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Dual variational approach to a quasilinear Schrödinger equation arising in plasma physics

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1. Introduction

In this paper, we consider the following quasilinear Schrödinger equation:

$$i\frac{\partial z}{\partial t}=-\Delta z-|z|^{p-1}z-\kappa\Delta(|z|^{\alpha})|z|^{\alpha-2}z, \quad (t,x)\in(0,\infty)\times\mathbb{R}^{N}, \tag{1.1}$$

where $\kappa>0,\ \alpha>1,\ N\geq 1$ and $p>1$. Equation (1.1) with $\alpha=2$ derives from a superfluid film equation in plasma physics, which was introduced in [**7**, **16**]. We are interested in the standing wave solution of the form: $z(t,x)=u(x)e^{i\lambda t},\ \lambda>0$. Then we obtain the following quasilinear elliptic problem:

$$-\Delta u+\lambda u-\kappa\Delta(|u|^{\alpha})|u|^{\alpha-2}u=|u|^{p-1}u \quad \text{in } \mathbb{R}^{N}. \tag{1.2}$$

When $\kappa=0$, (1.1) becomes well-studied Schrödinger equation:

$$i\frac{\partial z}{\partial t}=-\Delta z-|z|^{p-1}z,\ (t,x)\in(0,\infty)\times\mathbb{R}^{N}.$$

In this case, the standing waves associated with ground state solutions is orbitally stable if $1<p<1+\frac{4}{N}$ and unstable if $p\geq 1+\frac{4}{N}$ (see [**4**, **9**]). Especially the standing wave is unstable when $p=3$ and $N=2$. In [**7**], they stated that the quasilinear term stabilize the standing wave if $\kappa>0$. More precisely, they showed (by a formal calculation) that the standing wave is stable when $p=3$, $N=2$ and $\alpha=2$. Thus a natural question arises: stable range of p is actually larger than that of the case $\kappa=0$?

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Recently a result very close to this answer was given in [**11**]. They studied (1.1) for the case $\alpha = 2$, $\kappa = 1$ and showed: (i) Instability of standing waves by blow-up when $p > 3 + \frac{4}{N}$. (ii) Uniqueness of solutions of stationary problem when $N = 1$. (iii) Stability of solutions of the minimizing problem:

$$c(\lambda) = \inf\{\mathcal{E}(u); u \in X,\ \|u\|_{L^2}^2 = \lambda\},\ (X \text{ will be defined later})$$

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + 2|\nabla |u||^2 |u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx$$

if one of the conditions hold: $1 < p < 1 + \frac{4}{N}$, $\lambda > 0$ or $1 + \frac{4}{N} \le p < 3 + \frac{4}{N}$, λ is sufficiently large. Their result suggests that the stability holds for $1 < p < 3 + \frac{4}{N}$. However as pointed out in [**11**], this remains open because we don't know the uniqueness of ground state solutions. The purpose of this paper is to study the existence and the uniqueness of the ground state solution of (1.2).

To state the existence of a ground state solution, we use the following notation. Equation (1.2) has a variational structure, that is, one can obtain solutions of (1.2) as critical points of the associated functional J defined by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2\, dx + \frac{\kappa}{2\alpha}\int_{\mathbb{R}^N} |\nabla |u|^\alpha|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2\, dx + \frac{\alpha\kappa}{2}\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2}\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx. \end{aligned} \tag{1.3}$$

We remark that nonlinear functional $\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2}\, dx$ is not defined on all $H^1(\mathbb{R}^N)$ except for $N = 1$. Thus the natural function space for $N \ge 2$ is given by

$$X := \{u \in H^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2}\, dx < \infty\}. \tag{1.4}$$

We define the ground state energy level and the set of ground state solutions by

$$\begin{aligned} &m := \inf\{J(u);\ J'(u) = 0,\ u \in X \setminus \{0\}\}, \\ &\mathcal{G} := \{w \in X \setminus \{0\};\ J(w) = m,\ J'(w) = 0\}. \end{aligned}$$

Existence of a positive solution of (1.2) has been studied in [**1, 10, 17, 18, 21**]. In [**1**] and [**19**], they showed that if $\lambda > 0$, $\kappa > 0$, $\alpha > 1$ and

$$2\alpha - 1 \le p < \frac{(2\alpha-1)N+2}{N-2} \text{ for } N \ge 3,\ 2\alpha - 1 \le p < \infty \text{ for } N = 1, 2,$$

then (1.2) has *at least one* ground state solution which is positive, radially symmetric, decreasing with respect to $r = |x|$ and has the exponential decay by using the Nehari manifold. There are two remarkable points in their results. Firstly, they assumed $2\alpha-1 \le p$ to make use of the Nehari manifold. This seems to be rather technical. Secondly, their results are not sufficient to prove the uniqueness because the statement only says the positivity and radial symmetry hold for *at least one* ground state solution. To prove the uniqueness, we need more precise properties on ground state solutions. Actually we have the following result which generalizes the result in [**11**] for the case $\alpha = 2$.

Theorem 1.1. *Let* $\lambda > 0$, $\kappa > 0$, $\alpha > 1$ *and* $1 < p < \dfrac{(2\alpha-1)N+2}{N-2}$ *for* $N \ge 3$, $1 < p < \infty$ *for* $N = 1, 2$. *Then* $\mathcal{G} \neq \emptyset$ *and any* $w \in \mathcal{G}$ *satisfies the following properties:*

(i) $w \in C^2(\mathbb{R}^N, \mathbb{R})$.

(ii) $w(x) > 0$ *for all* $x \in \mathbb{R}^N$.

(iii) w *is radially symmetric:* $w(x) = w(|x|)$ *and decreases with respect to* $r = |x|$.

(iv) *There exist* c, $c' > 0$ *such that*

$$\lim_{|x|\to\infty} e^{\sqrt{\lambda}|x|}(|x|+1)^{\frac{N-1}{2}} w(x) = c, \quad \lim_{r\to\infty} e^{\sqrt{\lambda}r}(r+1)^{\frac{N-1}{2}} \frac{\partial w}{\partial r} = -c'.$$

As to the uniqueness of ground state solutions, we have the following results.

Theorem 1.2. *Assume* $N \ge 3$, $\alpha > 1$, $\alpha - 1 \le p < 3\alpha - 3$ *and*

$$\max\left\{\frac{\alpha-p}{\alpha(p-1)}, \frac{p}{\alpha(3\alpha-p-3)}\right\} \le \kappa\lambda^{\frac{2\alpha-2}{p-1}}.$$

Then the ground state solution of (1.2) *is unique.*

Corollary 1.3. *Suppose* $N \ge 3$, $\alpha = 2$, $1 < p < 3$ *and* $\max\{\frac{2-p}{2(p-1)}, \frac{p}{2(3-p)}\} \le \kappa\lambda^{\frac{2}{p-1}}$. *Then the ground state solution of* (1.2) *is unique.*

As we will see later, we need a stronger assumption when $N = 2$. We obtain the following sufficient conditions for the uniqueness of positive radial solutions in the case $N = 2$.

Theorem 1.4. *Suppose* $N = 2$, $\alpha > 2$ *and* $2\alpha - 1 \le p < 3\alpha - 3$. *Then*

(i) *For every fixed* $\kappa > 0$, *there exists* $\lambda_0 > 0$ *such that if* $\lambda \ge \lambda_0$, *then the ground state solution of* (1.2) *is unique.*

(ii) *For every fixed* $\lambda > 0$, *there exists* $\kappa_0 > 0$ *such that if* $\kappa \ge \kappa_0$, *then the ground state solution of* (1.2) *is unique.*

Since our problem is quasilinear, it is rather difficult to handle (1.2) directly. However problem (1.2) has a nice property, namely, one can we adapt *dual variational approach.*

More precisely, we convert our quasilinear equation into a semilinear equation by using a suitable translation f. We will see that the set of ground state solutions $\mathcal{G}$ has one-to-one correspondence to that of the semilinear problem. This enables us to prove the positivity and the radial symmetry of *any* ground state solution.

Moreover Theorem 1.1 implies that

$$\mathcal{G} \subset \{u \in X \cap C^2;\ u \text{ is a positive radial solution of (1.2)}\}.$$

Thus we can prove the uniqueness of ground state solutions if we could show the uniqueness of positive radial solutions of (1.2). We will also see that the set of positive radial solutions has one-to-one correspondence to that of the semilinear problem. This enables us to apply the uniqueness result [**3**, **12**, **15**, **20**, **22**] for semilinear elliptic equations.

This paper is organized as follows. In Section 2, we introduce the dual variational approach. In section 3, we study the existence of a ground state solution. Finally in Section 4, we study the uniqueness of ground state solutions.

Notation. Throughout this paper, we use the following notation:

$$\|v\|_{H^1}^2 = \int_{\mathbb{R}^N} |\nabla v|^2 + \lambda v^2 dx, \ v \in H^1(\mathbb{R}^N).$$

2. Dual variational approach

In this section, we introduce a dual variational approach. More precisely, we will see that quasilinear problem (1.2) can be converted into a semilinear problem and there is an one-to-one correspondence between two problems.

Let $\tilde{f}$ be a function defined by

$$\tilde{f}(s) := \int_0^s \sqrt{1 + \kappa\alpha t^{2\alpha-2}}\, dt.$$

Then $\tilde{f}$ is positive, monotone, convex and C^∞ on $(0, \infty)$. For $s < 0$, we put $\tilde{f}(s) = -\tilde{f}(-s)$.

Remark 2.1. *$\tilde{f}$ can be written by elliptic functions. When $\alpha = 2$, simply we have*

$$\tilde{f}(t) = \frac{\sinh^{-1}(\sqrt{2\kappa}t)}{2\sqrt{2\kappa}} + \frac{\sqrt{2\kappa}}{2} s\sqrt{s^2 + \frac{1}{2\kappa}}.$$

Since $\tilde{f}$ is monotone, we can define the inverse function f. Then f satisfies the following ODE:

$$f'(s) = \frac{1}{\sqrt{1 + \alpha\kappa f(s)^{2\alpha-2}}} \quad \text{on } s \in [0, \infty), \quad f(0) = 0. \tag{2.1}$$

From (2.1), we can observe that

$$f''(s) = -\kappa\alpha(\alpha-1)f^{2\alpha-3}(f')^4 = (\alpha-1)\frac{(f')^4}{f} - (\alpha-1)\frac{(f')^2}{f}, \tag{2.2}$$

$$f'''(s) = \frac{1}{f^2}\{4(\alpha-1)^2(f')^7 - (6\alpha-5)(\alpha-1)(f')^5 + (\alpha-1)(2\alpha-1)(f')^3\}. \tag{2.3}$$

Function f satisfies the following properties.

Lemma 2.2. *$f(s)$ satisfies the following properties:*

(i) $f(s) \le s$, $f'(s) \in (0,1]$, $f''(s) \le 0$ *for all* $s \ge 0$.

(ii) $\frac{1}{\alpha}f(s) \le sf'(s) \le f(s)$ *for all* $s \ge 0$.

(iii) $\left(\frac{f(s)}{sf'(s)}\right)' > 0$ *for all* $s > 0$.

Lemma 2.3. *It follows*

(i) $\lim_{s\to\infty} \frac{f(s)}{s^{\frac{1}{\alpha}}} = \left(\frac{\alpha}{\kappa}\right)^{\frac{1}{2\alpha}}$, $\lim_{s\to 0} \frac{f(s)}{s} = 1$.

(ii) $\lim_{s\to\infty} \frac{f'(s)}{s^{\frac{1-\alpha}{\alpha}}} = \frac{1}{\alpha}\left(\frac{\alpha}{\kappa}\right)^{\frac{1}{2\alpha}}$.

(iii) $\lim_{s\to\infty} \frac{f(s)}{sf'(s)} = \alpha$.

For the proof of Lemmas 2.2 and 2.3, we refer to [**1**, **2**].

Using the function f, we consider the following semilinear problem:

$$-\Delta v + \lambda f(v)f'(v) = |f(v)|^{p-1}f(v)f'(v) \quad \text{in } \mathbb{R}^N. \tag{2.4}$$

The functional associated to (2.4) is defined by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \lambda f(v)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx.$$

Lemma 2.4. *Assume* $1 < p < \frac{(2\alpha-1)N+2}{N-2}$ *for* $N \ge 3$, $1 < p < \infty$ *for* $N = 1, 2$. *Then* $I(v)$ *is well-defined on* $H^1(\mathbb{R}^N)$ *and of class* $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Proof. By (i) of Lemma 2.2, it follows

$$I(v) \le \int_{\mathbb{R}^N} |\nabla v|^2 + \lambda v^2\,dx + \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx.$$

By Lemma 2.3 (i), we have

$$f(v) \le C_1\chi_{|v|\le 1}|v| + C_2\chi_{|v|\ge 1}|v|^{\frac{1}{\alpha}}, \ f(v)^{p+1} \le C_1v^2 + C_2|v|^{\frac{p+1}{\alpha}}$$

where χ is the characteristic function. Since $\frac{p+1}{\alpha} < \frac{2N}{N-2}$, $I(v)$ is well-defined on $H^1(\mathbb{R}^N)$. In a standard way, we can show $I(v) \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. ∎

We have the following relation between (1.2) and (2.4), which was already shown in [**1**, **10**]. For the sake of completeness, we give the proof.

Lemma 2.5. *Suppose v is a nontrivial critical point of I and $v > 0$. Then $u = f(v)$ is a positive solution of* (1.2).

Proof. We can easily see that if $v \in H^1(\mathbb{R}^N)$ is a nontrivial critical point of $I(v)$, then v is a solution of (2.4). By standard elliptic regularity theory, we see that $v \in C^2(\mathbb{R}^N)$. Moreover $v > 0$ implies $u > 0$. Since $f \in C^\infty(0, \infty)$, we also have $u \in C^2(\mathbb{R}^N)$.

For $v = \tilde{f}(u)$, we have

$$\nabla v = \tilde{f}'(u)\nabla u, \quad \Delta v = \tilde{f}''(u)|\nabla u|^2 + \tilde{f}'(u)\Delta u.$$

From (2.1), it follows

$$\tilde{f}'(s) = \frac{1}{f'(\tilde{f}(s))} = \sqrt{1 + \kappa\alpha|f(\tilde{f}(s))|^{2\alpha-2}} = \sqrt{1 + \kappa\alpha|s|^{2\alpha-2}},$$
$$\tilde{f}''(s) = \frac{\kappa\alpha(\alpha-1)|s|^{2\alpha-4}s}{\sqrt{1+\kappa\alpha|s|^{2\alpha-2}}}.$$

Thus we have

$$\Delta v = \frac{\kappa\alpha(\alpha-1)|u|^{2\alpha-4}u}{\sqrt{1+\kappa\alpha|u|^{2\alpha-2}}}|\nabla u|^2 + \sqrt{1+\kappa\alpha|u|^{2\alpha-2}}\Delta u.$$

From (2.4), we can observe that u satisfies

$$-\Delta u - \kappa\alpha|u|^{2\alpha-2}\Delta u - \kappa\alpha(\alpha-1)|u|^{2\alpha-4}u|\nabla u|^2 + \lambda u = |u|^{p-1}u. \tag{2.5}$$

Now $u > 0$ implies $|u|^\alpha \in C^2$. Then it follows from

$$\begin{aligned}\Delta(|u|^\alpha) &= \operatorname{div}(\alpha|u|^{\alpha-2}u\nabla u)\\ &= \alpha|u|^{\alpha-2}u\Delta u + \nabla u \cdot \nabla(\alpha|u|^{\alpha-2}u)\\ &= \alpha|u|^{\alpha-2}u\Delta u + \alpha(\alpha-1)|u|^{\alpha-2}|\nabla u|^2\end{aligned}$$

that

$$\Delta(|u|^\alpha)|u|^{\alpha-2}u = \alpha|u|^{2\alpha-2}\Delta u + \alpha(\alpha-1)|u|^{2\alpha-4}u|\nabla u|^2. \tag{2.6}$$

Thus from (2.5) and (2.6), we see that if v is a nontrivial critical point of I and $v > 0$, then $u = f(v)$ is a positive solution of (1.2). ∎

Remark 2.6. *If $\alpha \geq 2$, then $f \in C^2[0,\infty)$ and $|f(v)|^\alpha \in C^2$ for any nontrivial critical point v of I. Thus Lemma 2.5 holds for any nontrivial critical point (possibly sign-changing) of I if $\alpha \geq 2$.*

Lemma 2.5 tells us that we have only to show the existence of a positive solution of (2.4) in order to find a positive solution of (1.2). However to show the existence of a ground state solution, we need more informations on the relation between (1.2) and (2.4). Actually we have the following relations.

Lemma 2.7.

(i) *It follows* $X = f(H^1(\mathbb{R}^N))$, *that is,* $X = \{f(v); v \in H^1(\mathbb{R}^N)\} =: Y$.

(ii) *For any* $v \in H^1(\mathbb{R}^N)$, *we put* $u = f(v)$. *Then it follows* $J(u) = I(v)$.

Proof. (i) First we show $Y \subset X$. For $v \in H^1(\mathbb{R}^N)$, we put $u = f(v)$. Then we have

$$|\nabla f(v)|^2 = |f'(v)|^2 |\nabla v|^2 = \frac{1}{1 + \kappa\alpha |f(v)|^{2\alpha-2}} |\nabla v|^2.$$

By (i) of Lemma 2.2 and (2.1), we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx + \kappa\alpha \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx = \int_{\mathbb{R}^N} |\nabla v|^2 + f(v)^2 \, dx \le C \|v\|_{H^1}^2 < \infty.$$

Thus it follows $Y \subset X$.

To show $X \subset Y$, it suffices to show $\tilde{f}(u) \in H^1(\mathbb{R}^N)$ for all $u \in X$. For $u \in X$, we put $v = \tilde{f}(u)$. Then it follows

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx = \int_{\mathbb{R}^N} |(\tilde{f})'(u)|^2 |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} (1 + \kappa\alpha |u|^{2\alpha-2}) |\nabla u|^2 \, dx < \infty.$$

Next by (i) of Lemma 2.3, it follows

$$\lim_{s \to 0} \frac{\tilde{f}(s)}{s} = 1, \quad \lim_{s \to \infty} \frac{\tilde{f}(s)}{s^\alpha} = c$$

for some $c > 0$. Thus there exist constants $C_1,\ C_2 > 0$ such that

$$|\tilde{f}(s)| \le C_1 \chi_{|s| \le 1} |s| + C_2 \chi_{|s| \ge 1} |s|^\alpha \quad \text{for all } s \in \mathbb{R}.$$

Then we have

$$|v|^2 \le C_1 \chi_{|u| \le 1} |u|^2 + C_2 \chi_{|u| \ge 1} |u|^{2\alpha} \le C_1 |u|^2 + C_2 |u|^{\frac{2N\alpha}{N-2}}.$$

By Sobolev's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |v|^2 \, dx &\le C_1 \int_{\mathbb{R}^N} |u|^2 \, dx + C_2 \int_{\mathbb{R}^N} |u|^{\frac{2N\alpha}{N-2}} \, dx \\ &\le C_1 \int_{\mathbb{R}^N} |u|^2 \, dx + C_2' \left(\int_{\mathbb{R}^N} \alpha^2 |\nabla u|^2 |u|^{2\alpha-2} \, dx \right)^{\frac{N}{N-2}} < \infty. \end{aligned}$$

Thus it follows $X \subset Y$ and hence $X = Y$.

(ii) We substitute $u = f(v)$ into $J(u)$. Then from (2.1), it follows

$$
\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla f(v)|^2 + \lambda f(v)^2\,dx + \frac{\kappa\alpha}{2}\int_{\mathbb{R}^N} |\nabla f(v)|^2 |f(v)|^{2\alpha-2}\,dx \\
&\qquad - \frac{1}{p+1}\int_{\mathbb{R}_N} |f(v)|^{p+1}\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2(1+\kappa\alpha|f(v)|^{2\alpha-2})|f'(v)|^2 + \lambda f(v)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \lambda f(v)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx \\
&= I(v).
\end{aligned}
$$

and we obtain (ii). ∎

Finally we give a relation on the sets of positive radial solutions of (1.2) and (2.4).

Lemma 2.8. *It follows*

$$
\begin{aligned}
&\{u \in X \cap C^2(\mathbb{R}^N)\,;\, J'(u) = 0,\ u > 0,\ u(x) = u(|x|)\} \\
&\qquad = f(\{v \in H^1 \cap C^2(\mathbb{R}^N)\,;\, I'(v) = 0,\ v > 0,\ v(x) = v(|x|)\}).
\end{aligned}
$$

Proof. By Lemma 2.5, we know that

$$\{u \in X \cap C^2(\mathbb{R}^N)\,;\, J'(u) = 0,\ u > 0\} \supseteq \{f(v)\,;\, I'(v) = 0,\ v > 0,\ v \in H^1 \cap C^2(\mathbb{R}^N)\}.$$

Suppose the equality does not hold. Then there exists $u_0 \in X$ such that u_0 is a positive solution of (1.2) but $u_0 \neq f(v)$ for any positive solution $v \in H^1(\mathbb{R}^N)$ of (2.4).

On the other hand by (i) of Lemma 2.7, we know if $u_0 \in X$, then there exists v_0 such that $u_0 = f(v_0)$. Since u_0 is a positive solution of (1.2), we can see that v_0 is a positive solution of (2.4). This is a contradiction. Thus we have

$$\{u \in X \cap C^2(\mathbb{R}^N)\,;\, J'(u) = 0,\ u > 0\} = \{f(v)\,;\, I'(v) = 0,\ v > 0,\ v \in H^1 \cap C^2(\mathbb{R}^N)\}.$$

Finally we can easily see that $u(x) = u(|x|)$ if and only if $v(x) = \tilde{f}(u(x))$ satisfies $v(x) = v(|x|)$. ∎

3. Existence of a ground state solution

Firstly we prepare the following Pohozaev-type identity.

Lemma 3.1. *Let $u \in X$ be a solution of* (1.2). *Then u satisfies the following identity:*

$$P(u) := \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u|^2 + \kappa\alpha|\nabla u|^2|u|^{2\alpha-2}\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx = 0. \tag{3.1}$$

Proof. For $t > 0$, we put $u_t(x) = u\left(\frac{x}{t}\right)$. Then we have

$$\int_{\mathbb{R}^N} |\nabla u_t|^2\,dx = t^{N-2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx, \quad \int_{\mathbb{R}^N} u_t^2\,dx = t^N\int_{\mathbb{R}^N} u^2\,dx,$$
$$\int_{\mathbb{R}^N} |\nabla u_t|^2|u_t|^{2\alpha-2}\,dx = t^{N-2}\int_{\mathbb{R}^N} |\nabla u|^2|u|^{2\alpha-2}\,dx, \quad \int_{\mathbb{R}^N} |u_t|^{p+1}\,dx = t^N\int_{\mathbb{R}^N} |u|^{p+1}\,dx.$$

If u is a solution of (1.2), then $\frac{d}{dt}J(u_t)\Big|_{t=1} = 0$. From this equality, we obtain (3.1). ∎

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We argue as in [**11**]. We define

$$g(s) = |f(s)|^{p-1}f(s)f'(s) - \lambda f(s)f'(s).$$

Then (2.4) can be written as

$$-\Delta v = g(v) \text{ in } \mathbb{R}^N.$$

By lemma 2.3 (i)-(ii), it follows

$$\lim_{s\to 0}\frac{g(s)}{s} = -\lambda < 0, \quad \lim_{s\to\infty}\frac{g(s)}{s^{\frac{N+2}{N-2}}} = 0.$$

Moreover let $\xi > f^{-1}((\frac{(p+1)\lambda}{2})^{\frac{1}{p-1}})$. Then

$$G(\xi) = \int_0^{\xi} g(s)ds = f(\xi)^2(\frac{1}{p+1}f(\xi)^{p-1} - \frac{\lambda}{2}) > 0.$$

Then by the results due to [**5**, **6**, **14**], there exists $\tilde{w} \in H^1(\mathbb{R}^N)$ such that $\tilde{w} > 0$, radial and

$$I(\tilde{w}) = \inf\{I(v)\,;\, I'(v) = 0,\ v \in H^1(\mathbb{R}^N)\backslash\{0\}\}.$$

By Lemma 2.5, $w = f(\tilde{w})$ satisfies $J'(w) = 0$. We claim that $w \in \mathcal{G}$.

Indeed we define

$$\tilde{P}(v) := \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} f(v)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx, \ v \in H^1(\mathbb{R}^N).$$

Then it follows $P(u) = \tilde{P}(v)$ for $u = f(v)$ where $P(u)$ was defined in (3.1). Moreover $\tilde{w}$ can be characterized by

$$\tilde{w} \in A := \{v \in H^1(\mathbb{R}^N)\,;\, \tilde{P}(v) = 0\}, \quad I(\tilde{w}) = \inf_{v \in A} I(v).$$

Now let $u \in X$ be a nontrivial critical point of J. Then from (3.1), we have

$$\begin{aligned} J(u) &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 + \alpha\kappa |\nabla u|^2 |u|^{2\alpha-2}\, dx \\ &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{\kappa}{\alpha N} \int_{\mathbb{R}^N} |\nabla |u|^\alpha|^2\, dx. \end{aligned}$$

By the pointwise inequality $|\nabla |u(x)|| \le |\nabla u(x)|$ a.e. $x \in \mathbb{R}^N$, it follows

$$J(u) \ge J(|u|) = I(\tilde{f}(|u|)).$$

If $N = 2$, then (3.1) implies $P(|u|) = 0$ and hence $\tilde{P}(\tilde{f}(|u|)) = 0$. Then we obtain

$$J(u) \ge I(\tilde{f}(|u|)) \ge I(\tilde{w}) = J(w).$$

If $N \ge 3$, we distinguish cases $P(|u|) = 0$ and $P(|u|) < 0$. If $P(|u|) = 0$, then we have $J(u) \ge J(w)$ as in the case $N = 2$. Suppose $P(|u|) = \tilde{P}(\tilde{f}(|u|)) < 0$. We put $\tilde{v} = \tilde{f}(|u|)$ and define $\tilde{v}_\theta(x) = \tilde{v}\left(\dfrac{x}{\theta}\right)$ for $\theta \in (0,1)$. We can see that there exists $\theta_0 \in (0,1)$ such that $\tilde{P}(\tilde{v}_{\theta_0}) = 0$, that is, $\tilde{v}_{\theta_0} \in A$. Then we have

$$\begin{aligned} I(\tilde{v}_{\theta_0}) &= \frac{\theta_0^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla \tilde{v}|^2\, dx \\ &= \frac{\theta_0^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla |u||^2 + \frac{\kappa}{\alpha} |\nabla |u|^\alpha|^2\, dx \\ &\le \frac{\theta_0^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\kappa}{\alpha} |\nabla |u|^\alpha|^2\, dx \\ &= \theta_0^{N-2} J(u) < J(u). \end{aligned}$$

Since $\tilde{v}_{\theta_0} \in A$, we obtain

$$J(u) > I(\tilde{v}_{\theta_0}) \ge I(\tilde{w}) = J(w).$$

This implies $w \in \mathcal{G}$.

Next we show that if $w \in \mathcal{G}$, then $|w| \in \mathcal{G}$. If $P(|w|) < 0$, then we get

$$J(w) > I(\tilde{v}_{\theta_0}) = J(w).$$

This is a contradiction. Thus it follows $P(|w|) = 0$. Arguing as above, we obtain $J(w) = J(|w|)$. This implies $|w| \in \mathcal{G}$.

Next we show properties (i)-(iv).

(i) We can see if $w \in \mathcal{G}$, then $w \in L^\infty_{loc}(\mathbb{R}^N)$. By the elliptic regularity theory, it follows $w \in C^2(\mathbb{R}^N)$.

(ii) Since $|w| \in \mathcal{G}$ for any $w \in \mathcal{G}$, it follows $w \geq 0$. We put $\tilde{w} = \tilde{f}(w)$. By the maximum principle, we have $\tilde{w} > 0$. This implies $w > 0$.

(iii) We observe that if $w \in \mathcal{G}$, then $\tilde{w} = \tilde{f}(w)$ is a ground state solution of (2.4). In fact for any nontrivial critical point u of J, $v = \tilde{f}(u)$ is a nontrivial critical point of I. Then by Lemma 2.7 (ii), we have

$$I(v) = J(u) \geq m = J(w) = I(\tilde{w}).$$

This implies $\tilde{w}$ is a ground state solution of (2.4).

By the result of [**8**], any ground state solution of (2.4) is radially symmetric and decreasing with respect to $r = |x|$. We can easily see that $w(x) = w(|x|)$ if and only if $\tilde{w}(x) = \tilde{f}(w(x))$ satisfies $\tilde{w}(x) = \tilde{w}(|x|)$. Thus claim (iii) holds.

(iv) Let $w \in \mathcal{G}$ and $\tilde{w} = \tilde{f}(w)$. From (ii), $\tilde{w}$ is a positive radial solution of (2.4). Then by the standard comparison principle, it follows

$$|D^k \tilde{w}(x)| \leq ce^{-\delta|x|} \quad \text{for all } \delta \in (0, \sqrt{\lambda}),\ x \in \mathbb{R}^N,\ |k| \leq 2 \text{ and some } c > 0.$$

By Lemma 2.2 (i), we can see that

$$|D^k w(x)| \leq ce^{-\delta|x|} \quad \text{for all } \delta \in (0, \sqrt{\lambda}),\ x \in \mathbb{R}^N,\ |k| \leq 2 \text{ and some } c > 0.$$

Thus for $|x| \gg 1$, we have

$$-\Delta w + \lambda w \leq ce^{-p\delta|x|} + ce^{-(2\alpha-1)\delta|x|}.$$

Since $2\alpha - 1 > 1$ and we can take δ arbitrarily close to $\sqrt{\lambda}$, it follows

$$-\Delta w + \lambda w = o(G(x)) \quad \text{as } |x| \to \infty,$$

where G is the fundamental solution of $-\Delta + \lambda I$. Then by Gidas-Ni-Nirenberg's asymptotic result [**13**], we obtain (iv). ∎

Finally in this section, we give the non-existence result for $p \geq \dfrac{(2\alpha-1)N+2}{N-2}$, which is an easy consequence of the Pohozaev identity.

Proposition 3.2. *Suppose* $p \geq \dfrac{(2\alpha-1)N+2}{N-2}$. *Then* (1.2) *has no nontrivial solution* $u \in X$.

Proof. Suppose $u \in X$ is a nontrivial solution of (1.2) and $p \geq \dfrac{(2\alpha-1)N+2}{N-2}$. From $J'(u)u = 0$, we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 + \kappa\alpha^2 |\nabla u|^2 |u|^{2\alpha-2}\, dx - \int_{\mathbb{R}^N} |u|^{p+1}\, dx = 0.$$

On the other hand, u satisfies (3.1). Then we obtain

$$\begin{aligned}\frac{(\alpha-1)(N-2)}{2\alpha}\int_{\mathbb{R}^N} |\nabla u|^2\, dx &+ \frac{(\alpha-1)N+2}{2\alpha}\int_{\mathbb{R}^N} \lambda u^2\, dx \\ &= \left(\frac{N}{p+1} - \frac{N-2}{2\alpha}\right)\int_{\mathbb{R}^N} |u|^{p+1}\, dx.\end{aligned}$$

Since the left hand side is positive, it follows

$$\frac{N}{p+1} - \frac{N-2}{2\alpha} > 0, \text{ that is, } p < \frac{(2\alpha-1)N+2}{N-2}.$$

This is a contradiction. ∎

4. Uniqueness of ground state solutions

In this section, we study the uniqueness of ground state solutions of (1.2). By Theorem 1.1, we know that

$$\mathcal{G} \subset \{u \in X \cap C^2;\ u \text{ is a positive radial solution of (1.2)}\}.$$

Moreover lemma 2.8 tells us that if (2.4) has at most one positive radial solution v, then (1.2) also has at most one positive radial solution $u = f(v)$. Thus we have only to study the uniqueness of the positive solution of semilinear problem (2.4). We put

$$g(s) = f(s)^p f'(s) - \lambda f(s) f'(s) \text{ for } s \geq 0 \text{ and } K_g(s) := \frac{sg'(s)}{g(s)}. \tag{4.1}$$

We apply the following uniqueness result due to Serrin and Tang [**22**].

Proposition 4.1 [**22**]. *Suppose that there exists* $b > 0$ *such that*

(i) *g is continuous on* $(0,\infty)$, $g(s) \leq 0$ *on* $(0,b]$ *and* $g(s) > 0$ *for* $s > b$.

(ii) $g \in C^1(b,\infty)$ *and* $K_g'(s) \leq 0$ *on* (b,∞).

Then the semilinear problem:

$$-\Delta u = g(u) \text{ in } \mathbb{R}^N,\ u > 0,\ u \to 0 \text{ as } |x| \to \infty,\ u(0) = \max u(x)$$

has at most one positive radial solution.

Now we can see that g defined in (4.1) is of the class $C^1[0,\infty)$ and

$$g(s) = 0 \Longleftrightarrow f^{p-1}(s) = \lambda \Longleftrightarrow s = f^{-1}(\lambda^{\frac{1}{p-1}}).$$

We put $b := f^{-1}(\lambda^{\frac{1}{p-1}})$. Since $(s-b)g(s) = (s-b)ff'(f^{p-1}-\lambda)$, we can see (i) of Proposition 4.1 holds.

Lemma 4.2. *Assume $\alpha - 1 \le p < 3\alpha - 3$ and*

$$\max\left\{\frac{\alpha-p}{\alpha(p-1)},\ \frac{p}{\alpha(3\alpha-p-3)}\right\} \le \kappa\lambda^{\frac{2\alpha-2}{p-1}}. \tag{4.2}$$

Then g satisfies (ii) of Proposition 4.1.

Proof. For a detail, we refer to [**2**]. We observe that

$$K_g'(s) = \frac{1}{g(s)^2}(g''(s)g(s)s + g'(s)g(s) - (g'(s))^2 s).$$

Thus we have only to show that $sg''g + g'g - s(g')^2 < 0$ for $s > b$. By direct computations, we have

$$\begin{aligned}
&sg''g + g'g - s(g')^2\\
&= (f^{p-1}-\lambda)^2\left(sf^2f'f''' + sf(f')^2f'' - s(f')^4 - sf^2(f'')^2 + f(f')^3 + f^2f'f''\right)\\
&\quad + (f^{p-1}-\lambda)(p-1)f^{p-1}(f')^2\left((p-2)s(f')^2 + sff'' + ff'\right)\\
&\quad - (p-1)^2sf^{2p-2}(f')^4.
\end{aligned}$$

Next we express $sg''g + g'g - s(g')^2$ in terms of f and f' and regard as a polynomial of $f^{p-1}-\lambda$. From (2.2) and (2.3), we have

$$\begin{aligned}
&sg''g + g'g - s(g')^2\\
&= (f^{p-1}-\lambda)^2\left((\alpha-1)f'^5\left(f - (4\alpha-p-3)sf' + 3(\alpha-1)sf'^3\right) - (p-\alpha+1)f'^3(\alpha sf' - f)\right)\\
&\quad + \lambda(p-1)(f^{p-1}-\lambda)\left(-(\alpha+p-1)sf'^4 + ff'^3 + (\alpha-1)sf'^6\right)\\
&\quad - \lambda^2(p-1)^2sf'^4\\
&=: (f^{p-1}-\lambda)^2H_1(s) + \lambda(p-1)(f^{p-1}-\lambda)H_2(s) - \lambda^2(p-1)^2sf'^4.
\end{aligned} \tag{4.3}$$

First we study the sign of $H_2(s)$. It follows

$$H_2(s) = f'^2\{-f'(\alpha s f' - f) - \left((p-1) - (\alpha-1)f'^2\right) s f'^2\}.$$

From Lemma 2.2 (ii), we know that $\alpha s f' - f \geq 0$ for all $s \geq 0$. Moreover from the fact that f' is decreasing, we have

$$(p-1) - (\alpha-1)f'^2 \geq p - 1 - (\alpha-1)f'(b)^2 = \frac{p - \alpha + \kappa\alpha(p-1)\lambda^{\frac{2\alpha-2}{p-1}}}{1 + \alpha\kappa\lambda^{\frac{2\alpha-2}{p-1}}}.$$

From (4.2), it follows $H_2(s) \leq 0$ for $s > b$.

Next we investigate the sign of $H_1(s)$. We have

$$\begin{aligned} H_1(s) &= (\alpha-1)f'^5\left(f - (4\alpha - p - 3)sf' + 3(\alpha-1)sf'^3\right) - (p - \alpha + 1)f'^3(\alpha s f' - f) \\ &=: (\alpha-1)f'^5 H_3(s) - (p - \alpha + 1)f'^3(\alpha s f' - f). \end{aligned}$$

Since $p - \alpha + 1 \geq 0$, it suffices to show that $H_3(s) \leq 0$ in order to prove $H_1(s) \leq 0$.

Now we observe that

$$H_3(s) = -(\alpha s f' - f) - sf'\left((3\alpha - p - 3) - 3(\alpha-1)f'^2\right).$$

From $p < 3\alpha - 3$ and the fact that f' is decreasing, we have

$$\begin{aligned} (3\alpha - p - 3) - 3(\alpha-1)f'(s)^2 &\geq (3\alpha - p - 3) - 3(\alpha-1)f'(b)^2 \\ &= (3\alpha - p - 3) - \frac{3(\alpha-1)}{1 + \alpha\lambda f(b)^{2\alpha-2}} \\ &= \frac{-p + \alpha(3\alpha - p - 3)\kappa\lambda^{\frac{2\alpha-2}{p-1}}}{1 + \alpha\kappa\lambda^{\frac{2\alpha-2}{p-1}}}. \end{aligned}$$

We see that

$$-p + \alpha(3\alpha - p - 3)\kappa\lambda^{\frac{2\alpha-2}{p-1}} \geq 0 \iff \frac{p}{\alpha(3\alpha - p - 3)} \leq \kappa\lambda^{\frac{2\alpha-2}{p-1}}.$$

Thus we have $H_3(s) \leq 0$ and hence $H_1(s) \leq 0$. From (4.3), we obtain $K_g'(s) < 0$ for $s > b$. ∎

By Lemma 4.2, we can apply Proposition 4.1. Hence we obtain the uniqueness of positive radial solutions of (1.2) when $N \geq 3$ and the proof of Theorem 1.2 is complete.

To prove the uniqueness of ground states solutions for $N = 2$, we apply the following uniqueness result due to Mcleod and Serrin [**20**].

Proposition 4.3 [20]. *Suppose that there exist $b > 0$ and $\tau \geq 1$ such that*

(i) $g \in C^1[0,\infty)$, $g(0)=0$, $g'(0) < 0$.

(ii) $g(s) < 0$ *for* $s \in (0,b)$, $g(s) > 0$ *for* $s \in (b,\infty)$.

(iii) $g'(b) > 0$.

(iv) $\left(\dfrac{g(s)}{s^\tau}\right)' > 0$ *for* $s > 0$, $s \neq b$.

(v) $\left(s\left(\dfrac{g(s)}{s^\tau}\right)'\right)' < 0$ *for* $s > b$.

Then the semilinear problem:

$$-\Delta u = g(u) \text{ in } \mathbb{R}^2,\ u > 0,\ u \to 0 \text{ as } |x| \to \infty,\ u(0) = \max u(x)$$

has at most one positive radial solution.

Now we can see g defined in (4.1) satisfies (i)-(iii) of Proposition 4.3.

Lemma 4.4. *Assume $2\alpha - 1 \leq p < \infty$. Then g defined in* (4.1) *satisfies (iv) of Proposition 4.3 with* $\tau = \dfrac{p+1-\alpha}{\alpha}$.

Proof. A direct calculation yields that

$$\begin{aligned} sg'(s) - \tau g(s) &= (p+1-\alpha)sf^{p-1}(f')^2 + (\alpha-1)sf^{p-1}(f')^4 \\ &\quad + (\alpha-2)\lambda s(f')^2 - (\alpha-1)\lambda s(f')^4 - \tau f^p f' + \tau\lambda f f' \\ &\geq (p+1-\alpha-\alpha\tau)sf^{p-1}(f')^2 + (\tau-1)\lambda s(f')^2 + (\alpha-1)sf^{p-1}(f')^4. \end{aligned}$$

Here we used $0 < f' \leq 1$ and Lemma 2.2. Choosing $\tau = \dfrac{p+1-\alpha}{\alpha}$, we obtain $\tau \geq 1$ and $sg' - \tau g > 0$ for $s > 0$, $s \neq b$. ∎

Lemma 4.5. *Suppose $\alpha > 2$, $2\alpha - 1 \leq p < 3\alpha - 3$ and let $\tau = \dfrac{p+1-\alpha}{\alpha}$ and $b = f^{-1}(\lambda^{\frac{1}{p-1}})$. Then*

(i) *For every fixed $\kappa > 0$, there exists $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$, g satisfies (v) of Proposition 4.3.*

(ii) *For every fixed $\lambda > 0$, there exists $\kappa_0 > 0$ such that if $\kappa \geq \kappa_0$, g satisfies (v) of Proposition 4.3.*

For the proof, we refer to [**2**].

By Lemmas 4.4-4.5, we can apply Proposition 4.3. Hence we obtain the uniqueness of positive radial solutions of (1.2) for $N = 2$ and the proof of Theorem 1.4 is complete.

Remark 4.6.

(i) *As mentioned in* [**15**], *(iv) and (v) of Proposition 4.3 imply (ii) of Proposition 4.1. This means that we need a stronger condition on the nonlinearity to show the uniqueness for $N = 2$.*

(ii) *The number $b = f^{-1}(\lambda^{\frac{1}{p-1}})$ depends on κ and λ. We can see $b \to \infty$ if either λ or κ goes to infinity. To obtain the uniqueness, we require that b is large.*

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