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Local Composition of Cellular Automata via Hit-or-Miss Transform

Mitsuhiko Fujio*

Abstract

The notion of *local composition* of cellular automata was introduced in [1] as an operator over local transition rules compatible with the composition of global transitions. A Mathematical morphological interpretation of local composition was given in [2] in terms of excitation patterns of local transition rules. In this article, by going ahead Mathematical morphological analysis, we describe the local composition via *hit-or-miss transform* which is known as a pattern detecting tool in Mathematical morphology [9, 11].

1 Introduction

A cellular automaton (CA for short) is a discrete dynamical system characterized by its locality and homogenuity. In fact, for every CA, its *transition funciton* over *configurations* is defined by a *local transition rules* so as to be translation invariant. Such a formalism of CA requires a group structure on the set of *sites* on which configurations are defined. CA's thus defined are completely determined by their local transition rules. In particular, any properties of CA's, in principle, should be described in terms of their local transition rules.

In [1], the author introduced the notion of *local compositon* of local transition rules and showed that the transition function corresponding to the local composition coincides with the composition of the transition funcitons corresponding to the composed local transition rules (more precisely, see Theorem 3). The description of local composition for local transition rules $\mathfrak{L}$ and $\mathfrak{M}$ given in [1] is in a conotative form for local patterns $\boldsymbol{c}$ on the composed support $V \odot W$ by using of the mapping $\sigma_{\boldsymbol{c}} : V \ni v \to v^{-1}\boldsymbol{c} \cap W \in \mathbf{2}^W$ as

$$\mathfrak{L} \diamond \mathfrak{M} = \left\{ \boldsymbol{c} \in \mathbf{2}^{V \odot W} \mid {\sigma_{\boldsymbol{c}}}^{-1}(\mathfrak{M}) \in \mathfrak{L} \right\},$$

where V and W are respectively the supports of the composed rules $\mathfrak{L}$ and $\mathfrak{M}$.

In the previous work [2], to clarify this description, the author rewrote the condition ${\sigma_{\boldsymbol{c}}}^{-1}(\mathfrak{M}) \in \mathfrak{L}$ in terms of Mathematical morphological (MM for short) operators concerning the mapping $\sigma_{\boldsymbol{c}}$. Namely, for any local pattern $\boldsymbol{c}$ on $V \odot W$,

$$\begin{aligned} &{\sigma_{\boldsymbol{c}}}^{-1}(\mathfrak{M}) \in \mathfrak{L} \Leftrightarrow \\ &\qquad \exists L \in \mathfrak{L} \text{ such that } \delta_{{}^t\sigma_{\boldsymbol{c}}}(L) \subseteq \mathfrak{M} \subseteq \varepsilon_{{}^t\sigma_{\boldsymbol{c}}}(L), \end{aligned}$$

where $\delta_{{}^t\sigma_{\boldsymbol{c}}}(L)$ and $\varepsilon_{{}^t\sigma_{\boldsymbol{c}}}(L)$ respectively denote the dilation and the erosion of L by the transposition ${}^t\sigma_{\boldsymbol{c}}$ of the local mapping $\sigma_{\boldsymbol{c}}$ regarded as a correspondence. A general frame-work of MM analysis for correspondences will be given in §3.

In this article, we give a more direct interpretation of ${\sigma_{\boldsymbol{c}}}^{-1}(\mathfrak{M})$ by using hit-or-miss transform as follows. The hit-or-miss transform of a set A by a pair of sets B and B' is defined by

$$A \circledast (B, B') = \varepsilon_B(A) \cap \varepsilon_{B'}(A^{\mathrm{C}}). \tag{1}$$

The resulting set $A \circledast (B, B')$ consists of those points for which B fits in the forground A and B' fits in the backgroound A^{C}. Consequently, if $B \cap B' \neq \varnothing$ then $A \circledast (B, B') = \varnothing$. By using hit-or-miss transform, ${\sigma_{\boldsymbol{c}}}^{-1}(\mathfrak{M})$ can be rewritten as $\bigcup_{M \in \mathfrak{M}} \boldsymbol{c} \circledast (M, W - M)$ and hence we have

$${\sigma_{\boldsymbol{c}}}^{-1}(\mathfrak{M}) \in \mathfrak{L} \Leftrightarrow \bigcup_{M \in \mathfrak{M}} \boldsymbol{c} \circledast (M, W - M) \in \mathfrak{L}.$$

This means that the local composition $\mathfrak{L} \diamond \mathfrak{M}$ consists of those local patterns $\boldsymbol{c}$ for which the neighbourhood at each point in W, both positively and negatively, fits some pattern in $\mathfrak{M}$ form a pattern in $\mathfrak{L}$.

*Deptartment of Systems Design and Informatics, Kyushu Institute of Technology

2 Cellular automata

In what follows, we denote by **2** the Boolean algebra consisting of two constants 0 and 1.

2.1 Cellular automata on groups

Notion of CA on groups was first treated as special cases for CA on graphs named *Caylay graphs* which represent groups [8, 7, 12]. General approaches are found in [1], [6] and [13]. Here we follow our previous investigation [1].

Let G be a group. By regarding its power set $\mathscr{C} = \mathbf{2}^G$ as the *configuration space*, a *local transition rule* with the support $V \subseteq G$ is defined as a family $\mathfrak{L}$ of subsets of V. The role of $\mathfrak{L}$ is explained as follows. Let c' be the evolution of a configuration $c \in \mathscr{C}$. Then the state at any site $g \in c'$ is determined according to whether the state pattern of c around g translated onto V around the unit element e by g^{-1} is in $\mathfrak{L}$ or not. More explicitly,

$$c \quad \mapsto \quad c' = \{ g \in G \mid g^{-1}c \cap V \in \mathfrak{L} \}.$$

Such a *transition function* is characterized as a tranformation $T : \mathscr{C} \to \mathscr{C}$ commutes with the G-action:

$$T(ac) \quad = \quad a(T(c)), \qquad (a \in G, c \in \mathscr{C}),$$

where the action of $a \in G$ on a configuration $c \in \mathscr{C}$ is given by $ac = \{ag \mid g \in c\}$. The usual infinite (resp. periodic with the size N) 1-dimensional CA is obtained by considering a symmetric interval $V = [-r, r]$ in the Abelian group $G = \mathbb{Z}$ (resp. $\mathbb{Z}_N$: the residue ring modulo N).

2.2 Algebraic structure

For a given set X and a Boolean algebra B we consider the pointwise Boolean algebra structure on the set B^X of all B-valued functions defined on X. Namely, it is defined by

$$\begin{aligned}(f \vee g)(x) &:= f(x) \vee g(x),\\ (f \wedge g)(x) &:= f(x) \wedge g(x),\\ (\neg f)(x) &:= \neg(f(x))\end{aligned}$$

for f, $g \in B^X$ and $x \in X$. Then we have

$$f \leqslant g \quad \Leftrightarrow \quad f(x) \leqslant g(x) \quad (\forall x \in X).$$

In case $B = \mathbf{2}$, the Boolean algebra structure of B^X coincides with that of set lattice and $\vee$, $\wedge$, $\neg$ and $\leqslant$, respectively become union, intersection, complementation and inclusion. The set $\mathcal{T}_G$ of all transition functions on G is a subalgebra of the Boolean algebra $[\mathbf{2}^G \to \mathbf{2}^G] = (\mathbf{2}^G)^{(\mathbf{2}^G)}$. On the other hand, the set $\mathcal{L}_V = [\mathbf{2}^V \to \mathbf{2}]$ of all local transition rules with the support V is regarded as a Boolean algebra as $\mathbf{2}^{(\mathbf{2}^V)}$. All of X^B, $\mathcal{T}_G$ and $\mathcal{L}_V$ are complete.

For any pair of subsets $V \subseteq W \subseteq G$, by virture of $\mathbf{2}^V \subseteq \mathbf{2}^W$, $\mathcal{L}_V$ can be naturally regarded as a sublattice of $\mathcal{L}_W$. We remark that the complementation depends on the algebra to which the element belongs. When the support should be explicitely indicated, we write as $\neg_V$ (see also the remark at the end of §2.3).

Algebraic properties of CA's on groups are investigated in [1] and [6].

2.3 Correspondence between local transition rules and transition functions

For a transition rule $\mathfrak{L} \in \mathcal{L}_V$ with the support V, we define a transition function on G by

$$T_{\mathfrak{L}}(c) \quad = \quad \{ g \in G \mid g^{-1}c \cap V \in \mathfrak{L} \} \quad (c \in \mathscr{C}).$$

It can be easily verified that $T_{\mathfrak{L}}$ commutes with the G-action.

To consider the converse correspondence, we introduce for transition functions a notion corresponding to that of supports of local transition rules. Let $T : \mathscr{C} \to \mathscr{C}$ be a transition function and $V \subseteq G$. We say that T is *local* at a site $g \in G$ on V if

$$c \cap V = c' \cap V \quad \Rightarrow \quad g \in T(c) \text{ iff } g \in T(c')$$

is satisfied for any configurations c, $c' \in \mathscr{C}$. We call V a *domain of locality* of T at g. We denote by $\mathcal{T}_{g,V}$ the set of all transition functions local at a site g on V by $\mathcal{T}_{g,V}$. $\mathcal{T}_{g,V}$ is a complete Boolean subalgebra of $\mathcal{T}_G$.

For simplicity, we denote by $\mathcal{T}_V$ instead of $\mathcal{T}_{e,V}$ for the unit element e. Then it can be easily shown that $\mathcal{T}_V = \mathcal{T}_{g,gV}$.

Now we assign a local transition function $\mathfrak{L}_T \in \mathcal{L}_V$ with the support V to each transition function

$T \in \mathcal{T}_V$ by

$$\mathfrak{L}_T = \left\{ \boldsymbol{c} \in \mathbf{2}^V \mid e \in T(\boldsymbol{c}) \right\}.$$

By virtue of commutativity of T with G-action,

$$\mathfrak{L}_T = \left\{ g^{-1}\boldsymbol{c} \in \mathscr{C} \mid \boldsymbol{c} \in \mathbf{2}^{gV},\, g \in T(\boldsymbol{c}) \right\}$$

holds for an arbitrary $g \in G$.

Then we have the following theorem and proposition.

Theorem 1 Both of the mappings $\mathcal{T}_V \ni T \mapsto \mathfrak{L}_T \in \mathcal{L}_V$ and $\mathcal{L}_V \ni \mathfrak{L} \mapsto T_{\mathfrak{L}} \in \mathcal{T}_V$ are isomorphisms of Boolean algebrals and each of them is the inverse of each other.

Proposition 2 Let $V \subseteq W \subseteq G$.

1. $\mathcal{L}_V$ is an ideal of $\mathcal{L}_W$, *i.e.*, $\mathcal{L}_V$ is a sublattice of $\mathcal{L}_W$ and if $\mathfrak{M} \subseteq \mathfrak{L}$ ($\mathfrak{L} \in \mathcal{L}_V$, $\mathfrak{M} \in \mathcal{L}_W$) then $\mathfrak{M} \in \mathcal{L}_V$.
2. $\mathcal{T}_V$ is a Boolean subalgebra of $\mathcal{T}_W$, *i.e.*, $\mathcal{L}_V$ is a sublattice of $\mathcal{L}_W$ and if $T \in \mathcal{T}_V$ then $\neg_V T = \neg_W T$.

It follows from this proposition that the two embeddings $\mathcal{L}_V \subseteq \mathcal{L}_W$ and $\mathcal{T}_V \subseteq \mathcal{T}_W$ are not compatible with the isomorphisms $\mathcal{L}_V \cong \mathcal{T}_V$ and $\mathcal{L}_W \cong \mathcal{T}_V$ established by Theorem1. For more precise, see [1]. Readers also find a proof of Theorem1 there.

2.4 Local composition of local transition rules

We define the *Mikowski product* of V, $W \subseteq G$ by

$$V \odot W = \{ vw \in G \mid v \in V,\, w \in W \}.$$

This is a non-commutative version of Minkowski addition $\oplus$ for Abelian groups [4]. Then we define the *local composition* of $\mathfrak{L} \in \mathcal{L}_V$ and $\mathfrak{M} \in \mathcal{L}_W$ by

$$\mathfrak{L} \diamond \mathfrak{M} = \left\{ \boldsymbol{c} \in \mathbf{2}^{V \odot W} \mid \sigma_{\boldsymbol{c}}{}^{-1}(\mathfrak{M}) \in \mathfrak{L} \right\}. \quad (2)$$

Here $\sigma_{\boldsymbol{c}} : V \to \mathbf{2}^W$ is the mapping defined by

$$\sigma_{\boldsymbol{c}}(v) = v^{-1}\boldsymbol{c} \cap W \quad (3)$$

for $\boldsymbol{c} \in \mathbf{2}^{V \odot W}$ and $\sigma_{\boldsymbol{c}}{}^{-1}(\mathfrak{M})$ denotes the inverse image of the set $\mathfrak{M} \subseteq \mathbf{2}^W$ with respect to this mapping. Finally, $\mathfrak{L} \diamond \mathfrak{M}$ is the set of all configurations $\boldsymbol{c}$ such that this inverse image coincides with a member of $\mathfrak{L}$. By definition, $\mathfrak{L} \diamond \mathfrak{M} \in \mathcal{L}_{V \odot W}$.

The following theorem ensures us to call this as local composition[1]:

Theorem 3 Let $\mathfrak{L} \in \mathcal{L}_V$, $\mathfrak{M} \in \mathcal{L}_W$. Then the composition $T_{\mathfrak{L}} \circ T_{\mathfrak{M}}$ of $T_{\mathfrak{L}} \in \mathcal{T}_V$ and $T_{\mathfrak{M}} \in \mathcal{T}_W$ is the unique transition function that is local at e on $V \odot W$ satisfying

$$T_{\mathfrak{L}} \circ T_{\mathfrak{M}} = T_{\mathfrak{L} \diamond \mathfrak{M}}.$$

3 Mathematical morphology

MM was first introduced as an analyzing tool of image processing by shape and has been developed as a systematic non-linear analysis methodology [9, 10]. Theoretically, it is founded on complete lattices [10, 5] and its methodology has been extended there [4].

In this section, we recall minimal requisites for MM. For precise description and general references, readers should see [10], [4]

3.1 Binary relations and correspondences

In what follows, we regard a binary relation $R \subseteq X \times A$ as a correspondence of X into A by

$$R : X \ni x \mapsto \{ a \in A \mid (x, a) \in R \} \subseteq A$$

and vise versa. The *transpose* ${}^tR$ of R is given by

$${}^tR = \{ (a, x) \mid a \in A,\, x \in X,\, (x, a) \in R \} \subseteq A \times X$$

or, in terms of correspondence,

$${}^tR : A \ni a \mapsto \{ x \in X \mid (x, a) \in R \} \subseteq X.$$

As well as for ordinary mappings, we denote the *image* of an element $x \in X$ under R by $R(x)$ and that of a subset $Y \subseteq X$ by $R(Y) = \bigcup_{y \in Y} R(y)$. The usual set theoretical inverse image of a subset $B \subseteq A$ can be expressed as $R^{-1}(B) = {}^tR(B)$ in our notation.

3.2 Dilation and erosion

Dilation and erosion are the fundamental operators in MM. Notions of dilation and erosion were extended to complete lattices and general properties are investigated [10, 5, 4]. Here we adopt a slightly general definition[3].

Let X, A be partially ordered sets. A mapping $\delta : X \to A$ is called an *algebraic dilation* or for short, a *dilation* iff for any family $\{x_\lambda\} \subseteq X$ which admits the supremum $\bigvee_\lambda x_\lambda$ in X, the family $\{\delta(x_\lambda)\}$ also admits the supremum $\bigvee_\lambda \delta(x_\lambda)$ in A and $\bigvee_\lambda \delta(x_\lambda) = \delta\left(\bigvee_\lambda x_\lambda\right)$ is satisfied. Namely, every dilation preserves supremum. Dually, a mapping $\varepsilon : X \to A$ is called an *algebraic erosion* or for short, an *erosion* iff for any family $\{x_\lambda\} \subseteq X$ which admits the infimum $\bigwedge_\lambda x_\lambda \in X$, the family $\{\varepsilon(x_\lambda)\}$ also admits the infimum and $\bigwedge_\lambda \varepsilon(x_\lambda) = \varepsilon\left(\bigwedge_\lambda x_\lambda\right)$ is satisfied. Dually to dilation, every erosion preserves infimum.

Proposition 4 Every dilation or erosion is monotone.

Example 3.1 (MM of set lattices) Let $R \subseteq X \times A$ be a binary relation. We define the following set operators from $\mathbf{2}^A$ into $\mathbf{2}^X$ (notice that the direction is opposed):

$$\delta_R(B) = \{x \in X \mid R(x) \cap B \neq \varnothing\}, \quad (4)$$

$$\varepsilon_R(B) = \{x \in X \mid R(x) \subseteq B\} \quad (5)$$

for $B \in \mathbf{2}^A$. Then $\delta_R : \mathbf{2}^A \to \mathbf{2}^X$ is a dilation and $\varepsilon_R : \mathbf{2}^A \to \mathbf{2}^X$ is an erosion. We call δ_R and ε_R the *set dilation* and the *set erosion* defined by R respectively.

By considering the transpose ${}^tR$ of R, we also obtain operators $\delta_{{}^tR} : \mathbf{2}^X \to \mathbf{2}^A$, $\varepsilon_{{}^tR} : \mathbf{2}^X \to \mathbf{2}^A$.

We note that any dilation and erosion between set lattices are obtained in this way. In fact, for any dilation $\delta : \mathbf{2}^X \to \mathbf{2}^A$, since any set lattice is "*atomic*" and δ preserves supremum by definition, we have $\delta(Y) = \delta\left(\bigcup_{y \in Y}\{y\}\right) = \bigcup_{y \in Y} \delta(\{y\})$. Thus the effect of δ on any set $Y \subseteq X$ is determined by its effect on each singleton. By taking the correspondence $R : X \ni x \mapsto \delta(\{x\}) \subseteq A$, we have $\delta = \delta_{{}^tR}$. For any erosion $E : \mathbf{2}^X \to \mathbf{2}^A$, since its dual $\overline{\varepsilon}(Y) = \left(\varepsilon(Y^{\mathrm{C}})\right)^{\mathrm{C}}$ is a dilation, there exists a binary relation R such that $\overline{\varepsilon} = \delta_{{}^tR}$. Then it can be verified that $\varepsilon = \overline{\delta_{{}^tR}} = \varepsilon_{{}^tR}$.

Relationships among "duality", "transposition" and "adjunction" will be investigated in §3.4.

As a special case, consider a mapping $\psi : X \to A$. In this case, since the image of $x \in X$ by ψ is a singleton $\{\psi(x)\}$, we have $\{\psi(x)\} \cap B \neq \varnothing \Leftrightarrow \psi(x) \in B \Leftrightarrow \{\psi(x)\} \subseteq B$ for $B \in \mathbf{2}^A$. Hence

$$\delta_\psi(B) = \varepsilon_\psi(B) = \psi^{-1}(B). \quad (6)$$

On the other hand, since its transpose ${}^t\psi(a)$ as a biary relation of $a \in A$ is ${}^t\psi(a) = \{x \in X \mid a = \psi(x)\} = \psi^{-1}(\{a\})$, its dilation and erosion are respectively expressed as

$$\delta_{{}^t\psi}(Y) = \{a \in A \mid \exists y \in Y, \psi(y) = a\} = \psi(Y) \quad (7)$$

and

$$\varepsilon_{{}^t\psi}(Y) = \{a \in A \mid \psi^{-1}(a) \subseteq Y\} = \overline{\psi}(Y). \quad (8)$$

Example 3.2 (MM on groups) Let G be a group. For $H \subseteq G$, the *symmetry* $\widetilde{H}$ of H is defined by $\widetilde{H} = \{g \in G \mid g^{-1} \in H\}$.

The set dilation and the set erosion defined by the binary relation

$$R_H = \{(g, g') \in G \times G \mid g^{-1}g' \in H\}$$

are respectively called the *dilation* and the *erosion by H on the right.* We note that for $a \in G$, $R_H(a) = aH$ and hence

$$\delta_{R_H}(K) = \bigcup_{h \in H} Kh^{-1}, \quad \varepsilon_{R_H}(K) = \bigcap_{h \in H} Kh^{-1}.$$

The dilation $\delta_{R_H}(K)$ of K by H on the right coincides with the Minkowski product $K \odot \widetilde{H}$ of K and $\widetilde{H}$. On the other hand, the erosion $\varepsilon_{R_H}(K)$ of K by H on the right is denoted by $K \oslash \widetilde{H}$ and called the *Minkowski quotient* of K by $\widetilde{H}$ on the right. Furthermore, since ${}^tR_H = R_{\widetilde{H}}$, we have $\delta_{{}^tR_H} = \delta_{R_{\widetilde{H}}}$ and $\varepsilon_{{}^tR_H} = \varepsilon_{R_{\widetilde{H}}}$.

Simlarly, by considering the binary relation

$$L_H = \{(g, g') \in G \times G \mid g'g^{-1} \in H\}$$

we obtain the dilation δ_{L_H} and the erosion ε_{L_H} by H on the left. Notice that the dilation $\delta_{L_H}(K)$ of K by H is equal to $\widetilde{H} \odot K$. On the other hand, the erosion $\varepsilon_{L_H}(K)$ of K by H on the left is denoted by $\widetilde{H} \otimes K$ and called the *Minkowski quotient* of K by $\widetilde{H}$ on the left.

3.3 Adjoint

The notion of adjoint plays the fundamental roll in MM analysis. It is defined as follows. Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be mappings. The pair (f, g) is called an *adjoint* of X into A iff for $\forall x \in X$, $\forall a \in A$

$$f(x) \leqslant a \Leftrightarrow x \leqslant g(a) \tag{9}$$

is satisfied. Then f is called the *lower adjoint* of g and g is called the *upper adjoint* of f.

Relations between MM operators and adjoint pairs are given by the following two propositions.

Proposition 5 Let X, A be partially ordered sets and (f, g) be an adjoint of X into A. Then f is a dilation and g is an erosion.

The converse of Proposition 5 holds under the assumtpions of completeness:

Proposition 6 Let X, A be partially ordered sets.

1. When A is a complete $\bigvee$-semi lattice, for a mapping $f : X \to A$ to be a dilation, it is necessary and sufficient that f is monotone and the pair (f, g) is an adjoint for the mapping defined by $g(a) = \bigvee f^{-1}\{b \in A \mid b \leqslant a\}$.
2. When X is a complete $\bigwedge$-semi lattice, for a mapping $g : A \to X$ to be an erosion, it is necessary and sufficient that g is monotone and the pair (f, g) is an adjoint for the mapping defined by $f(x) = \bigwedge g^{-1}\{y \in X \mid x \leqslant y\}$.

Example 3.3 (Adjoint of set lattices) Let $R \subseteq X \times A$ be a binary relation. Then the pair $(\delta_{{}^tR}, \varepsilon_R)$ is an adjoint of $\mathbf{2}^X$ into $\mathbf{2}^A$. In fact, for $Y \in \mathbf{2}^X$, $B \in \mathbf{2}^A$,

$$\begin{aligned}
&\delta_{{}^tR}(Y) \subseteq B \\
&\Leftrightarrow \forall a \in A, \forall x \in X\,(x \in Y \Rightarrow (a \in R(x) \Rightarrow a \in B)) \\
&\qquad\qquad\qquad\qquad \Leftrightarrow Y \subseteq \varepsilon_R(B).
\end{aligned}$$

Similarly, the pair $(\delta_R, \varepsilon_{{}^tR})$ is an adjoint of $\mathbf{2}^A$ into $\mathbf{2}^X$.

In particular, for a group G, we have for H, J, $K \in \mathbf{2}^G$

$$H \subseteq \tilde{J} \otimes K \Leftrightarrow J \odot H \subseteq K \Leftrightarrow J \subseteq K \oslash \tilde{H}.$$

3.4 Involutions in Boolean lattices

3.4.1 Duality, transposition and adjunction

For dilations and erosions of Boolean lattices, there are three sorts of involutive transformations of operators, namely, *duality*, *transposition* and *adjunction* [10, 4]. More precisely, if μ is an MM operator and τ is one of these three transformations then $\tau(\tau(\mu)) = \mu$ is satisfied.

Let X, A be Boolean lattices and $\delta : X \to A$ be a dilation. Then its *dual* $\overline{\delta} : X \to A$, *transpose* ${}^t\delta : A \to X$ and *adjoint* $\delta^* : A \to X$ are respectively defined by

$$\begin{array}{rcll}
\overline{\delta}(x) & = & \neg(\delta(\neg x)) & (x \in X), \\
{}^t\delta(a) \bigwedge x = 0 & \Leftrightarrow & a \bigwedge \delta(x) = 0 & (x \in X,\, a \in A), \\
x \leqslant \delta^*(a) & \Leftrightarrow & \delta(x) \leqslant a & (x \in X,\, a \in A).
\end{array}$$

Although transpose and adjoint are implicitly defined, they are uniquely determined if they exist for a given dilation.

Similarly, for an erosion $\varepsilon : X \to A$, its *dual* $\overline{\varepsilon} : X \to A$, *transpose* ${}^t\varepsilon : A \to X$ and *adjoint* $\varepsilon^* : A \to X$ are respectively defined by

$$\begin{array}{rcll}
\overline{\varepsilon}(x) & = & \neg(\varepsilon(\neg x)) & (x \in X), \\
{}^t\varepsilon(a) \bigvee x = 1 & \Leftrightarrow & a \bigvee \varepsilon(x) = 1 & (x \in X,\, a \in A), \\
\varepsilon^*(a) \leqslant x & \Leftrightarrow & a \leqslant \varepsilon(x) & (x \in X,\, a \in A).
\end{array}$$

Proposition 7 Let X, A be Boolean lattices.

1. For a dilation $\delta : X \to A$ it has an adjoint δ^* iff it has a transpose ${}^t\delta$.
2. For an erosion $\varepsilon : X \to A$ it has an adjoint ε^* iff it has a transpose ${}^t\varepsilon$.

Proposition 8 Let X, A be Boolean lattices.

1. For a dilation $\delta : X \to A$, the dual $\overline{\delta}$ and the adjoint δ^* are erosions and the transpose ${}^t\delta$ is a dilation.
2. For an erosion $\varepsilon : X \to A$, the dual $\overline{\varepsilon}$ and the adjoint ε^* are dilations and the transpose ${}^t\varepsilon$ is an erosion.

3.4.2 Interrelations among involutions

All of the operator transformations defined above are involutive. On the other hand, successive applications of operators are independent of order. For example, $\overline{({}^t\delta)}$ and ${}^t(\overline{\delta})$ coincide and are equal to δ^* and so on.

Example 3.4 (Involutions of set lattices) In case of a binary relation R, for the operators $\delta = \delta_R$ and $\varepsilon = \varepsilon_R$, the results of involutions are expressed in more explicit forms:

$$\begin{array}{lll}
\overline{\delta_R} = \varepsilon_R, & {}^t\delta_R = \delta_{{}^tR}, & (\delta_R)^* = \varepsilon_{{}^tR}, \\
\overline{\varepsilon_R} = \delta_R, & {}^t\varepsilon_R = \varepsilon_{{}^tR}, & (\varepsilon_R)^* = \delta_{{}^tR}.
\end{array}$$

By virtue of these equalities, we only have to employ 4 operators among them, for example δ_R, ε_R, $\delta_{{}^tR}$ and $\varepsilon_{{}^tR}$. The relations are diagrammatically represented as follows:

$$\begin{tikzcd} \delta_R \arrow[r, leftrightarrow, "-"] \arrow[d, leftrightarrow, "t"'] \arrow[dr, leftrightarrow, "*"] & \varepsilon_R \arrow[d, leftrightarrow, "t"] \arrow[dl, leftrightarrow] \\ \delta_{{}^tR} \arrow[r, leftrightarrow, "-"'] & \varepsilon_{{}^tR} \end{tikzcd}$$

Figure 1: Rerations among involutive operations on set lattice morphologies

4 Main result

As an MM interpretation of local composition, the following theorem is obtained in [2].

Theorem 9 For $\mathbf{c} \in \mathbf{2}^{V \cdot W}$, $\mathbf{c} \in \mathfrak{L} \diamond \mathfrak{M}$ iff there exists a $L_c \in \mathfrak{L}$ such that

$$\delta_{{}^t\sigma_c}(L_c) \subseteq \mathfrak{M} \subseteq \varepsilon_{{}^t\sigma_c}(L_c) \tag{10}$$

To give a further interpretation of local composition, we consider the hit-or-miss transform on G. On G, (1) can be expressed as

$$X \circledast (B, B') = (X \oslash \widetilde{B}) \cap (X^{\mathrm{C}} \oslash \widetilde{B'}). \tag{11}$$

Here we consider the morphology on the right. Then our main result can be stated as

Theorem 10 (Main result) For $\mathbf{c} \in \mathbf{2}^{V \cdot W}$, $\mathbf{c} \in \mathfrak{L} \diamond \mathfrak{M}$ iff

$$\bigcup_{M \in \mathfrak{M}} \mathbf{c} \circledast (M, W - M) \in \mathfrak{L}. \tag{12}$$

Proof. By virtue of (2), it suffices to show that

$$\sigma_c{}^{-1}(\mathfrak{M}) = \bigcup_{M \in \mathfrak{M}} \mathbf{c} \circledast (M, W - M). \tag{13}$$

Since σ_c is a mapping of V into $\mathbf{2}^W$, $\sigma_c{}^{-1}(\mathfrak{M}) \subseteq V$. Furthermore, by the definition of inverse image, for $v \in V$,

$$\begin{aligned} v \in \sigma_c{}^{-1}(\mathfrak{M}) &\Leftrightarrow \sigma_c(v) \in \mathfrak{M} \\ &\Leftrightarrow \exists M \in \mathfrak{M} \text{ such that } v^{-1}\mathbf{c} \cap W = M. \end{aligned}$$

Now we decompose the condition $v^{-1}\mathbf{c} \cap W = M$ into two parts

$$M \subseteq v^{-1}\mathbf{c} \cap W. \qquad v^{-1}\mathbf{c} \cap W \subseteq M. \tag{14}$$

By noticing that $M \subseteq W$,

$$\begin{aligned} 1^{\text{st}} \text{ part of (14)} &\Leftrightarrow M \subseteq v^{-1}\mathbf{c} \\ &\Leftrightarrow vM \subseteq \mathbf{c} \\ &\Leftrightarrow \{v\} \subseteq \mathbf{c} \oslash \widetilde{M}. \end{aligned}$$

Hence the 1^{st} part of (14) is equivalent to that $v \in \mathbf{c} \oslash \widetilde{M}$. On the other hand, by using the adjunction $X \cap Y \subseteq Z \Leftrightarrow X \subseteq Z \cup Y^{\mathrm{C}}$,

$$\begin{aligned} 2^{\text{nd}} \text{ part of (14)} &\Leftrightarrow v^{-1}\mathbf{c} \subseteq M \cup W^{\mathrm{C}} = (W - M)^{\mathrm{C}} \\ &\Leftrightarrow \tilde{\mathbf{c}}v \subseteq \widetilde{(W - M)}^{\mathrm{C}} \\ &\Leftrightarrow \{v\} \subseteq \mathbf{c} \obslash \left(\widetilde{(W - M)}^{\mathrm{C}}\right). \end{aligned}$$

By using dualities,

$$\begin{aligned} \mathbf{c} \obslash \left(\widetilde{(W - M)}^{\mathrm{C}}\right) &= \left(\mathbf{c} \odot \widetilde{(W - M)}\right)^{\mathrm{C}} \\ &= \mathbf{c}^{\mathrm{C}} \oslash \widetilde{(W - M)}. \end{aligned}$$

Thus the 2^{nd} part of (14) is equivalent to that $v \in \mathbf{c}^{\mathrm{C}} \oslash \widetilde{(W - M)}$. By combining these results, the condition $v^{-1}\mathbf{c} \cap W = M$ is equivalent to

$$\begin{aligned} v &\in (\mathbf{c} \oslash \widetilde{M}) \cap (\mathbf{c}^{\mathrm{C}} \oslash \widetilde{(W - M)}) \\ &= \mathbf{c} \circledast (M, W - M). \end{aligned}$$

This establishes the theorem. *q.e.d.*

By this theorem, the local composition $\mathfrak{L} \diamond \mathfrak{M}$ contains the all local patterns $\mathbf{c}$ consisting of those points where M fits $\mathbf{c}$ and $W - M$ fits $\mathbf{c}^{\mathrm{C}}$ fits some pattern M in $\mathfrak{M}$ form a pattern in $\mathfrak{L}$.

The main theorem is straightfowardly applied to the local decomposition problem. Namely, for given local transition rules $\mathfrak{M} \in \mathcal{L}_W$ and $\mathfrak{K} \in \mathcal{L}_{V \odot W}$, when is $\mathfrak{K}$ decomposed into $\mathfrak{L} \diamond \mathfrak{M}$ for some $\mathfrak{L} \in \mathcal{L}_V$? From main theorem, it follows that $\bigcup_{M \in \mathfrak{M}} K \circledast (M, W - M)$ must be a member of $\mathfrak{L}$ for each $K \in \mathfrak{K}$ and such subsets constitute $\mathfrak{L}$. Notice that $\bigcup_{M \in \mathfrak{M}} K \circledast (M, W - M)$ may not be included in V. Thus we have

Cororally 11 *Let $\mathfrak{M} \in \mathcal{L}_W$ and $\mathfrak{K} \in \mathcal{L}_{V \odot W}$. For $\mathfrak{K}$ to be equal to $\mathfrak{L} \diamond \mathfrak{M}$ for some $\mathfrak{L} \in \mathcal{L}_V$, it is neccesary and sufficient that*

$$\bigcup_{M \in \mathfrak{M}} K \circledast (M, W - M) \subseteq V$$

for all $K \in \mathfrak{K}$. Then the left factor $\mathfrak{L}$ is given by

$$\mathfrak{L} = \left\{ \bigcup_{M \in \mathfrak{M}} K \circledast (M, W - M) \;\middle|\; K \in \mathfrak{K} \right\}.$$

Notice that the notion of local decomposition should be defined modulo equivalence. That is, two local transition rules $\mathfrak{K}$ and $\mathfrak{K}'$ may define the same transition function $T_{\mathfrak{K}} = T_{\mathfrak{K}'}$. In this case, we call $\mathfrak{K}$ is equivalent to $\mathfrak{K}'$ and denote by $\mathfrak{K} \sim \mathfrak{K}'$. For this reason, we say that $\mathfrak{L} \diamond \mathfrak{M}$ is a local decmoposition of $\mathfrak{K}$ iff $\mathfrak{K} \sim \mathfrak{L} \diamond \mathfrak{W}$. The cororally gives a sufficient condition for $\mathfrak{K}$ to be locally decmoposed into $\mathfrak{L} \diamond \mathfrak{M}$.

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