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NORMAL FUNCTIONS AND THE GHC

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In this note, based on the first author's talk at the symposium on "Hodge theory and algebraic geometry" (June 29 - July 3, 2009) in honor of Professor S. Usui, we revisit the link between singularities of admissible normal functions and the Hodge conjecture established in [GG],[BFNP],[dCM] (also see [KP1]), and describe how it extends to the generalized Hodge conjecture. Proofs may be found in [KP2], where the results will also be placed in a richer context. We thank RIMS and the organizers Asakura, Ikeda, and Kato for a wonderful conference, and anticipate that the developments on Néron models and zero loci of normal functions which were taking place (or which we learned about) during that week will allow (GG) (and perhaps (2) and (3)) to be established in new cases. We also thank A. Beilinson for the letter [Be] which inspired us to work out this story, and J. Lewis and M. Saito for helpful discussions.

Given a smooth projective variety X of dimension $2m$ over $\mathbb{C}$ and a very ample line bundle $L \to X$, we define the primitive cohomology by

$$H^{2m}(X)_L^{prim} := \ker\{H^{2m}(X) \stackrel{\cup c_1(L)}{\longrightarrow} H^{2m+2}(X)\}$$

and the primitive Hodge classes by

$$Hg^m(X)_L^{prim} := \mathrm{Hom}_{MHS}(\mathbb{Z}(0), H^{2m}(X, \mathbb{Z}(m))_L^{prim}).$$

Now take

$$\mathcal{X} = \{(x, [\sigma]) \in X \times \mathcal{S} \mid \sigma(x) = 0\}$$

to be the incidence variety associated to X and $\mathcal{S} = \mathbb{P}H^0(X, \mathcal{O}(L))$. Let $\rho : \mathcal{X} \to \mathcal{S}$ denote projection onto the second factor and $X^\vee$ be the discriminant locus of ρ. Then, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{S}\backslash X^\vee & = & \mathcal{S}^{sm} & \stackrel{\jmath}{\hookrightarrow} & \mathcal{S} \\ & & \uparrow \rho^{sm} & & \uparrow \rho \\ \rho^{-1}(\mathcal{S}^{sm}) & = & \mathcal{X}^{sm} & \hookrightarrow & \mathcal{X} \end{array} ,$$

and the pullbacks of primitive cohomology classes to fibres of ρ^{sm} are zero. The same is not true for their pullbacks to singular fibres of ρ.

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Let $\mathcal{H}$ be the variation of pure Hodge structure of weight -1 over $\mathcal{S}^{sm}$ with underlying local system $R^{2m-1}(\rho^{sm})_*\mathbb{Z}_{\mathcal{X}^{sm}}(m)$. Let $\mathrm{ANF}(\mathcal{S}^{sm},\mathcal{H})$ denote the group of admissible normal functions [S] associated to $\mathcal{H}$ over $\mathcal{S}^{sm}$. Then, via Deligne cohomology (see [S], remark 1.7), we obtain an Abel-Jacobi map

$$\mathrm{AJ} : Hg^m(X)_L^{prim} \to \mathrm{ANF}(\mathcal{S}^{sm},\mathcal{H})/J^m(X).$$

Here, an element η of the intermediate Jacobian $J^m(X)$ defines an admissible normal function $\nu(s) = \imath_s^*(\eta)$ via the induced map $\imath_s^* : J^m(X) \to J^m(X_s)$ defined by inclusion of the fiber $X_s = \rho^{-1}(s)$ into X.

By [GG][BFNP][dCM], for each point $s \in \mathcal{S}$, we have a singularity class map

$$\mathrm{sing}_s : \mathrm{ANF}(\mathcal{S}^{sm},\mathcal{H})/J^m(X) \to \mathrm{Hom}(\mathbb{Q}(0), \imath_s^* R^1 \jmath_{!*}\mathcal{H}) \cong (IH_s^1(\mathcal{H}))_{\mathbb{Q}}^{(0,0)},$$

induced by $\imath_s^* \circ R\jmath_{!*}$ in the category of mixed Hodge modules. An element ν of $\mathrm{ANF}(\mathcal{S}^{sm},\mathcal{H})/J^m(X)$ is said to be singular at the point $s \in \mathcal{S}$ if $\mathrm{sing}_s(\nu) \neq 0$. Clearly, the target of sing_s vanishes if $s \in \mathcal{S}^{sm}$, and hence the singular locus of $\nu \in \mathrm{ANF}(\mathcal{S}^{sm},\mathcal{H})/J^m(X)$ is contained in $X^\vee$. As a consequence of admissibility, the target of sing_s also vanishes if s is a smooth point of $X^\vee$. In what follows, we shall say that an admissible normal function is singular if its singular locus is non-empty.

Where necessary, the dependence of the above constructs upon the choice of the very ample line bundle L will be signified by adjoining a subscript L. In particular, a primitive Hodge class ζ with respect to L defines an admissible normal function $\nu_{\zeta,L} = \mathrm{AJ}_L(\zeta)$. The theorem of Griffiths and Green states that the Hodge conjecture is equivalent to the following (conjectural) statement:

$$\textbf{(GG)} \qquad \begin{array}{c} \forall (X, L, \zeta) \text{ as above, } \exists\, a \in \mathbb{N} \text{ such that } \nu_{\zeta,L^a} \\ \text{is singular on } \mathcal{S}_{L^a}. \end{array}$$

In keeping with the RIMS workshop, in the above discussion we retained an integral structure on $\mathcal{H}$ in order to obtain true normal functions. In what follows, we shall write down the analog of **(GG)** for the generalized Hodge conjecture. For this, we work over $\mathbb{Q}$, hence obtaining $\mathbb{Q}$-normal functions. We will also replace $\mathcal{H}$ with its quotient by the constant subvariation consisting of the image of $H^{2m-1}(X,\mathbb{Q}(m))$ in the cohomology of the fibers.

As the first step in this program, we need to rephrase the GHC: given a polarized Hodge structure H, denote by $F^K_{hdg}H$ the largest sub-Hodge structure of H with complexification contained in $F^K H_{\mathbb{C}}$. For each $K \in \mathbb{Z}_{\leq 0}$, define a set of 4-tuples

$$\Xi_K := \left\{ (m, X, L, \zeta) \;\middle|\; \begin{array}{c} m \in \mathbb{Z}_{\geq -K} \\ X = \text{smooth proj. var. } /\mathbb{C} \text{ of dim. } 2m \\ L = \text{very ample line bundle } /X \\ \zeta \in F^K_{hdg} H^{2m}(X, \mathbb{Q}(m))^{prim}_L \end{array} \right\},$$

noting that $F^0_{hdg} H^{2m}(X, \mathbb{Q}(m))^{prim}_L \cong Hg^m(X)^{prim}_L$. Then

$$\textbf{(GHC}_{\textbf{K}}\textbf{)} \qquad \forall (m, X, L, \zeta) \in \Xi_K,\ \exists\ \mathcal{W} \underset{\substack{\text{codim.}\\ m+K}}{\overset{\imath_{\mathcal{W}}}{\hookrightarrow}} X \text{ such that}$$
$$0 \neq \imath^*_{\mathcal{W}}[\zeta] \in H^{2m}(\mathcal{W}, \mathbb{Q}(m))$$

is equivalent to the (usual) statement that all such ζ belong to the $(m+K)^{\underline{\text{th}}}$ coniveau filtrand. Write Ξ^+_K if the condition $m \geq -K$ is replaced by $m > -K$.

Now one has the "tautological extension"

$$0 \to \mathcal{H} \longrightarrow \tilde{U} \longrightarrow \underline{H^1(\mathcal{S}^{sm}, \mathcal{H})} \to 0$$

of sheaves $/\mathcal{S}^{sm}$, which we may pull back to

$$(1) \qquad 0 \to \mathcal{H} \longrightarrow U \longrightarrow \underline{H^{2m}(X, \mathbb{Q}(m))^{prim}_L} \to 0.$$

In fact, we may regard (1) as an extension in the category of admissible VMHS (or MHM) on $\mathcal{S}^{sm}$. Applying $\jmath_{!*}$ and taking the long-exact cohomology sequence over $\mathcal{S}$ yields the connecting homomorphism

$$\theta : H^{2m}(X, \mathbb{Q}(m))^{prim}_L \to IH^1(\mathcal{H}),$$

a morphism of (pure) Hodge structure whose composition with restriction to s

$$IH^1(\mathcal{H}) \xrightarrow{|_s} Gr^W_0 IH^1_s(\mathcal{H})$$

extends $\text{sing}_s \circ \text{AJ}$ to all primitive *cohomology* classes. Again signifying the dependence of these constructions on L by adjoining a subscript, we have the main

Theorem. *For each $K \in \mathbb{Z}_{\leq 0}$,* $\textbf{(GHC}_{\textbf{K}}\textbf{)}$ *is equivalent to the statement*

(2) $\forall (m, X, L, \zeta) \in \Xi^+_K$, $\exists\, a \in \mathbb{N}$ *and* $s \in X^{\vee}_{L^a}$ *such that* $\theta_{L^a}(\zeta)|_s \neq 0$.

For $K = 0$ this essentially reproduces the Griffiths-Green result.

There are two directions one can take this: refining the information contained in θ by substrata of $X^{\vee}$; and relating it back to normal functions, which we do first. Pulling (1) back on the right by the inclusion of a sub-Hodge structure $V \subseteq H^{2m}(X, \mathbb{Q}(m))_L^{prim}$ yields an extension

$$0 \to \mathcal{H} \longrightarrow U_V \longrightarrow V \to 0$$

in the category of admissible VMHS over $\mathcal{S}^{sm}$, or equivalently a normal function

$$\nu_V \in ANF(\mathcal{S}, \mathcal{H} \otimes V^\vee).$$

Write $\tilde{\Xi}_K^{(+)}$ for the analogue of $\Xi_K^{(+)}$ with ζ replaced by an *irreducible* sub-Hodge structure V of $F_{hdg}^K H^{2m}(X, \mathbb{Q}(m))_L^{prim}$.

Corollary. $(\mathbf{GHC_K})$ *is equivalent to*

(3) $\forall (m, X, L, V) \in \tilde{\Xi}_K^+, \exists\, a \in \mathbb{N}$ *such that* ν_{V,L^a} *is singular on* $\mathcal{S}_{L^a}$.

Consider the decreasing filtration $\mathfrak{F}^K X^\vee$ on the discriminant locus (or "dual variety") defined by

$$s \in \mathfrak{F}^K X^\vee \iff \dim(\mathrm{sing}(X_s)) \leq -2K,$$

with $\mathfrak{F}^{-m+1}X^\vee = X^\vee$ and $\mathfrak{F}^1 X^\vee = \emptyset$. Here $\mathrm{sing}(X_s)$ means the subvariety of X_s consisting of its singular points. Writing $\{s_i\}$ for a system of representative points in the substrata of $\mathfrak{F}^K X^\vee$, we "refine" θ by the morphisms

$$Res_L^K := \left(\oplus_{s_i \in \mathfrak{F}^K X_L^\vee} \theta_L(\cdot)|_{s_i}\right) : \; H^{2m}(X, \mathbb{Q}(m))_L^{prim} \to \oplus_i Gr_0^W IH^1_{s_i}(\mathcal{H})$$

of Hodge structures. We then have (for each $K \in \mathbb{Z} \cap [-m+1, 0]$) two sub-Hodge structures, $\ker(Res_L^K)$ and F_{hdg}^K, of the middle primitive cohomology of X.

Question. *Assume* $(\mathbf{GHC_K})$. *Then for each* (m, X, L) *(with* $m > -K$*), does* $a \gg 0 \implies$

(4) $$H^{2m}(X, \mathbb{Q}(m))_L^{prim} = \ker(Res_{L^a}^K) \oplus F_{hdg}^K \,?$$

That $\ker(Res^K) + F_{hdg}^K$ spans all of H^{2m} would be implied by the statement

(5) $$s \in \mathfrak{F}^K X^\vee \implies \mathrm{level}(IH^1_s(\mathcal{H})) \leq -2K,$$

since then the target Hodge structure of Res^K would equal its own F_{hdg}^K. Furthermore, could one show (given any $\zeta \in F_{hdg}^K$) that

(6) the $\mathcal{W}$ in $(\mathbf{GHC_K})$ can be taken to be smooth,

there would exist $s \in \mathfrak{F}^K X^\vee$ with $\mathcal{W} \subset X_s$. This leads easily to $\zeta \notin \ker(Res^K)$, so that $\ker(Res^K) \cap F_{hdg}^K = \{0\}$. So proving (4) involves checking (5) and (6) (or analogous statements).

We take a look at the extreme cases. First, for $K = -m+1$, the Theorem will do in lieu of (6) (since $\mathfrak{F}^{-m+1}$ is all of $X^{\vee}$), and (5) is automatic. For $K = 0$, all three statements hold, using §4 of [Th] for (6), at least provided we replace $\mathfrak{F}^{0}X^{\vee}$ by the (smaller) locus of fibres with ODP singularities. Some similar modification of $\mathfrak{F}^{\bullet}$ may be required to make (4) work for all K. The analogy to automorphic forms (if we think of $\ker(Res^{K})$ as "$|K|$-cusp forms") seems tantalizing enough that this should be worked out.

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