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Refined Young inequality with Kantorovich constant

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1 Introduction

Throughout this note, A, B are positive operators on a Hilbert space, we use the following notations: $A\nabla_{\mu}B=(1-\mu)A+\mu B$, $A\sharp_{\mu}B=A^{1/2}(A^{-1/2}BA^{-1/2})^{\mu}A^{1/2}$, and $A!_{\mu}B=((1-\mu)A^{-1}+\mu B^{-1})^{-1}$, see F. Kubo and T. Ando [6]. When $\mu=1/2$ we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively. The Kontorovich constant is defined as $K(t,2)=\frac{(t+1)^{2}}{4t}$ for $t>0$, while the Specht ratio [9] is denoted by

$$S(t)=\frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}}\quad \text{for}\quad t>0, t\neq 1;\quad \text{and}\quad S(1)=\lim_{t\to 1}S(t)=1.$$

We start from the famous Young inequality:

$$a\nabla_{\mu}b\geq a^{1-\mu}b^{\mu} \tag{1}$$

for positive numbers a, b and $\mu\in[0,1]$. The inequality (1) is also called a weighted arithmetic-geometric mean inequality and its reverse inequality was given in [10] with the Specht ratio as follows:

$$a\nabla_{\mu}b\leq S(h)a^{1-\mu}b^{\mu} \tag{2}$$

for all $\mu\in[0,1]$, where $0<m\leq a, b\leq M$ and $h=\frac{M}{m}$.

Recently, an improvement of the inequality (1) was given in [2] as follows:

Theorem F For $a, b>0$, if $\mu\in[0,1]$, $r=\min\{\mu, 1-\mu\}$ and $h=\frac{b}{a}$, then

$$a\nabla_{\mu}b\geq S(h^{r})a^{1-\mu}b^{\mu}. \tag{3}$$

Based on this, the refined weighted arithmetic-geometric operator mean inequality is given by

$$A\nabla_{\mu}B\geq S(h^{r})A\sharp_{\mu}B. \tag{4}$$

See [3, 4] for recent developments of the improved Young inequality. See also [5] for another type of improvement for the classical Young inequality.

In this short paper, we improve the inequality (3) via the Kantorovich constant as follows:

$$a\nabla_\mu b \geq K(h,2)^r a^{1-\mu}b^\mu$$

for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$ and $h = \frac{b}{a}$. It admits an operator extension

$$A\nabla_\mu B \geq K(h,2)^r A\sharp_\mu B$$

for positive operators A, B on a Hilbert space. While we provide a new viewpoint and method which is different from that of the refinement given in [2].

2 Refinement of Young Inequalities

First of all, we cite a refinement of the weighted arithmetic-geometric mean inequality for n positive numbers, which was shown by Pečarić et.al., see [7; Theorem 1, P.717] and also [1, 8].

Lemma 1. Let $x_1, \cdots, x_n$ belong to a fixed closed interval $I = [a,b]$ with $a < b$, $p_1, \cdots, p_n \geq 0$ with $\sum\limits_{i=1}^{n} p_i = 1$ and $\lambda = \min\{p_1, \cdots, p_n\}$. If f is a convex function on I, then

$$\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \geq n\lambda \Big[\sum_{i=1}^{n} \frac{1}{n} f(x_i) - f(\frac{1}{n}\sum_{i=1}^{n} x_i) \Big]. \tag{5}$$

We will use lemma 1 as the following form by applying $f(x) = -\log x$:

Corollary 2. If $x_i \in [a,b]$, $0 < a < b$, $p_1, \cdots, p_n \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and $\lambda = \min\{p_1, \cdots, p_n\}$, then

$$\frac{\sum_{i=1}^{n} p_i x_i}{\prod_{i=1}^{n} x_i^{p_i}} \geq \left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i^{\frac{1}{n}}} \right)^{n\lambda}. \tag{6}$$

The case $n = 2$ in (6) is simplified to the following one, which is a loose extension of [2].

Corollary 3. If $a, b > 0$, $\mu \in [0,1]$, then

$$a\nabla_\mu b \geq K(h,2)^r a^{1-\mu}b^\mu, \tag{7}$$

where $r=\min\{\mu,1-\mu\}$ and $h=\frac{b}{a}$.

Replacing a, b by a^{-1}, b^{-1}, respectively, we have the counterpart of (7) itself.

Corollary 4. If $a,b>0$ and $\mu\in[0,1]$, then

$$a^{1-\mu}b^{\mu}\geq K(h,2)^{r}a!_{\mu}b. \tag{8}$$

Furthermore Corollary 3 implies Theorem F because of the following fact.

Lemma 5. If $t>0$ and $0\leq r\leq\frac{1}{2}$, then

$$K(t,2)^{r}\geq S(t^{r}). \tag{9}$$

To prove Lemma 5, we need the following lemma.

Lemma 6. ([2] Lemma 2.3) If $t>0$ and $t\neq 1$, then

$$\frac{t^{\frac{t}{t-1}}}{e}\leq\frac{t^2+1}{t+1}. \tag{10}$$

Proof. We give it a proof for convenience. By taking logarithms in (10), it is enough to prove that $f(t)=\log(t^2+1)-\log(t+1)-\frac{t}{t-1}\log t+1\geq 0$ for $t>0$ and $t\neq 1$.

Since $f'(t)=\frac{2t}{t^2+1}-\frac{1}{t+1}-\frac{1}{t-1}++\frac{\log t}{(t-1)^2}=\frac{4t}{t^4-1}+\frac{\log t}{(t-1)^2}$, it follows that $f'(t)\leq 0$ for $0<t<1$ and $f'(t)\geq 0$ for $t>1$. Thus we have $f(t)\geq\lim_{t\to 1}f(t)=0$ for all $t>0$ with $t\neq 1$. □

Proof of Lemma 5. If $t=1$, then it is easy to get $S(1)=1=K(1,2)$.

If $t>0$ and $t\neq 1$, then, logarithmic-arithmetic mean inequality implies

$$\frac{t^r-1}{\log t^r}\leq\frac{t^r+1}{2}\quad\text{for}\quad 0\leq r\leq\frac{1}{2}.$$

Combining with (10) we have

$$S(t^r)=\frac{t^{r\frac{1}{t^r-1}}}{e}\frac{t^r-1}{\log t^r}=\frac{1}{t^r}\frac{t^{r\frac{t^r}{t^r-1}}}{e}\frac{t^r-1}{\log t^r}\leq\frac{1}{t^r}\frac{t^{2r}+1}{t^r+1}\frac{t^r+1}{2}=\frac{t^{2r}+1}{2t^r}.$$

Since $f(x)=x^{2r}(x\geq 0)$ is concave for $0\leq r\leq\frac{1}{2}$, it follows that

$$\frac{t^{2r}+1}{2}\leq\left(\frac{t+1}{2}\right)^{2r}=\left[\frac{(t+1)^2}{4}\right]^{r}.$$

Hence we have

$$S(t^r)\leq\frac{t^{2r}+1}{2}\frac{1}{t^r}\leq\left[\frac{(t+1)^2}{4t}\right]^{r}=K(t,2)^{r}.$$ □

3 Applications to Operator Young Inequality

Theorem 7. Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either of the following conditions:

(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$

(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$.

Then

$$A\nabla_\mu B \geq K(h,2)^r A\sharp_\mu B \tag{11}$$

for all $\mu \in [0,1]$, where $r = \min\{\mu, 1-\mu\}$, $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$.

Proof. From Corollary 3, we have

$$(1-\mu) + \mu x \geq K(x,2)^r x^\mu$$

for any $x > 0$. And hence

$$(1-\mu)I + \mu X \geq \min_{h \leq x \leq h'} K(x,2)^r X^\mu$$

for the positive operator X such that $0 < hI \leq X \leq h'I$.

Substituting $A^{-1/2}BA^{-1/2}$ for X in the above inequality we have:

In the case of (i), $1 < h = \frac{M}{m} \leq A^{-1/2}BA^{-1/2} \leq \frac{M'}{m'} = h'$, we have

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geq \min_{h \leq x \leq h'} K(x,2)^r (A^{-1/2}BA^{-1/2})^\mu.$$

It is easy to check that $K(x,2)$ is an increasing function for $x > 1$, then

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geq K(h,2)^r (A^{-1/2}BA^{-1/2})^\mu. \tag{12}$$

In the case of (ii), we have $0 < 1/h' \leq A^{-1/2}BA^{-1/2} \leq 1/h < 1$, then

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geq \min_{1/h' \leq x \leq 1/h} K(x,2)^r (A^{-1/2}BA^{-1/2})^\mu.$$

Since $K(x,2)$ is a decreasing function for $0 < x < 1$, we have

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geq K(1/h,2)^r (A^{-1/2}BA^{-1/2})^\mu. \tag{13}$$

Multiplying both sides by $A^{1/2}$ to inequality (12) and (13) and using $K(1/h,2)=K(h,2)$ for $h>0$, we obtain the refined arithmetic-geometric operator mean inequality. □

By replacing A, B by A^{-1}, B^{-1}, respectively, then the noncommutative geometric-harmonic mean inequality can be obtained as follows:

Theorem 8. Assume the conditions as in Theorem 7. Then

$$A\sharp_\mu B \geq K(h,2)^r A!_\mu B. \tag{14}$$

From Lemma 5, it's easy to get the following

Corollary 9. [2] Assume the conditions as in Theorem 7. Then

$$A\nabla_\mu B \geq S(h^r) A\sharp_\mu B. \tag{15}$$

In the remainder, we focus on extending the refined weighted arithmetic-harmonic mean inequality to an operator version for another type of improvement.

Lemma 10. If $x_1,\cdots,x_n>0$ and $p_1,\cdots,p_n\geq 0$ with $\sum\limits_{i=1}^{n} p_i=1$, then

$$\sum_{i=1}^{n} p_i x_i^{-1} - \left(\sum_{i=1}^{n} p_i x_i\right)^{-1} \geq n\lambda\left[\sum_{i=1}^{n}\frac{1}{n}x_i^{-1} - \left(\sum_{i=1}^{n}\frac{1}{n}x_i\right)^{-1}\right], \tag{16}$$

where $\lambda=\min\{p_1,p_2,\cdots p_n\}$.

Proof. Let $f(x)=x^{-1}$ in lemma 1, then the desired inequality is obtained. □

Theorem 11. If $\mu\in[0,1]$, A and B are positive operators, then

$$A\nabla_\mu B \geq A!_\mu B + 2r(A\nabla B - A!B), \tag{17}$$

where $r=\min\{\mu,1-\mu\}$.

Proof. From the case $n=2$ in Lemma 10, we have, for $x>0$ and $\mu\in[0,1]$,

$$(1-\mu)+\mu x^{-1} - ((1-\mu)+\mu x)^{-1} \geq 2r\left[\frac{1+x^{-1}}{2} - \left(\frac{1+x}{2}\right)^{-1}\right].$$

Thus it follows that

$$(1-\mu)I+\mu T^{-1} \geq ((1-\mu)I+\mu T)^{-1} + 2r\left[\frac{I+T^{-1}}{2} - \left(\frac{I+T}{2}\right)^{-1}\right] \tag{18}$$

for a strictly positive operator T and $\mu \in [0,1]$.

We may assume that A, B are invertible. Put $T = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ in (18), then

$$(1-\mu)I + \mu(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1} \geq ((1-\mu)I + \mu A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1} \\ +2r\left[\frac{I + (A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1}}{2} - \left(\frac{I + A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}}{2}\right)^{-1}\right].$$

Multiplying both sides by $A^{\frac{1}{2}}$ we have

$$(1-\mu)A + \mu B \geq ((1-\mu)A^{-1} + \mu B^{-1})^{-1} + 2r\left[\frac{A+B}{2} - \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right],$$

so that

$$A\nabla_{\mu}B \geq A!_{\mu}B + 2r(A\nabla B - A!B). \qquad \square$$

References

[1] J. Barić, M. Matić, and J. Pečarić, On the bounds for the normalized Jensen functional and Jensen-Steffensen inequality. Math. Inequal. Appl., 12(2009), 413-432.

[2] S. Furuichi, Refined Young inequalities with Specht's ratio, ArXiv:1004. 0581v2.

[3] S. Furuichi, On refined Young inequalities and reverse inequalities, ArXiv:1001. 0535v2.

[4] T. Furuta, The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators, Amer. Math. Monthly, 108(2001), 68-69.

[5] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices, J. Math. Anal. Appl., 36(2010), 262-269.

[6] F. Kubo and T. Ando, Means of positive operators, Math. Ann., 264(1980), 205-224.

[7] D.S. Mitrinović, J. Pečarić and A.M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[8] S. Simić, On a new converse of Jensen inequality. Publ. de L'institut Math., 85(2009), 107-110.

[9] W. Specht, Zer Theorie der elementaren Mittel, Math. Z., 74(1960), 91-98.

[10] M. Tominaga, Specht's ratio in the Young inequality, Sci. Math. Japon., 55(2002), 583-588.