数理解析研究所講究録
第 1753 巻 2011 年 58-66

Martingale Morrey-Campanato spaces

大阪教育大学　教育学部　中井 英一 (Eiichi Nakai)
Department of Mathematics
Osaka Kyoiku University
大阪教育大学　教育学部　貞末 岳 (Gaku Sadasue)
Department of Mathematics
Osaka Kyoiku University

1 Introduction

In this paper, we introduce Morrey and Campanato spaces of martingales, and state some basic properties of these spaces. We give only outline of the proofs of these properties. This paper is an announcement of the authors' recent results. The details will be given in the authors' forthcoming paper [6].

We consider a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$, where $\{\mathcal{F}_n\}_{n\geq 0}$ is a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms.

We state definitions and notation in the next section. In Section 3, we give basic properties of martingale Morrey and Campanato spaces and compare these spaces with martingale Lipschitz spaces by Weisz [11].

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, is dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2000 *Mathematics Subject Classification.* Primary 46E30, 60G46; Secondary 42B35, 26A33.
Key words and phrases. martingale, Campanato space, Morrey space.
The first author was supported by Grant-in-Aid for Scientific Research (C), No. 20540167, Japan Society for the Promotion of Science.

2 Definitions and notation

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{\mathcal{F}_n\}_{n\geq 0}$ a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. In this paper we always suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called atom, more precisely $(\mathcal{F}_n, P)$-atom, if any $A \subset B$, $A \in \mathcal{F}_n$, satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$.

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively.

It is known that, if $p \in (1, \infty)$, then any L_p-bounded martingale converges in L_p. Moreover, if $f \in L_p$, $p \in [1, \infty)$, then $(f_n)_{n\geq 0}$ with $f_n = E_n f$ is an L_p-bounded martingale and converges to f in L_p (see for example [7]). For this reason a function $f \in L_1$ and the corresponding martingale $(f_n)_{n\geq 0}$ with $f_n = E_n f$ will be denoted by the same symbol f.

Let $\mathcal{M}$ be the set of all martingale such that $f_0 = 0$. For $p \in [1, \infty]$, let L_p^0 be the set of all $f \in L_p$ such that $E_0 f = 0$. For any $f \in L_p^0$, let $f_n = E_n f$. Then $(f_n)_{n\geq 0}$ is an L_p-bounded martingale in $\mathcal{M}$. In this case we regard as $L_p^0 \subset \mathcal{M}$.

Now we introduce martingale Morrey space $L_{p,\lambda}$ and martingale Campanato spaces $\mathcal{L}_{p,\lambda}$.

Definition 2.1. Let $p \in [1, \infty)$ and $\lambda \in (-\infty, \infty)$. For $f \in L_1$, let

$$\|f\|_{L_{p,\lambda}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p \, dP \right)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p \, dP \right)^{1/p},$$

and define

$$L_{p,\lambda} = \{f \in L_p^0 : \|f\|_{L_{p,\lambda}} < \infty\}, \quad \mathcal{L}_{p,\lambda} = \{f \in L_p^0 : \|f\|_{\mathcal{L}_{p,\lambda}} < \infty\}.$$

Then functionals $\|f\|_{L_{p,\lambda}}$ and $\|f\|_{\mathcal{L}_{p,\lambda}}$ are norms.

The stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \tag{2.1}$$

holds for all nonnegative martingales $(f_n)_{n\geq 0}$.

Remark 2.1. In general, $L_{p,\lambda} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}} \le 2\|f\|_{L_{p,\lambda}}$. Actually, for any $B \in A(\mathcal{F}_n)$,

$$\begin{aligned}
&\left(\frac{1}{P(B)}\int_B |f - E_n f|^p \, dP\right)^{1/p} \\
&\le \left(\frac{1}{P(B)}\int_B |f|^p \, dP\right)^{1/p} + \left(\frac{1}{P(B)}\int_B |E_n f|^p \, dP\right)^{1/p} \\
&\le 2\left(\frac{1}{P(B)}\int_B |f|^p \, dP\right)^{1/p}.
\end{aligned}$$

Moreover, if $\{\mathcal{F}_n\}_{n\ge 0}$ is regular and $\lambda < 0$, then we can prove that $L_{p,\lambda} = \mathcal{L}_{p,\lambda}$ with equivalent norms (Theorem 3.1 and Remark 3.1).

Remark 2.2. By the definition, if $0 \ge \lambda' \ge \lambda$, then we have that $L_\infty^0 \subset L_{p,\lambda'} \subset L_{p,\lambda}$ with $\|f\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda'}} \le \|f\|_{L_\infty}$ and $L_\infty^0 \subset \mathcal{L}_{p,\lambda'} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}} \le \|f\|_{\mathcal{L}_{p,\lambda'}} \le 2\|f\|_{L_\infty}$. If $\lambda \le -1/p$, then $L_p^0 \subset L_{p,\lambda} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}}/2 \le \|f\|_{L_{p,\lambda}} \le \|f\|_{L_p}$.

Remark 2.3. By the definition and Remark 2.1, if $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $L_{p,\lambda} \subset \mathcal{L}_{p,\lambda} \subset L_p^0$ with $\|f\|_{L_p} \le \|f\|_{\mathcal{L}_{p,\lambda}} \le 2\|f\|_{L_{p,\lambda}}$.

Definition 2.2. Let $\mathrm{BMO} = \mathcal{L}_{1,0}$ and $\mathrm{Lip}_\alpha = \mathcal{L}_{1,\alpha}$ if $\alpha > 0$.

Our definitions of BMO and Lip_α are different from ones by Weisz [11]. To compare both we give another definition of martingale Morrey and Campanato spaces.

Definition 2.3. Let $p \in [1,\infty)$ and $\lambda \in (-\infty,\infty)$. For $f \in L_1$, let

$$\|f\|_{L_{p,\lambda,\mathcal{F}}} = \sup_{n\ge 0}\sup_{B\in\mathcal{F}_n} \frac{1}{P(B)^\lambda}\left(\frac{1}{P(B)}\int_B |f|^p \, dP\right)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} = \sup_{n\ge 0}\sup_{B\in\mathcal{F}_n} \frac{1}{P(B)^\lambda}\left(\frac{1}{P(B)}\int_B |f - E_n f|^p \, dP\right)^{1/p},$$

and define

$$L_{p,\lambda,\mathcal{F}} = \{f \in L_p^0 : \|f\|_{L_{p,\lambda,\mathcal{F}}} < \infty\}, \quad \mathcal{L}_{p,\lambda,\mathcal{F}} = \{f \in L_p^0 : \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} < \infty\}.$$

Remark 2.4. By the definitions we have the relations $L_{p,\lambda,\mathcal{F}} \subset L_{p,\lambda}$ with $\|f\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda,\mathcal{F}}}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}} \le \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}}$. If $\lambda \ge 0$, then we can prove that $L_{p,\lambda,\mathcal{F}} = L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} = \mathcal{L}_{p,\lambda}$ with the same norms, respectively (see Proposition 3.4). If $-1/p < \lambda < 0$, then $L_{p,\lambda,\mathcal{F}} \subsetneqq L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subsetneqq \mathcal{L}_{p,\lambda}$ in general (see Proposition 3.5).

Remark 2.5. It is known that, if $\{\mathcal{F}_n\}_{n\geq 0}$ is regular and $\lambda \geq 0$, then $\mathcal{L}_{1,\lambda,\mathcal{F}} = \mathcal{L}_{p,\lambda,\mathcal{F}}$ with $\|f\|_{\mathcal{L}_{1,\lambda,\mathcal{F}}} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq C_p\|f\|_{\mathcal{L}_{1,\lambda,\mathcal{F}}}$ for each $p \in [1,\infty)$ (see for example [12]).

We also define weak Morrey spaces.

Definition 2.4. For $p \in [1,\infty)$ and $\lambda \in (-\infty,\infty)$, Let

$$\|f\|_{WL_{p,\lambda}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \sup_{t>0} \left(\frac{P(B\cap\{f>t\})}{P(B)}\right)^{1/p}$$

for measurable functions f, and define

$$WL_{p,\lambda} = \{f \in L_1^0 : \|f\|_{WL_{p,\lambda}} < \infty\}.$$

3 Basic properties of Morrey and Campanato spaces

In this section we give basic properties of Morrey and Campanato spaces. The following theorem gives the relation between Morrey and Campanato spaces.

Theorem 3.1. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, $\mathcal{F}_0 = \{\emptyset,\Omega\}$ and $(\Omega,\mathcal{F},P)$ be nonatomic. Let $p \in [1,\infty)$.*

(i) *If $\lambda \leq -1/p$, then $L_{p,\lambda} = \mathcal{L}_{p,\lambda} = L_p^0$ and*

$$\frac{1}{2}\|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} = \|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda}}.$$

(ii) *If $-1/p < \lambda < 0$, then $L_\infty^0 \subsetneqq L_{p,\lambda} = \mathcal{L}_{p,\lambda} \subsetneqq L_p^0$ and*

$$\|f\|_{L_p} \leq \|f\|_{L_{p,\lambda}} \leq \|f\|_{L_\infty}, \quad \frac{1}{2}\|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} \leq C\|f\|_{\mathcal{L}_{p,\lambda}}.$$

(iii) *If $\lambda = 0$, then $L_\infty^0 = L_{p,0} \subsetneqq \mathcal{L}_{p,0} = \mathrm{BMO}$ and*

$$\|f\|_{L_{p,0}} = \|f\|_{L_\infty}, \quad \|f\|_{\mathrm{BMO}} \leq \|f\|_{\mathcal{L}_{p,0}} \leq C_p\|f\|_{\mathrm{BMO}}.$$

(iv) *If $\lambda > 0$, then $\{0\} = L_{p,\lambda} \subsetneqq \mathcal{L}_{p,\lambda} = \mathrm{Lip}_\lambda$ and*

$$\|f\|_{\mathrm{Lip}_\lambda} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq C_p\|f\|_{\mathrm{Lip}_\lambda}.$$

Remark 3.1. We can prove (i) without the assumption that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular or that $(\Omega, \mathcal{F}, P)$ is nonatomic. In (ii), we can prove that $L_{p,\lambda} = \mathcal{L}_{p,\lambda}$ and $\frac{1}{2}\|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} \leq C\|f\|_{\mathcal{L}_{p,\lambda}}$ in (ii) without the assumption that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ or that $(\Omega, \mathcal{F}, P)$ is nonatomic.

To prove the theorem we need a lemma and two propositions.

Lemma 3.2. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n),$$

has the following property: For each $n \geq 1$,

$$B_n = B_{n-1} \text{ or } (1+1/R)P(B_n) \leq P(B_{n-1}) \leq RP(B_n),$$

where R is the constant in (2.1).

Remark 3.2. Since $B_n \in A(\mathcal{F}_n)$ is an $(\mathcal{F}_n, P)$-atom, we always interpret $B_n \supset B_{n-1}$ as $B_n \cup A \supset B_{n-1}$ for some $A \in \mathcal{F}_n$ with $P(A) = 0$.

Remark 3.3. By the lemma we see that, there exists m such that $B_m = B_n$ for all $n \geq m$, if and only if $\lim_{n\to\infty} P(B_n) > 0$.

Outline of the proof of Lemma 3.2. For the inequality $P(B_{n-1}) \leq RP(B_n)$, we consider the set $\tilde{B}_{n-1} = \{E_{n-1}[\chi_{B_n}] \geq 1/R\}$. Then, we can show that $P(B_{n-1}) \leq P(\tilde{B}_{n-1}) \leq RP(B_n)$.

We can show the part $B_n = B_{n-1}$ or $(1+1/R)P(B_n) \leq P(B_{n-1})$, by the fact that $E_{n-1}[\chi_{B_{n-1}\setminus B_n}] = \frac{P(B_{n-1}\setminus B_n)}{P(B_{n-1})}\chi_{B_{n-1}}$. □

Proposition 3.3. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, $1 \leq p < \infty$ and $\lambda > -1/p$.*

(i) *For a sequence $B_0 \supset B_1 \supset \cdots \supset B_k \supset \cdots$, $B_k \in A(\mathcal{F}_k)$, let $f_0 = 0$ and*

$$f_n = \sum_{k=1}^{n} P(B_k)^\lambda \left(\frac{P(B_{k-1})}{P(B_k)}\chi_{B_k} - \chi_{B_{k-1}}\right), \quad n \geq 1. \tag{3.1}$$

Then $f = (f_n)_{n\geq 0}$ is a martingale in $\mathcal{M}$ and in $\mathcal{L}_{p,\lambda}$.

(ii) *Let $0 \geq \lambda' > \lambda > -1/p$. If there exists a sequence $B_0 \supset B_1 \supset \cdots \supset B_k \supset \cdots$, $B_k \in A(\mathcal{F}_k)$ and $\lim_{k\to\infty} P(B_k) = 0$, then $L_\infty^0 \subsetneqq \mathcal{L}_{p,\lambda'} \subsetneqq \mathcal{L}_{p,\lambda}$. If $\mathcal{F}_0 = \{\emptyset, \Omega\}$ also, then $L_\infty^0 \subsetneqq \mathcal{L}_{p,\lambda'} \subsetneqq \mathcal{L}_{p,\lambda} \subsetneqq L_p^0$.*

Outline of the proof. (i) The sequence $(f_n)_{n\geq 0}$ belongs to $\mathcal{M}$ is easily verified.

By Lemma 3.2, we can take a sequence of integers $0 = k_0 < k_1 < \cdots < k_j < \cdots$ such that

$$f_n = \sum_{k_j \leq n} P(B_{k_j})^\lambda \left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right)$$

and $(1 + 1/R)P(B_{k_j}) \leq P(B_{k_{j-1}}) \leq RP(B_{k_j})$. Then, we can show that $(f_n)_{n\geq 0}$ converges in L_p and the limit f belongs to $\mathcal{L}_{p,\lambda}$.

(ii) By Remark 2.2, we need only to show that $\mathcal{L}_{p,0} \setminus L_\infty^0 \neq \emptyset$ and $\mathcal{L}_{p,\lambda} \setminus \mathcal{L}_{p,\lambda'} \neq \emptyset$.

If $\lambda = 0$, then we can show that the limit f above is not bounded and have $\mathcal{L}_{p,0} \setminus L_\infty^0 \neq \emptyset$.

If $0 \geq \lambda' > \lambda > -1/p$, then we can show that $f \in \mathcal{L}_{p,\lambda}$ and $f \notin \mathcal{L}_{p,\lambda'}$ and have $\mathcal{L}_{p,\lambda} \setminus \mathcal{L}_{p,\lambda'} \neq \emptyset$.

□

Proposition 3.4. *Let* $1 \leq p < \infty$.

(i) *If* $\mathcal{F}_0 = \{\emptyset, \Omega\}$ *and* $\lambda \leq -1/p$, *then* $L_p^0 = L_{p,\lambda} = L_{p,\lambda,\mathcal{F}} = \mathcal{L}_{p,\lambda} = \mathcal{L}_{p,\lambda,\mathcal{F}}$ *with*

$$\|f\|_{L_{p,\lambda,\mathcal{F}}} = \|f\|_{L_{p,\lambda}} = \|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq 2\|f\|_{L_{p,\lambda,\mathcal{F}}}.$$

(ii) *If* $\lambda \geq 0$, *then* $L_{p,\lambda} = L_{p,\lambda,\mathcal{F}}$ *and* $\mathcal{L}_{p,\lambda} = \mathcal{L}_{p,\lambda,\mathcal{F}}$ *with the same norms, respectively.*

Outline of the proof. (i) If $\lambda \leq -1/p$, then $P(B)^{-\lambda-1/p} \leq P(\Omega)^{-\lambda-1/p}$ for any $B \in A(\mathcal{F}_n)$ or $B \in \mathcal{F}_n$. From this, we have $\|f\|_{L_{p,\lambda,\mathcal{F}}} = \|f\|_{L_{p,\lambda}} = \|f\|_{L_p}$. The rest are deduced from Remark 2.1 and Remark 2.4.

(ii) By Remark 2.4 we need to show only $\|f\|_{L_{p,\lambda,\mathcal{F}}} \leq \|f\|_{L_{p,\lambda}}$. We can show this by the assumption that each $\mathcal{F}_n$ is generated by $A(\mathcal{F}_n)$. □

Outline of the proof of Theorem 3.1. (i) We have the conclusion by Proposition 3.4. (ii) By Proposition 3.3 and Remark 2.1, we only need to prove $\|f\|_{L_{q,\lambda}} \leq C\|f\|_{\mathcal{L}_{q,\lambda}}$. For any $f \in \mathcal{L}_{p,\lambda}$ and any $B \in A(\mathcal{F}_n)$,

$$\begin{aligned}\left(\frac{1}{P(B)}\int_B |f|^p\,dP\right)^{1/p} &\leq \left(\frac{1}{P(B)}\int_B |f - E_n f|^p\,dP\right)^{1/p} + \left|\frac{1}{P(B)}\int_B f(\omega)\,dP\right| \\ &\leq P(B)^\lambda \|f\|_{\mathcal{L}_{p,\lambda}} + \left|\frac{1}{P(B)}\int_B f(\omega)\,dP\right|, \quad \text{a.s. } \omega \in B.\end{aligned}$$

since

$$E_n f = \frac{1}{P(B)} \int_B f(\omega)\, dP \quad \text{on } B.$$

By Lemma 3.2 we can choose $B_{k_j} \in A(\mathcal{F}_{k_j})$, $0 = k_0 < k_1 < \cdots < k_m \le n$, such that $B_{k_0} \supset B_{k_1} \supset B_{k_2} \supset \cdots \supset B_{k_m} = B$ and that $(1+1/R)P(B_{k_j}) \le P(B_{k_{j-1}}) \le RP(B_{k_j})$. Then, we can show

$$\left| \frac{1}{P(B)} \int_B f(\omega)\, dP \right| \sim P(B)^{\lambda}\, \|f\|_{\mathcal{L}_{p,\lambda}}.$$

Therefore we have $\|f\|_{L_{p,\lambda}} \lesssim \|f\|_{\mathcal{L}_{p,\lambda}}$.

(iii) By Remark 2.2 and Proposition 3.3, we only need to show that $\|f\|_{L_\infty} \le \|f\|_{L_{p,0}}$. We can show this by the assumptions that $\mathcal{F}$ is generated by $\cup_n \mathcal{F}_n$ and that each $\mathcal{F}_n$ is generated by $A(\mathcal{F}_n)$.

(iv) We can show $L_{p,\lambda} = \{0\}$ by the assumptions that $\mathcal{F}$ is generated by $\cup_n \mathcal{F}_n$, each $\mathcal{F}_n$ is generated by $A(\mathcal{F}_n)$, and $\mathcal{F}$ is nonatomic.

The rests are obtained from Proposition 3.3, Proposition 3.4 and Remark 2.5. $\square$

We show that $L_{p,\lambda,\mathcal{F}} \subsetneqq L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subsetneqq \mathcal{L}_{p,\lambda}$ in general by an example.

Proposition 3.5. *Let $(\Omega, \mathcal{F}, P)$ be as follows:*

$$\Omega = [0,1), \quad A(\mathcal{F}_n) = \{I_{n,j} = [j2^{-n}, (j+1)2^{-n}) : j = 0, 1, \cdots, 2^n - 1\}$$
$$\mathcal{F}_n = \sigma(A(\mathcal{F}_n)), \quad \mathcal{F} = \sigma(\cup_n \mathcal{F}_n), \quad P = \text{ the Lebesgue measure.}$$

If $-1/p < \lambda < 0$, then $L_{p,\lambda,\mathcal{F}} \subsetneqq L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subsetneqq \mathcal{L}_{p,\lambda}$.

Outline of the proof. We only need to construct f such that

$$f \in L_{p,\lambda} \setminus L_{p,\lambda,\mathcal{F}} \quad \text{and} \quad f \in \mathcal{L}_{p,\lambda} \setminus \mathcal{L}_{p,\lambda,\mathcal{F}}. \tag{3.2}$$

Step 1: Denote the characteristic function of $I_{n,j}$ by $\chi_{n,j}$ and let

$$f_n = \sum_{j=0}^{2^n-1} f_{n+m,2^m j}, \quad f_{n,j} = P(I_{n,j})^{\lambda}(\chi_{n+1,2j} - \chi_{n+1,2j+1}),$$

where we choose m such that $P(I_{n+m,0})^{p\lambda+1} \le P(I_{n,0})$. Then, we can show

$$f_n \in L_{p,\lambda} \cap \mathcal{L}_{p,\lambda} \quad \text{and} \quad \|f_n\|_{L_{p,\lambda}} = \|f_n\|_{\mathcal{L}_{p,\lambda}} = 1$$

and $\|f_n\|_{L_{p,\lambda,\mathcal{F}}}, \|f_n\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \geq 2^{-n\lambda} \to \infty$ as $n \to \infty$.

Step 2: Let f_n be as in Step 1. Then, we can show $\|f_n\chi_{k,\ell}\|_{L_{p,\lambda,\mathcal{F}}}, \|f_n\chi_{k,\ell}\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \geq 2^{(-n+k)\lambda}$.

Step 3: Let f_n be as in Step 1 and let

$$f = \sum_{n=1}^{\infty} 2^{n\lambda/2} f_{2n}\chi_{n,1}.$$

Then we have

$$\|f\|_{L_{p,\lambda}}, \|f\|_{\mathcal{L}_{p,\lambda}} \leq \sum_{n=1}^{\infty} 2^{n\lambda/2} = \frac{2^{\lambda/2}}{1-2^{\lambda/2}}.$$

On the other hand, we have that

$$\|f\|_{L_{p,\lambda,\mathcal{F}}}, \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \geq 2^{n\lambda/2} \times 2^{(-2n+n)\lambda} = 2^{-n\lambda/2}$$

for all n. This shows (3.2). □

At the end of this section we prove the relation of $\|f\|_{L_{1,\lambda}}$ and $\|f\|_{WL_p}$.

Proposition 3.6. *If $1 < p < \infty$ and $-1/p = \lambda$, then*

$$\|f\|_{L_{1,\lambda}} \leq C\|f\|_{WL_p}.$$

Outline of the proof. Let $\|f\|_{WL_p} = 1$. For $B \in A(\mathcal{F}_n)$, let $\eta = P(B)^\lambda = P(B)^{-1/p}$ and

$$f = f^\eta + f_\eta, \quad f^\eta(\omega) = \begin{cases} f(\omega), & |f(\omega)| > \eta, \\ 0, & |f(\omega)| \leq \eta. \end{cases}$$

Then by $P(|f| > t) \leq t^{-p}$ and Hölder's inequality we have

$$\frac{1}{P(B)^{\lambda+1}} \int_B |f^\eta(\omega)|\, dP \leq \frac{p}{p-1}$$

and

$$\frac{1}{P(B)^{\lambda+1}} \int_B |f_\eta(\omega)|\, dP \leq 1.$$

□

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