数理解析研究所講究録
第 1746 巻 2011 年 58-68

Stationary problem of a prey-predator cross-diffusion system with a protection zone

早稲田大学・基幹理工学部　大枝 和浩 (Kazuhiro Oeda)
Department of Applied Mathematics,
Waseda University
E-mail: k-oeda@aoni.waseda.jp

1 Introduction

The main purpose of this article is to make a resume of recent results obtained by the author [10]. In this article, we study the following Lotka-Volterra prey-predator model:

$$
\text{(P)} \begin{cases} u_t = \Delta[(1+k\rho(x)v)u] + u(\lambda - u - b(x)v), & (x,t) \in \Omega \times (0,\infty), \\ v_t = \Delta v + v(\mu + cu - v), & (x,t) \in \Omega \setminus \overline{\Omega}_0 \times (0,\infty), \\ \partial_n u = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \partial_n v = 0, & (x,t) \in \partial(\Omega \setminus \overline{\Omega}_0) \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \\ v(x,0) = v_0(x) \geq 0, & x \in \Omega \setminus \overline{\Omega}_0, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N (N \leq 3)$ with smooth boundary $\partial\Omega$ and Ω_0 is a subdomain of Ω with smooth boundary $\partial\Omega_0$; n is the outward unit normal vector on the boundary and $\partial_n = \partial/\partial n$; $k \geq 0$, $\lambda > 0$, $c > 0$ and $\mu \in \mathbb{R}$ are all constants; $\rho > 0$ and $b > 0$ in $\Omega \setminus \Omega_0$, whereas $\rho = b = 0$ in Ω_0 because v is not defined in Ω_0. In addition, we make the following assumption: if $N = 2$ or 3, then $\overline{\Omega}_0 \subset \Omega$; if $N = 1$ and $\Omega = (a_1, a_2)$ for $a_1 < a_2$, then $\Omega_0 = (a_1, a)$ or $\Omega_0 = (a, a_2)$ for some $a \in (a_1, a_2)$. In (P), unknown functions $u(x,t)$ and $v(x,t)$ denote the population densities of prey and predator respectively; λ and μ denote the intrinsic growth rates of the respective species; $b(x)$ and c denote the coefficients of prey-predator interaction; the zero-flux boundary condition means that no individuals cross the boundary.

In the first equation of (P), $k\Delta[\rho(x)vu]$ is usually referred to as a cross-diffusion term which was originally proposed by Shigesada et al. [13]. The cross-diffusion term $k\Delta[\rho(x)vu]$ means that the movement of the prey species is affected by population pressure from the predator species and the cross-diffusion coefficient k denotes the sensitivity of the prey species to population pressure from the predator species.

In (P), the predator species cannot enter the subregion Ω_0 of the habitat Ω, while the prey species can enter and leave Ω_0 freely. Namely, Ω_0 is a predation-free zone for the prey species and such a subregion Ω_0 is called a protection zone. One can think that there is a barrier along $\partial\Omega_0$ that blocks the predator but not the prey (see [2]–[4] for further details). In the case where cross-diffusion is absent, Du et al. [2]–[4] have studied the effects of a protection zone on Lotka-Volterra competition model [2], Leslie prey-predator model [3], and Holling type II prey-predator model [4] respectively. They have proved that if the size of the protection zone is larger than a certain critical patch size, which is common to three models, then a fundamental change occurs in the dynamical behavior of each of three models.

Let $\Omega_1 := \Omega \setminus \overline{\Omega}_0$. The stationary problem associated with (P) is given by

$$
\text{(SP)} \begin{cases} \Delta[(1+k\rho(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega_1, \\ \partial_n u = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega_1. \end{cases}
$$

When $\Omega_0 = \emptyset$, there are some studies on prey-predator models with cross-diffusion analogous to (SP) (see e.g. [5], [6], [14]).

In this article, we study the following two subjects: first, we study the effects of cross-diffusion on the existence and non-existence of positive solutions of (SP), and secondly, we study the asymptotic behavior of positive solutions of (SP) as $k \to \infty$. From an ecological viewpoint, a positive solution of (SP) means a coexistence state of the two species. From now on, we always assume that

$$
\rho(x) = \chi_{\Omega\setminus\Omega_0}(x) := \begin{cases} 1 & \text{if } x \in \Omega \setminus \Omega_0, \\ 0 & \text{if } x \in \Omega_0, \end{cases} \quad \text{and } b(x) = \begin{cases} \beta & \text{if } x \in \Omega \setminus \Omega_0, \\ 0 & \text{if } x \in \Omega_0, \end{cases}
$$

where β is a positive constant.

This article is organized as follows. In Section 2, we will state the main results of this article. In Section 3, we will state a priori estimates of positive solutions. Moreover, we will study the local bifurcation of positive solutions from semitrivial solutions. In Section 4, we will prove our main results except for Theorem 2.5.

2 Main results

We define

$$
U = (1 + k\rho(x)v)u. \tag{2.1}
$$

Then (SP) is rewritten in the following form:

$$
\text{(EP)}\begin{cases}
\Delta U + \dfrac{U}{1+k\rho(x)v}\left(\lambda - \dfrac{U}{1+k\rho(x)v} - b(x)v\right) = 0 & \text{in } \Omega,\\
\Delta v + v\left(\mu + \dfrac{cU}{1+kv} - v\right) = 0 & \text{in } \Omega_1,\\
\partial_n U = 0 & \text{on } \partial\Omega,\\
\partial_n v = 0 & \text{on } \partial\Omega_1.
\end{cases}
$$

Define

$$E = C_n^1(\overline{\Omega}) \times C_n^1(\overline{\Omega}_1), \tag{2.2}$$

where $C_n^1(\overline{O}) = \{w \in C^1(\overline{O}) : \partial_n w = 0 \text{ on } \partial O\}$. We say that (u,v) is a positive solution of (SP) if $(U,v) \in E$ is a positive solution of (EP) and u is defined by (2.1).

Let $\lambda_1^D(\Omega_0)$ be the first eigenvalue of $-\Delta$ over Ω_0 with the homogeneous Dirichlet boundary condition (the boundary condition should be replaced by $\phi(a) = \phi'(a_i) = 0$ for $i = 1$ or 2 if $N = 1$, but we use the same symbol $\lambda_1^D(\Omega_0)$). For $q \in L^\infty(O)$, we denote by $\lambda_1^N(q, O)$ the first eigenvalue of $-\Delta + q$ over O with the homogeneous Neumann boundary condition. Before stating our main results, we state the following lemma.

Lemma 2.1. *For any fixed k and Ω_0, there exists a continuous and strictly increasing function $\lambda^*(\mu)$ with respect to $\mu \geq 0$ such that $\lambda^*(0) = 0$, $\lambda^*(\mu) < \beta\mu$ for any $\mu > 0$, $\lim_{\mu\to\infty}\lambda^*(\mu) \leq \lambda_1^D(\Omega_0)$ and*

$$\left\{(\lambda,\mu) \in [0,\infty)^2 : \lambda_1^N\left(\frac{b(x)\mu - \lambda}{1+k\rho(x)\mu}, \Omega\right) = 0\right\} = \{(\lambda^*(\mu),\mu) : \mu \geq 0\}.$$

Our first result is the following theorem concerning the existence of coexistence states of (SP) with fixed k and Ω_0.

Theorem 2.2. *The following results hold true:*

(i) *Suppose that $\mu \geq 0$. Then* (SP) *has at least one positive solution if and only if $\lambda > \lambda^*(\mu)$.*

(ii) *Suppose that $\mu < 0$. Then* (SP) *has at least one positive solution if $\lambda > -\mu/c$.*

Hereafter, We write $\lambda^*(\mu, k, \Omega_0)$ instead of $\lambda^*(\mu)$ to state the dependence on k and Ω_0 explicitly. Moreover, we define $\lambda_\infty^*(k,\Omega_0) := \lim_{\mu\to\infty}\lambda^*(\mu,k,\Omega_0) \leq \lambda_1^D(\Omega_0)$.

When $\Omega_0 = \emptyset$, it is known that for any $k \geq 0$, (SP) has no positive solution if $\lambda \leq \beta\mu$. On the other hand, Lemma 2.1 and part (i) of Theorem 2.2 assert that when $\Omega_0 \neq \emptyset$, (SP) has at least one positive solution for any $\mu > 0$ if $\lambda \geq \lambda_\infty^*(k,\Omega_0)$. Namely, we can regard $\lambda_\infty^*(k,\Omega_0)$ as a threshold prey growth rate for the survival of the prey species. Here, we see from [4] that $\lambda_\infty^*(0,\Omega_0)$ is given by $\lambda_1^D(\Omega_0)$. Then it is interesting to study the dependence of the threshold prey growth rate $\lambda_\infty^*(k,\Omega_0)$ on k and Ω_0 and the following theorem holds.

Theorem 2.3. *The following results hold true:*

(i) *Suppose that $\mu > 0$. Then $\lambda^*(\mu, k, \Omega_0)$ is strictly decreasing with respect to k.*

(ii) *For any $k > 0$, it holds that*

$$\lambda^*_\infty(k, \Omega_0) = \inf_{\{\phi\in H^1(\Omega):\int_{\Omega_0}\phi^2dx>0\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2dx + \frac{\beta}{k}\int_{\Omega\setminus\Omega_0}\phi^2dx}{\displaystyle\int_{\Omega_0}\phi^2dx} \leq \frac{\beta|\Omega\setminus\Omega_0|}{k|\Omega_0|}.$$

Part (i) of Theorem 2.3 means that when $\mu > 0$, the coexistence region become larger as k increases, and part (ii) of Theorem 2.3 means that the threshold prey growth rate $\lambda^*_\infty(k, \Omega_0)$ decreases to 0 as $k \to \infty$ or Ω_0 is enlarged to the entire Ω. Namely, in the limiting case where $k \to \infty$ or Ω_0 is enlarged to Ω, the prey species can coexist with the predator species regardless of the values of $\lambda > 0$ and $\mu > 0$. This is in sharp contrast to the no cross-diffusion case, where the threshold prey growth rate $\lambda_1^D(\Omega_0)$ satisfies $\lambda_1^D(\Omega_0) \geq \lambda_1^D(\Omega) > 0$ for any $\Omega_0 \subset \Omega$. Therefore, we can say that the cross-diffusion for the prey has beneficial effects on the survival of the prey species when a protection zone is present.

Concerning the asymptotic behavior of positive solutions of (SP) as $k \to \infty$, the following theorem holds.

Theorem 2.4. *Let (u_k, v_k) be any positive solution of* (SP) *for each k.*

(i) *Suppose that $\mu \geq 0$. Then*

$$\lim_{k\to\infty}(u_k, u_k, v_k) = (\lambda, 0, \mu) \ \ in\ C^1(\Omega_0) \times C^1(\overline{\Omega}_1) \times C^1(\overline{\Omega}_1).$$

Moreover, $\lim_{k\to\infty} kv_k = \infty$ uniformly in $\overline{\Omega}_1$ even when $\mu = 0$.

(ii) *Suppose that $\lambda > -\mu/c > 0$ and let $\{k_i\}_{i=1}^\infty$ be any sequence with $\lim_{i\to\infty} k_i = \infty$. Then, by passing to a subsequence if necessary,*

$$\lim_{i\to\infty} u_{k_i} = \bar{u} \ \ uniformly\ in\ \overline{\Omega}, \quad \lim_{i\to\infty}(v_{k_i}, k_i v_{k_i}) = (0, \bar{w}) \ \ in\ C^1(\overline{\Omega}_1)^2,$$

where $(\bar{u}, \bar{w})$ is a positive solution of

$$\begin{cases} \Delta[(1+\rho(x)\bar{w})\bar{u}] + \bar{u}(\lambda - \bar{u}) = 0 & in\ \Omega, \\ \Delta\bar{w} + \bar{w}(\mu + c\bar{u}) = 0 & in\ \Omega_1, \\ \partial_n\bar{u} = 0 & on\ \partial\Omega, \\ \partial_n\bar{w} = 0 & on\ \partial\Omega_1. \end{cases} \tag{2.3}$$

Part (i) of Theorem 2.4 means that when $\mu \geq 0$, the prey species concentrates in the protection zone as $k \to \infty$ and when $\mu > 0$ in particular, the two species become spatially segregated as $k \to \infty$.

We can analyze the bifurcation structure of positive solutions of the limiting system (2.3).

Theorem 2.5. *The set of positive solutions of* (2.3) *with bifurcation parameter* μ *contains an unbounded connected set* Γ *in* $\mathbb{R} \times L^\infty(\Omega) \times C^1(\overline{\Omega}_1)$ *satisfying the following properties:*

(i) Γ *bifurcates from* $\{(\mu, \bar{u}, \bar{w}) = (\mu, \lambda, 0) : \mu \in \mathbb{R}\}$ *at* $\mu = -c\lambda$,

(ii) $(-c\lambda, 0) \subset \{\mu : (\mu, \bar{u}, \bar{w}) \in \Gamma\} \subset (\tilde{\mu}, 0)$ *for some* $\tilde{\mu} \in (-\infty, -c\lambda]$,

(iii) $\lim_{\mu \to 0} \bar{u}_\mu = \lambda$ *in* $C^1(\Omega_0)$ *and* $\lim_{\mu \to 0}(\bar{u}_\mu, \bar{w}_\mu) = (0, \infty)$ *uniformly in* $\overline{\Omega}_1$, *where* $(\mu, \bar{u}_\mu, \bar{w}_\mu) \in \Gamma$.

We remark that (iii) of Theorem 2.5 is compatible with (i) of Theorem 2.4.

3 A priori estimates and local bifurcation

3.1 A priori estimates of positive solutions

By combining L^2-estimates of positive solutions of (EP) with Harnack inequality (see [7] and [9]), we can prove the following a priori estimates of positive solutions.

Lemma 3.1. *Let* $\theta \in (0, 1)$. *Then there exists a positive constant* C *independent of* k *such that any positive solution* (U, v) *of* (EP) *satisfies*

$$\|U\|_{C^{1,\theta}(\overline{\Omega})} \leq C \quad \text{and} \quad \|v\|_{C^{1,\theta}(\overline{\Omega}_1)} \leq C.$$

3.2 Local bifurcation from semitrivial solutions

In this subsection, we regard λ as a bifurcation parameter in order to obtain a branch of positive solutions which bifurcates from the semitrivial solution curve

$$\Gamma_U = \{(\lambda, U, v) = (\lambda, \lambda, 0) : \lambda > 0\} \quad \text{or} \quad \Gamma_v = \{(\lambda, U, v) = (\lambda, 0, \mu) : \lambda > 0\}.$$

For $p > N$, we define

$$X_1 = W^{2,p}_n(\Omega) \times W^{2,p}_n(\Omega_1) \quad \text{and} \quad X_2 = L^p(\Omega) \times L^p(\Omega_1),$$

where $W^{2,p}_n(O) = \{w \in W^{2,p}(O) : \partial_n w = 0 \text{ on } \partial O\}$. We note that $X_1 \subset E$ by the Sobolev embedding theorem, where E is the Banach space defined by (2.2).

We first consider the local bifurcation from Γ_v for any fixed $\mu > 0$. Let $\lambda^* = \lambda^*(\mu)$ be the positive number defined in Lemma 2.1 and let ϕ^* be a positive solution of

$$-\Delta\phi^* + \frac{b(x)\mu - \lambda^*}{1 + k\rho(x)\mu}\phi^* = 0 \text{ in } \Omega, \quad \partial_n\phi^* = 0 \text{ on } \partial\Omega.$$

We also define

$$\psi^* = (-\Delta + \mu I)^{-1}_{\Omega_1}\left[\frac{c\mu}{1 + k\mu}\phi^*\right],$$

where I is the identity mapping and $(-\Delta + \mu I)^{-1}_{\Omega_1}$ is the inverse operator of $-\Delta + \mu I$ over Ω_1 subject to the homogeneous Neumann boundary condition. Then we can prove the following proposition by applying the local bifurcation theorem of Crandall and Rabinowitz [1] to (EP).

Proposition 3.2. *Assume that $\mu > 0$. Positive solutions of* (EP) *bifurcate from Γ_v if and only if $\lambda = \lambda^*$. To be precise, all positive solutions of* (EP) *near $(\lambda^*, 0, \mu) \in \mathbb{R} \times X_1$ can be expressed as*

$$\hat{\Gamma}_\delta = \{(\lambda, U, v) = (\lambda(s), s(\phi^* + U(s)), \mu + s(\psi^* + v(s))) : s \in (0, \delta)\}$$

for some $\delta > 0$. Here $(\lambda(s), U(s), v(s))$ is a smooth function with respect to s and satisfies $(\lambda(0), U(0), v(0)) = (\lambda^*, 0, 0)$ and $\int_\Omega U(s)\phi^* dx = 0$. Furthermore, $\lambda'(0) > 0$.

Proof. We only prove $\lambda'(0) > 0$. Define a mapping $F : \mathbb{R} \times X_1 \to X_2$ by

$$F(\lambda, U, v) = \begin{pmatrix} \Delta U + \dfrac{U}{1 + k\rho(x)v}\left(\lambda - \dfrac{U}{1 + k\rho(x)v} - b(x)v\right) \\ \Delta v + v\left(\mu + \dfrac{cU}{1 + kv} - v\right) \end{pmatrix}.$$

Then we can verify that

$$\operatorname{Ker} F_{(U,v)}(\lambda^*, 0, \mu) = \operatorname{span}\{(\phi^*, \psi^*)\}.$$

Using the direction formula of bifurcation (see [12]), we have

$$\lambda'(0) = -\frac{\langle F_{(U,v)(U,v)}(\lambda^*, 0, \mu)[\phi^*, \psi^*]^2, l_1\rangle}{2\langle F_{\lambda(U,v)}(\lambda^*, 0, \mu)[\phi^*, \psi^*], l_1\rangle},$$

where l_1 is the linear functional on X_2 defined by $\langle[\phi, \psi], l_1\rangle := \int_\Omega \phi\phi^* dx$. By simple calculations, we obtain

$$F_{(U,v)(U,v)}(\lambda^*, 0, \mu)[\phi^*, \psi^*]^2 = 2\begin{pmatrix} -\dfrac{(\phi^*)^2}{(1 + k\rho(x)\mu)^2} - \dfrac{b(x) + k\rho(x)\lambda^*}{(1 + k\rho(x)\mu)^2}\phi^*\psi^* \\ \dfrac{c}{(1 + k\mu)^2}\phi^*\psi^* - (\psi^*)^2 \end{pmatrix}$$

and

$$F_{\lambda(U,v)}(\lambda^*, 0, \mu)[\phi^*, \psi^*] = \begin{pmatrix} \dfrac{\phi^*}{1 + k\rho(x)\mu} \\ 0 \end{pmatrix}.$$

Hence

$$\lambda'(0) = \int_\Omega \frac{(\phi^*)^3 + (b(x) + k\rho(x)\lambda^*)(\phi^*)^2\psi^*}{(1 + k\rho(x)\mu)^2} dx \Big/ \int_\Omega \frac{(\phi^*)^2}{1 + k\rho(x)\mu} dx > 0.$$

$\square$

Next we consider the local bifurcation from Γ_U for any fixed $\mu < 0$. We define

$$\phi_* = \left(-\Delta + \frac{-\mu}{c} I\right)_\Omega^{-1} \left[-\frac{\mu}{c}\left(-\frac{k\rho(x)\mu}{c} - b(x)\right)\right]. \tag{3.1}$$

Then we can prove the following proposition.

Proposition 3.3. *Assume that $\mu < 0$. Positive solutions of* (EP) *bifurcate from Γ_U if and only if $\lambda = -\mu/c$. To be precise, all positive solutions of* (EP) *near $(-\mu/c, -\mu/c, 0) \in \mathbb{R} \times X_1$ can be expressed as*

$$\left\{(\lambda, U, v) = \left(\tilde{\lambda}(s), \tilde{\lambda}(s) + s(\phi_* + \tilde{U}(s)), s(1 + \tilde{v}(s))\right) : s \in (0, \tilde{\delta})\right\}$$

for some $\tilde{\delta} > 0$. Here $(\tilde{\lambda}(s), \tilde{U}(s), \tilde{v}(s))$ is a smooth function with respect to s and satisfies $(\tilde{\lambda}(0), \tilde{U}(0), \tilde{v}(0)) = (-\mu/c, 0, 0)$ and $\int_{\Omega_1} \tilde{v}(s)dx = 0$. Furthermore, $\tilde{\lambda}'(0) > 0$.

Proof. We only prove $\tilde{\lambda}'(0) > 0$. We can verify that

$$\operatorname{Ker} F_{(U,v)}(-\mu/c, -\mu/c, 0) = \operatorname{span}\{(\phi_*, 1)\}.$$

Moreover, we see that

$$\tilde{\lambda}'(0) = -\frac{\langle F_{(U,v)(U,v)}(-\mu/c, -\mu/c, 0)[\phi_*, 1]^2, l_2\rangle}{2\langle F_{\lambda(U,v)}(-\mu/c, -\mu/c, 0)[\phi_*, 1], l_2\rangle},$$

where l_2 is the linear functional on X_2 defined by $\langle [\phi, \psi], l_2\rangle := \int_{\Omega_1} \psi dx$. By simple calculations, we have

$$\begin{aligned} &F_{(U,v)(U,v)}(-\mu/c, -\mu/c, 0)[\phi_*, 1]^2 \\ &= 2\begin{pmatrix} -\phi_*^2 + \{3k\rho(x)(-\mu/c) - b(x)\}\phi_* + k\rho(x)(-\mu/c)\{b(x) - 2k\rho(x)(-\mu/c)\} \\ c\phi_* - \{ck\rho(x)(-\mu/c) + 1\} \end{pmatrix} \end{aligned}$$

and

$$F_{\lambda(U,v)}(-\mu/c, -\mu/c, 0)[\phi_*, 1] = \begin{pmatrix} -\phi_* - \dfrac{2k\rho(x)\mu}{c} - b(x) \\ c \end{pmatrix}.$$

We notice from (3.1) that

$$c\phi_* - \{ck\rho(x)(-\mu/c) + 1\} = -\frac{c^2}{\mu}\Delta\phi_* - cb(x) - 1.$$

Thus

$$\tilde{\lambda}'(0) = -\frac{\displaystyle\int_{\Omega_1} [c\phi_* - \{ck\rho(x)(-\mu/c) + 1\}]dx}{\displaystyle\int_{\Omega_1} cdx} = \frac{\displaystyle\int_{\Omega_1} (cb(x) + 1)dx}{c|\Omega_1|} > 0.$$

□

4 Proof of main results

4.1 Proof of Theorem 2.2

We first consider the case $\mu > 0$. By virtue of the strong maximum principle and the global bifurcation theory of Rabinowitz (see [8] and [11]), we can show that $\hat{\Gamma}_\delta$ in Proposition 3.2 is extended to an unbounded connected set of positive solutions of (EP) in $\mathbb{R} \times E$. Moreover, we can easily show that if $\lambda \leq \lambda^*(\mu)$, then (EP) has no positive solution. It thus follows from Lemma 3.1 that (EP) has at least one positive solution if and only if $\lambda > \lambda^*(\mu)$. Thus the proof for the case $\mu > 0$ is complete.

We can discuss the case $\mu < 0$ in a similar manner and so omit the proof. Hence it only remains to discuss the case $\mu = 0$. Fix any $\lambda > 0$. By virtue of the above result, we can take a sequence $\{(\mu_i, U_i, v_i)\}_{i=1}^{\infty}$ such that (U_i, v_i) is a positive solution of (EP) with $\mu = \mu_i$ and $\lim_{i\to\infty} \mu_i = 0$. Since $\{\mu_i\}_{i=1}^{\infty}$ is a bounded sequence, it follows from Lemma 3.1 that there exists a subsequence, still denoted by $\{(\mu_i, U_i, v_i)\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} (U_i, v_i) = (U_\infty, v_\infty) \text{ in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$$

for a pair of non-negative functions $(U_\infty, v_\infty) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$. By $\lim_{i\to\infty} \mu_i = 0$, (U_∞, v_∞) is a non-negative solution of (EP) with $\mu = 0$. Then we can verify from the strong maximum principle that $U_\infty > 0$ in $\overline{\Omega}$ and $v_\infty > 0$ in $\overline{\Omega}_1$. This means the existence of a positive solution of (EP) with $\mu = 0$ for any fixed $\lambda > 0$. We have thus proved Theorem 2.2.

4.2 Proof of Theorem 2.3

We only prove part (ii). For any $\mu \geq 0$, let ϕ_μ be a unique positive solution of

$$-\Delta\phi_\mu + \frac{b(x)\mu - \lambda^*(\mu, k, \Omega_0)}{1 + k\rho(x)\mu}\phi_\mu = 0 \text{ in } \Omega, \quad \partial_n\phi_\mu = 0 \text{ on } \partial\Omega \tag{4.1}$$

satisfying $\int_\Omega \phi_\mu^2 dx = 1$. Multiplying the above differential equation by ϕ_μ and integrating the resulting expression over Ω, we see from Lemma 2.1 that

$$\int_\Omega |\nabla\phi_\mu|^2 dx = \int_\Omega \frac{\lambda^*(\mu, k, \Omega_0) - b(x)\mu}{1 + k\rho(x)\mu}\phi_\mu^2 dx \leq \lambda_1^D(\Omega_0).$$

Thus $\{\phi_\mu\}_{\mu \geq 0}$ is bounded in $H^1(\Omega)$ and so there exists a sequence $\{\mu_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\mu_i = \infty$ such that $\lim_{i\to\infty}\phi_{\mu_i} = \phi_\infty$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ for some non-negative function $\phi_\infty \in H^1(\Omega)$ satisfying $\int_\Omega \phi_\infty^2 dx = 1$. Moreover, we find from (4.1) that

$$\int_\Omega \left(\nabla\phi_{\mu_i} \cdot \nabla\psi + \frac{b(x)\mu_i - \lambda^*(\mu_i, k, \Omega_0)}{1 + k\rho(x)\mu_i}\phi_{\mu_i}\psi \right) dx = 0$$

for any $\psi \in H^1(\Omega)$. Letting $i \to \infty$ in the above equation, we have

$$\int_\Omega \nabla\phi_\infty \cdot \nabla\psi dx + \frac{\beta}{k}\int_{\Omega\setminus\Omega_0} \phi_\infty\psi dx - \lambda_\infty^*(k, \Omega_0)\int_{\Omega_0} \phi_\infty\psi dx = 0$$

for any $\psi \in H^1(\Omega)$, where $\lambda_\infty^*(k, \Omega_0) = \lim_{\mu\to\infty}\lambda^*(\mu, k, \Omega_0)$. Namely, ϕ_∞ is a weak solution of

$$-\Delta\phi_\infty + \frac{\beta}{k}\chi_{\Omega\setminus\Omega_0}\phi_\infty - \lambda_\infty^*(k, \Omega_0)\chi_{\Omega_0}\phi_\infty = 0 \text{ in } \Omega, \quad \partial_n\phi_\infty = 0 \text{ on } \partial\Omega.$$

Since $\phi_\infty \geq 0$ in Ω and $\int_\Omega \phi_\infty^2 dx = 1$, we see $\phi_\infty > 0$ in $\overline{\Omega}$ by the strong maximum principle. This means that $\eta = \lambda_\infty^*(k, \Omega_0)$ is the first eigenvalue of

$$-\Delta\phi + \frac{\beta}{k}\chi_{\Omega\setminus\Omega_0}\phi = \eta\chi_{\Omega_0}\phi \text{ in } \Omega, \quad \partial_n\phi = 0 \text{ on } \partial\Omega.$$

Therefore, by the variational characterization of the first eigenvalue, we have

$$\lambda_\infty^*(k, \Omega_0) = \inf_{\{\phi\in H^1(\Omega):\int_{\Omega_0}\phi^2 dx>0\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2 dx + \frac{\beta}{k}\int_{\Omega\setminus\Omega_0} \phi^2 dx}{\displaystyle\int_{\Omega_0} \phi^2 dx} \leq \frac{\beta|\Omega\setminus\Omega_0|}{k|\Omega_0|},$$

where the last inequality is obtained by setting $\phi \equiv 1$ in Ω.

4.3 Proof of Theorem 2.4

Theorem 2.4 is proven by combining the following three lemmas.

Lemma 4.1. *Let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of* (SP) *with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$, and set $U_{k_i} := (1 + k_i\rho(x)v_{k_i})u_{k_i}$. Then, by passing to a subsequence if necessary,*

$$\lim_{i\to\infty} (U_{k_i}, v_{k_i}) = (\bar{U}, \max\{\mu, 0\}) \text{ in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$$

for some non-negative function $\bar{U} \in C^1(\overline{\Omega})$.

Lemma 4.2. *Suppose $\lambda > -\mu/c \geq 0$ and let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of* (SP) *with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$. If $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded, then $\mu < 0$ and by passing to a subsequence if necessary,*

$$\lim_{i\to\infty} u_{k_i} = \bar{u} \text{ uniformly in } \overline{\Omega} \quad \text{and} \quad \lim_{i\to\infty} k_i v_{k_i} = \bar{w} \text{ in } C^1(\overline{\Omega}_1),$$

where $(\bar{u}, \bar{w})$ is a positive solution of (2.3).

Lemma 4.3. *Assume that $\mu = 0$ and let $\{(k_i, u_{k_i}, v_{k_i})\}_{i=1}^{\infty}$ be any sequence such that (u_{k_i}, v_{k_i}) is a positive solution of* (SP) *with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$. Then $\{\min_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is unbounded.*

References

[1] M.G. Crandall, P.H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321–340.

[2] Y. Du, X. Liang, A diffusive competition model with a protection zone, J. Differential Equations 244 (2008) 61–86.

[3] Y. Du, R. Peng, M.X. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, J. Differential Equations 246 (2009) 3932–3956.

[4] Y. Du, J. Shi, A diffusive predator-prey model with a protection zone, J. Differential Equations 229 (2006) 63–91.

[5] K. Kuto, Bifurcation branch of stationary solutions for a Lotka–Volterra cross-diffusion system in a spatially heterogeneous environment, Nonlinear Anal. Real World Appl. 10 (2009) 943–965.

[6] K. Kuto, Stability and Hopf bifurcation of coexistence steady-states to an SKT model in spatially heterogeneous environment, Discrete Contin. Dyn. Syst. 24 (2009) 489–509.

[7] C.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1988) 1–27.

[8] J. López-Gómez, Spectral Theory and Nonlinear Functional Analysis, Research Notes in Mathematics, vol.426, Chapman and Hall/CRC, Boca Raton, FL, 2001.

[9] Y. Lou, W.M. Ni, Diffusion vs cross-diffusion: An elliptic approach, J. Differential Equations 154 (1999) 157–190.

[10] K. Oeda, Effect of cross-diffusion on the stationary problem of a prey-predator model with a protection zone, to appear in J. Differential Equations.

[11] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.

[12] J. Shi, Persistence and bifurcation of degenerate solutions, J. Funct. Anal. 169 (1999) 494–531.

[13] N. Shigesada, K. Kawasaki, E. Teramoto, Spatial segregation of interacting species, J. Theoret. Biol. 79 (1979) 83–99.

[14] X.Z. Zeng, Non-constant positive steady states of a prey-predator system with cross-diffusions, J. Math. Anal. Appl. 332 (2007) 989–1009.