数理解析研究所講究録
第 1753 巻 2011 年 23-28

Some inequalities concerning geometric constants of Banach spaces

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Let X be a real Banach space with dim $X \geq 2$. The closed unit ball and unit sphere of X are denoted by B_X and S_X, respectively. We shall consider the following constants:

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : \ x \in S_X, y \in B_X\right\}, \tag{1}$$

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{4} : \ x, y \in S_X\right\}, \tag{2}$$

$$C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2+\|y\|^2} : \ x \in S_X, y \in B_X\right\}, \tag{3}$$

$$C'_Z(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{2} : \ x, y \in S_X\right\}, \tag{4}$$

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : \ x, y \in S_X\}, \tag{5}$$

$$\rho_X(1) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \ x, y \in S_X\right\}. \tag{6}$$

Properties and relations concerning these constants have been studied by many authors. In particular the *James constant* $J(X)$ and the *von Neumann-Jordan constant* $C_{NJ}(X)$ have been most widely treated. Recall that a Banach space X is *uniformly non-square* provided $J(X) < 2$ or equivalently $C_{NJ}(X) < 2$. The constant $C'_{NJ}(X)$ may be considered as the unitary version of $C_{NJ}(X)$ ([2],[6]). The constant $C_Z(X)$ was introduced by Zbăganu [11], who conjectured that $C_{NJ}(X) = C_Z(X)$ for all Banach spaces X, but in general these two constants are different.

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The constant $C'_Z(X)$ may be considered as the unitary version of $C_Z(X)$ ([6]). Then we easily have

$$\frac{J(X)^2}{2} \le C'_Z(X) \le C_Z(X) \le C_{NJ}(X), \tag{7}$$

$$\frac{J(X)^2}{2} \le C'_Z(X) \le \frac{(1+\rho_X(1))^2}{2} \le C'_{NJ}(X) \le C_{NJ}(X). \tag{8}$$

There are many Banach spaces X for which all the terms in (7) and (8) coincide. On the other hand, there exists a Banach space X such that $C_Z(X) < (1 + \rho_X(1))^2/2$ and $C_Z(X^*) > C'_{NJ}(X^*)$, where X^* is the dual space of X ([6, Example 3]).

We shall start with the estimate of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and $\rho_X(1)$, which was recently proved in Kato and Takahashi [5]. In the following, let $f(u,v) = u - v + \sqrt{(u-v-1)^2+v^2}$. Then it is easy to see that for all $u_2 \ge u_1 \ge 1$ and $v_2 \ge v_1 \ge u_1 - 1$,

$$f(u_1,v_1) \le f(u_2,v_2),$$

where equality holds only when $u_1 = u_2$ and $v_1 = v_2$ ([5, Lemma 3.6]).

Theorem 1 ([5, Theorem 3.4]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \le f(C'_{NJ}(X), \rho_X(1)) \tag{9}$$

or

$$C_{NJ}(X) \le C'_{NJ}(X) - \rho_X(1) + \sqrt{(C'_{NJ}(X) - \rho_X(1) - 1)^2 + \rho_X(1)^2}.$$

If X is not uniformly non-square, we have equality in (9) as $C_{NJ}(X) = C'_{NJ}(X) = 2$ and $\rho_X(1) = 1$. The inequality (9) also attains equality with both of the Day-James ℓ_∞-ℓ_1 and ℓ_2-ℓ_∞ spaces, which are uniformly non-square ([5, Remark 3.5]). Let us mention that Theorem 1 yields some previous results concerning the estimates of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and by $\rho_X(1)$. To see this we need the following inequalities

$$\frac{(1+\rho_X(1))^2}{2} \le C'_{NJ}(X) \le 1 + \rho_X(1)^2, \tag{10}$$

see [5, Proposition 3.2]. Since $\rho_X(1) \le \sqrt{2C'_{NJ}(X)} - 1$, by (9) we have

Corollary 1 ([2, Theorem 1]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \le f(C'_{NJ}(X), \sqrt{2C'_{NJ}(X)} - 1),$$

which is written as

$$C_{NJ}(X) \le 1 + \left(\sqrt{2C'_{NJ}(X)} - 1\right)^2. \tag{11}$$

Since $C'_{NJ}(X) \le 1 + \rho_X(1)^2$, by (9) we also have

Corollary 2 ([7, Theorem 2]). *Let X be a Banach space. Then*

$$C_{NJ}(X) \le f(1 + \rho_X(1)^2, \rho_X(1)),$$

which is written as

$$C_{NJ}(X) \le 1 + \rho_X(1)\left(\sqrt{(1-\rho_X(1))^2 + 1} - (1 - \rho_X(1))\right). \tag{12}$$

We shall present the estimate of $C_{NJ}(X)$ by $C'_Z(X)$. To do this we need the estimates of $\rho_X(1)$ and $C'_{NJ}(X)$ by $C'_Z(X)$.

Proposition 1. *For any Banach space X,*

$$\rho_X(1) \le \frac{C'_Z(X)}{2}, \tag{13}$$

$$C'_{NJ}(X) \le 1 + \frac{C'_Z(X)^2}{4}. \tag{14}$$

In (13) and (14) equalities are attained if X is not uniformly non-square. If X is the ℓ_2-ℓ_1 space, we also have equalities in (13) and (14) as $\rho_X(1) = 1/\sqrt{2}$, $C'_{NJ}(X) = 3/2$ and $C'_Z(X) = \sqrt{2}$ ([6, Example 3]).

By (9), (13) and (14) we have

Theorem 2. *Let X be a Banach space. Then*

$$C_{NJ}(X) \le f(1 + C'_Z(X)^2/4, C'_Z(X)/2),$$

which is written as

$$C_{NJ}(X) \le \frac{C'_Z(X)^2}{4} + 1 + \frac{C'_Z(X)}{4}\left(\sqrt{C'_Z(X)^2 - 4C'_Z(X) + 8} - 2\right). \tag{15}$$

Now we shall present the estimate of $C_Z(X)$ by $C'_Z(X)$ and $\rho_X(1)$, which is similar to Theorem 1.

Theorem 3. *Let X be a Banach space. Then*

$$C_Z(X) \le f(C'_Z(X), \rho_X(1)) \tag{16}$$

or

$$C_Z(X) \le C'_Z(X) - \rho_X(1) + \sqrt{(C'_Z(X) - \rho_X(1) - 1)^2 + \rho_X(1)^2}.$$

Since $C'_Z(X) \le (1 + \rho_X(1))^2/2$, by (16) we have

Corollary 3. *For any Banach space X*

$$C_Z(X) \le f((1 + \rho_X(1))^2/2, \rho_X(1)) = 1 + \rho_X(1)^2. \tag{17}$$

Remark 1. The estimates given by Theorem 3 and Corollary 3 are sharp. It should be noted that if $C_Z(X) = 1 + \rho_X(1)^2$, we have equality in (16). It is clear that if X is not uniformly non-square, we have equality in (17) as $C_Z(X) = 2$ and $\rho_X(1) = 1$. The inequality (17) also attains equality with both of the Day-James ℓ_∞-ℓ_1 and ℓ_2-ℓ_∞ spaces, which are uniformly non-square. In fact, $C_Z(X) = 5/4$, $\rho_X(1) = 1/2$ if X is the ℓ_∞-ℓ_1 space ([6, Example 4]), and $C_Z(X) = 3/2$, $\rho_X(1) = 1/\sqrt{2}$ if X is the ℓ_2-ℓ_∞ space ([6, Example 3]).

By (13) and (16) we have

Theorem 4. *Let X be a Banach space. Then*

$$C_Z(X) \le f(C'_Z(X), C'_Z(X)/2),$$

which is written as

$$C_Z(X) \le \frac{1}{2}\left\{C'_Z(X) + \sqrt{(C'_Z(X) - 2)^2 + C'_Z(X)^2}\right\}. \tag{18}$$

Finally we shall present the estimates of $C'_Z(X)$ and $C_Z(X)$ by $J(X)$.

Theorem 5. *Let X be a Banach space. Then*

$$C'_Z(X) \le 4(1 - 1/J(X)). \tag{19}$$

The estimate (19) yields that $\rho_X(1) \le C'_Z(X)/2 \le 2(1 - 1/J(X))$ ([7, Theorem 1]).

By Theorems 2 and 5 we shall have the quite simple inequality $C_{NJ}(X) \le J(X)$, which was proved in 2009 by Takahashi and Kato [7] (see also [5, 8, 10]).

Corollary 4. *Let X be a Banach space. Then*

$$C_{NJ}(X) \le \frac{4}{4 - C'_Z(X)} \le J(X). \tag{20}$$

By (14), (15), (18) and (19) we have

Theorem 6. *Let X be a Banach space and let $J = J(X)$. Then*

$$
\begin{aligned}
C_{NJ}(X) &\leq \left(\frac{\sqrt{2}(J-1)+\sqrt{(J-1)^2+1}}{J}\right)^2, && (21)\\
C'_{NJ}(X) &\leq 1+4\left(1-\frac{1}{J}\right)^2, && (22)\\
C_Z(X) &\leq \frac{2(J-1)+\sqrt{5J^2-12J+8}}{J}. && (23)
\end{aligned}
$$

Remark 2. Let $J = J(X) < 2$. Then it is easy to see that

$$\frac{2(J-1)+\sqrt{5J^2-12J+8}}{J} < 1+4(1-1/J)^2 < \left(\frac{\sqrt{2}(J-1)+\sqrt{(J-1)^2+1}}{J}\right)^2.$$

The estimates (21) and (22) was proved by Kato and Takahashi [5], see also Wang [8]. On the other hand, the estimate (23) is new.

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