t) \geq 0$ *and* $L_{tf(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $n \in \mathbb{N}$ *and all* $t_1,\dots,t_n \in (0,\infty)$.

Moreover, if the above conditions are satisfied, then $\lim_{t\to\infty} f(t)/t$ *and* $\lim_{t\searrow 0} tf(t)$ *exist in* $(-\infty,\infty]$ *and* $[0,\infty)$*, respectively.*

Theorem 2.3. *Let f be a real C^1 function on $(0,\infty)$. For each $n \in \mathbb{N}$ consider the following conditions:*

(a)$'_n$ *f is n-monotone on* $(0,\infty)$*;*

(b)$'_n$ $\limsup_{t\to\infty} f(t)/t < +\infty$, $\limsup_{t\to\infty} f(t) > -\infty$, *and* $L_f(t_1,\dots,t_n)$ *is c.p.d. for all* $t_1,\dots,t_n \in (0,\infty)$*;*

(c)$'_n$ $\liminf_{t\searrow 0} tf(t) \le 0$, $\limsup_{t\to\infty} f(t) > -\infty$, *and* $L_{tf(t)}(t_1,\dots,t_n)$ *is c.n.d. for all* $t_1,\dots,t_n \in (0,\infty)$*;*

(d)$'_n$ $\liminf_{t\searrow 0} tf(t) \le 0$, $\limsup_{t\searrow 0} t^2 f(t) \ge 0$, *and* $L_{t^2 f(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $t_1,\dots,t_n \in (0,\infty)$.

Then for every $n \in \mathbb{N}$ *the following implications hold:*

$$\text{(a)}'_n \Longrightarrow \text{(b)}'_n \text{ if } n \ge 2, \quad \text{(b)}'_{4n+1} \Longrightarrow \text{(a)}'_n, \quad \text{(a)}'_{2n+2} \Longrightarrow \text{(c)}'_n, \quad \text{(c)}'_{2n+1} \Longrightarrow \text{(a)}'_n,$$

$$\text{(a)}'_n \Longrightarrow \text{(d)}'_n \text{ if } n \ge 2, \quad \text{(c)}'_{2n+1} \Longrightarrow \text{(d)}'_n, \quad \text{(d)}'_{2n+1} \Longrightarrow \text{(c)}'_n.$$

Corollary 2.4. *Let f be a real C^1 function on* $(0,\infty)$. *Then the following conditions are equivalent:*

(a)$'$ *f is operator monotone on* $(0,\infty)$*;*

(b)$'$ $\limsup_{t\to\infty} f(t)/t < +\infty$, $\limsup_{t\to\infty} f(t) > -\infty$, *and* $L_f(t_1,\dots,t_n)$ *is c.p.d. for all* $n \in \mathbb{N}$ *and all* $t_1,\dots,t_n \in (0,\infty)$*;*

(c)$'$ $\liminf_{t\searrow 0} tf(t) \le 0$, $\limsup_{t\to\infty} f(t) > -\infty$, *and* $L_{tf(t)}(t_1,\dots,t_n)$ *is c.n.d. for all* $n \in \mathbb{N}$ *and all* $t_1,\dots,t_n \in (0,\infty)$*;*

(d)$'$ $\liminf_{t\searrow 0} tf(t) \le 0$, $\limsup_{t\searrow 0} t^2 f(t) \ge 0$, *and* $L_{t^2 f(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $n \in \mathbb{N}$ *and all* $t_1,\dots,t_n \in (0,\infty)$.

Moreover, if the above conditions are satisfied, then $\lim_{t\to\infty} f(t)/t$, $\lim_{t\to\infty} f(t)$, *and* $\lim_{t\searrow 0} tf(t)$ *exist in* $[0,\infty)$, $(-\infty,\infty]$, *and* $(-\infty,0]$, *respectively, and* $\lim_{t\searrow 0} t^\alpha f(t) = 0$ *for any* $\alpha > 1$.

Proposition 2.5. *Consider the power functions t^α on* $(0,\infty)$, *where* $\alpha \in \mathbb{R}$. *Then:*

(1) *t^α is 2-monotone if and only if* $0 \le \alpha \le 1$, *or equivalently, t^α is operator monotone. Moreover, $-t^\alpha$ is 2-monotone if and only if* $-1 \le \alpha \le 0$.

(2) *t^α is 2-convex if and only if either* $-1 \le \alpha \le 0$ *or* $1 \le \alpha \le 2$, *or equivalently, t^α is operator convex.*

(3) *$L_{t^\alpha}(t_1,t_2)$ is c.p.d. for all $t_1,t_2 \in (0,\infty)$ if and only if either* $0 \le \alpha \le 1$ *or* $\alpha \ge 2$.

(4) *$L_{t^\alpha}(t_1,t_2)$ is c.n.d. for all $t_1,t_2 \in (0,\infty)$ if and only if either* $\alpha \le 0$ *or* $1 \le \alpha \le 2$.

(5) $L_{t^\alpha}(t_1, t_2, t_3)$ *is c.p.d. for all* $t_1, t_2, t_3 \in (0, \infty)$ *if and only if either* $0 \le \alpha \le 1$ *or* $2 \le \alpha \le 3$.

(6) $L_{t^\alpha}(t_1, t_2, t_3)$ *is c.n.d. for all* $t_1, t_2, t_3 \in (0, \infty)$ *if and only if either* $-1 \le \alpha \le 0$ *or* $1 \le \alpha \le 2$.

Theorem 2.6. *Let* f *be a real* C^1 *function on* (a, b) *where* $-\infty < a < b < \infty$. *For each* $n \in \mathbb{N}$ *consider the following conditions:*

$(\alpha)_n$ f *is* n*-monotone on* (a, b);

$(\beta)_n$ $\limsup_{t \nearrow b}(b-t)f(t) < +\infty$, $\limsup_{t \nearrow b} f(t) > -\infty$, *and* $L_{(b-t)^2 f(t)}(t_1, \dots, t_n)$ *is c.p.d. for all* $t_1, \dots, t_n \in (a, b)$;

$(\gamma)_n$ $\liminf_{t \searrow a}(t-a)f(t) \le 0$, $\limsup_{t \nearrow b} f(t) > -\infty$, *and* $L_{(t-a)(b-t)f(t)}(t_1, \dots, t_n)$ *is c.n.d. for all* $t_1, \dots, t_n \in (a, b)$;

$(\delta)_n$ $\liminf_{t \searrow a}(t-a)f(t) \le 0$, $\limsup_{t \searrow a}(t-a)^2 f(t) \ge 0$, *and* $L_{(t-a)^2 f(t)}(t_1, \dots, t_n)$ *is c.p.d. for all* $t_1, \dots, t_n \in (a, b)$.

Then for every $n \in \mathbb{N}$ *the following implications hold:*

$$(\alpha)_n \Longrightarrow (\beta)_n \text{ if } n \ge 2, \quad (\beta)_{4n+1} \Longrightarrow (\alpha)_n, \quad (\alpha)_{2n+2} \Longrightarrow (\gamma)_n, \quad (\gamma)_{2n+1} \Longrightarrow (\alpha)_n,$$

$$(\alpha)_n \Longrightarrow (\delta)_n \text{ if } n \ge 2, \quad (\gamma)_{2n+1} \Longrightarrow (\delta)_n, \quad (\delta)_{2n+1} \Longrightarrow (\gamma)_n.$$

Corollary 2.7. *Let* f *be a real* C^1 *function on* (a, b) *where* $-\infty < a < b < \infty$. *Then the following conditions are equivalent:*

(α) f *is operator monotone on* (a, b);

(β) $\limsup_{t \nearrow b}(b-t)f(t) < +\infty$, $\limsup_{t \nearrow b} f(t) > -\infty$, *and* $L_{(b-t)^2 f(t)}(t_1, \dots, t_n)$ *is c.p.d. for all* $n \in \mathbb{N}$ *and all* $t_1, \dots, t_n \in (a, b)$;

(γ) $\liminf_{t \searrow a}(t-a)f(t) \le 0$, $\limsup_{t \nearrow b} f(t) > -\infty$, *and* $L_{(t-a)(b-t)f(t)}(t_1, \dots, t_n)$ *is c.n.d. for all* $n \in \mathbb{N}$ *and all* $t_1, \dots, t_n \in (a, b)$;

(δ) $\liminf_{t \searrow a}(t-a)f(t) \le 0$, $\limsup_{t \searrow a}(t-a)^2 f(t) \ge 0$, *and* $L_{(t-a)^2 f(t)}(t_1, \dots, t_n)$ *is c.p.d. for all* $n \in \mathbb{N}$ *and all* $t_1, \dots, t_n \in (a, b)$.

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