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Fixed Point Theorems and Convergence Theorems for New Nonlinear Operators in Banach Spaces

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Abstract. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. In this article, we extend this class of generalized hybrid mappings in a Hilbert space to more wide classes of nonlinear mappings in a Hilbert space and a Banach space. Then, we prove fixed point theorems and convergence theorems for these classes of nonlinear mappings in a Hilbert space and a Banach space.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Then, an equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C.$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \quad \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;
(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following theorem appears implicitly in Blum and Oettli [3].

Theorem 1.1. *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r>0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

The following theorem was also given in Combettes and Hirstoaga [8].

Theorem 1.2. *Assume that $f: C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r>0$ and $x \in H$, define a mapping $T_r: H \to C$ as follows:*

$$T_r x=\left\{z \in C: f(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;
(2) T_r is a firmly nonexpansive mapping, i.e., for all $x,y \in H$,

$$\|T_r x-T_r y\|^2 \leq \langle T_r x-T_r y, x-y\rangle;$$

(3) $F(T_r)=EP(f)$;
(4) $EP(f)$ is closed and convex.

The following three nonlinear mappings are deduced from a firmly nonexpansive mapping T_r in a Hilbert space. A mapping $T: C \to H$ is said to be nonexpansive, nonspreading [20], and hybrid [32] if

$$\|Tx-Ty\| \leq \|x-y\|,$$

$$2\|Tx-Ty\|^2 \leq \|Tx-y\|^2+\|Ty-x\|^2$$

and

$$3\|Tx-Ty\|^2 \leq \|x-y\|^2+\|Tx-y\|^2+\|Ty-x\|^2$$

for all $x,y \in C$, respectively. Motivated by these mappings, Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a class of nonlinear mappings called λ-hybrid in a Hilbert space. Kocourek, Takahashi and Yao [17] also introduced a more wide class of nonlinear mappings containing the class of λ-hybrid mappings: A mapping $T: C \to H$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for all $x,y \in C$. They proved the following fixed point theorem and nonlinear ergodic theorem in a Hilbert space; see Kocourek, Takahashi and Yao [17].

Theorem 1.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T: C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Theorem 1.4. *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T: C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x=\frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

In this article, we extend the class of generalized hybrid mappings in a Hilbert space to more wide classes of nonlinear mappings in a Hilbert space and a Banach space. Then, we prove fixed point theorems and convergence theorems for these classes of nonlinear mappings in a Hilbert space and a Banach space.

2 Preliminaries

Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. From [31], we know the following basic equalities. For $x, y, u, v \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

and

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y\in C}\|x-y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [31] for more details.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty closed convex subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [16]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.3) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.3) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.3) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [28, 29]. The following results are also in [28, 29].

Theorem 2.1. *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle,$$

where $j \in Jy$.

Theorem 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.4}$$

for $x, y \in E$, where J is the duality mapping of E. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.5}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Further, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.6}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.7}$$

The following result was proved by Xu [39].

Theorem 2.3. *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [28].

3 New Classes of Nonlinear Operators in Hilbert Spaces

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $S : C \to H$ is called super hybrid [17] if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Sx-Sy\|^2 &+ (1-\alpha+\gamma)\|x-Sy\|^2 \\ &\le (\beta+(\beta-\alpha)\gamma)\|Sx-y\|^2 + (1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2 \\ &\quad + (\alpha-\beta)\gamma\|x-Sx\|^2 + \gamma\|y-Sy\|^2 \end{aligned} \tag{3.1}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta,\ \gamma)$-super hybrid mapping. We notice that an $(\alpha,\ \beta,\ 0)$-super hybrid mapping is $(\alpha,\ \beta)$-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. In fact, let us consider a super hybrid mapping S with $\alpha = 1$, $\beta = 0$ and $\gamma = 1$. Then, we have

$$\|Sx-Sy\|^2 + \|x-Sy\|^2 \le -\|Sx-y\|^2 + 3\|x-y\|^2 + \|x-Sx\|^2 + \|y-Sy\|^2$$

for all $x, y \in C$. This is equivalent to

$$\|Sx-Sy\|^2 + 2\langle x-y, Sx-Sy\rangle \le 3\|x-y\|^2$$

for all $x, y \in C$. In the case of $H = \mathbb{R}$, consider $Sx = 2 - 2x$ for all $x \in \mathbb{R}$. Then,

$$\begin{aligned} |Sx-Sy|^2 &+ 2\langle x-y, Sx-Sy\rangle \\ &= |2-2x-(2-2y)|^2 + 2\langle x-y, 2-2x-(2-2y)\rangle \\ &= 4|x-y|^2 + 4\langle x-y, y-x\rangle \\ &= 0 \le 3|x-y|^2 \end{aligned}$$

for all $x, y \in \mathbb{R}$. Hence S is super hybrid and $F(S) \ne \emptyset$. However, S is not quasi-nonexpansive. Furthermore, we have that

$$Tx = \frac{1}{2}Sx + \frac{1}{2}x = \frac{1}{2}(2-2x) + \frac{1}{2}x = 1 - \frac{1}{2}x$$

and hence T is nonexpansive. In general, we have the following theorem for generalized hybrid mappings and supper hybrid mappings; see Takahashi, Yao and Kocourek [38].

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \ne -1$. Let S and T be mappings of C into H such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, S is $(\alpha,\ \beta,\ \gamma)$-super hybrid if and only if T is $(\alpha,\ \beta)$-generalized hybrid. In this case, $F(S) = F(T)$.*

Using Theorems 3.1 and 1.3, we have the following fixed point theorem [17] for super hybrid mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an (α, β, γ)-super hybrid mapping. Then, S has a fixed point in C.*

Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers. Then, $U : C \to H$ is called an (α, β, γ)-extended hybrid mapping [11] if

$$\begin{aligned}\alpha(1+\gamma)\|Ux-Uy\|^2 &+ (1-\alpha(1+\gamma))\|x-Uy\|^2 \\ &\leq (\beta+\alpha\gamma)\|Ux-y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\ &\quad - (\alpha-\beta)\gamma\|x-Ux\|^2 - \gamma\|y-Uy\|^2\end{aligned}$$

for all $x, y \in C$. We call such a mapping an (α, β, r)-extended hybrid mapping. Putting $\gamma = \frac{-r}{1+r}$ in (3.1) with $1+r>0$, we get that for all $x, y \in C$,

$$\begin{aligned}\alpha\|Sx-Sy\|^2 &+ (1-\alpha+\frac{-r}{1+r})\|x-Sy\|^2 \\ &\leq (\beta+(\beta-\alpha)\frac{-r}{1+r})\|Sx-y\|^2 + (1-\beta-(\beta-\alpha-1)\frac{-r}{1+r})\|x-y\|^2 \\ &\quad + (\alpha-\beta)\frac{-r}{1+r}\|x-Sx\|^2 + \frac{-r}{1+r}\|y-Sy\|^2.\end{aligned}$$

From $1+r>0$, we have

$$\begin{aligned}\alpha(1+r)\|Sx-Sy\|^2 &+ (1+r-\alpha(1+r)-r)\|x-Sy\|^2 \\ &\leq \big(\beta(1+r)-(\beta-\alpha)r\big)\|Sx-y\|^2 + \big(1+r-\beta(1+r) \\ &\quad + (\beta-\alpha-1)r\big)\|x-y\|^2 - (\alpha-\beta)r\|x-Sx\|^2 - r\|y-Sy\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha(1+r)\|Sx-Sy\|^2 &+ (1-\alpha(1+r))\|x-Sy\|^2 \\ &\leq \big(\beta+\alpha r\big)\|Sx-y\|^2 + \big(1-(\beta+\alpha r)\|x-y\|^2 \\ &\quad - (\alpha-\beta)r\|x-Sx\|^2 - r\|y-Sy\|^2.\end{aligned}$$

This implies that S is extended hybrid. The following theorem is in [11].

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$. Then, for $1+\gamma>0$, $T : C \to H$ is an (α, β)-generalized hybrid mapping if and only if $U : C \to H$ is an (α, β, γ)- extended hybrid mapping.*

Using Theorems 3.2 and 3.3, we can prove a fixed point theorem [11] for generalized hybrid nonself-mappings in a Hilbert space.

Theorem 3.4. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α and β be real numbers. Let T be an (α, β)-generalized hybrid mapping with $\alpha - \beta \geq 0$ of C into H. Suppose that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y-x)$ for some $y \in C$ and $t \in \mathbb{R}$ with $1 \leq t \leq m$. Then, T has a fixed point in C.*

4 Convergence Theorems in Hilbert Spaces

In this section, using the technique developed by Takahashi [26], we prove a nonlinear ergodic theorem of Baillon's type [2] for super hybrid mappings in a Hilbert space. Before proving it, we need the following lemma [11].

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping from C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$. Then, $\lim_{n\to\infty} \|S_n x - TS_n x\| = 0$. In particular, if C is bounded, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|S_n x - TS_n x\| = 0.$$

Using Lemma 4.1, we can prove the following nonlinear ergodic theorem [11].

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and γ be real numbers with $\gamma \geq 0$ and let $S : C \to C$ be an (α, β, γ)-super hybrid mapping with $F(S) \neq \emptyset$ and let P be the mertic projection of H onto $F(S)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=1}^{n}(\frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I)^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n\to\infty} PT^n x$ and $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$.

We can also prove the following strong convergence theorems [11] of Halpern's type for super hybrid mappings in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let γ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be a mapping such that*

$$\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq (1 + 2\gamma)\|x - y\|^2$$

for all $x, y \in C$. Let $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers such that

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad and \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Suppose $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C\{\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n\}, \quad n \in \mathbb{N}.$$

If $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element v of $F(S)$, where $v = P_{F(S)}u$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be a (α, β, γ)-super hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(S)$. Suppose $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}(\frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I)^k x_n \end{cases}$$

for all $n \in \mathbb{N}$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to Pu.

5 Fixed Point Theorems in Banach Spaces

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then, a mapping $T : C \to E$ is said to be firmly nonexpansive [6] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$. It is known that the resolvent of an accretive operator in a Banach space is a firmly nonexpansive mapping; see [6] and [7]. Using Theorem 2.1, we have that for any $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 \leq \langle x - y, j\rangle &\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\
&\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\
&\Longleftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2 \\
&\Longleftrightarrow \|Tx - Ty\| \leq \|x - y\|.
\end{aligned}$$

This implies that T is nonexpansive. We also have that for any $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 \leq \langle x - y, j\rangle &\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\
&\Longleftrightarrow 0 \leq 2\langle x - Tx, j\rangle + 2\langle Ty - y, j\rangle \\
&\Longrightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\
&\Longleftrightarrow 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\
&\Longleftrightarrow 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.
\end{aligned}$$

This implies that T is a nonspreading mapping in the sense of norm. Furthermore we have that for any $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 \leq \langle x - y, j\rangle &\Longleftrightarrow 0 \leq 4\langle x - Tx - (y - Ty), j\rangle \\
&\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle + 2\langle x - Tx - (y - Ty), j\rangle \\
&\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\
&\Longleftrightarrow 3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Ty\|^2 + \|y - Tx\|^2.
\end{aligned}$$

This implies that T is a hybrid mapping in the sense of norm. Thus, it is natural that we extend a generalized hybrid mapping in a Hilbert space by Kocourek, Takahashi and Yao [17] to Banach spaces as follows: Let E be a Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to E$ is called generalized hybrid [13] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{5.1}$$

for all $x, y \in C$. We may also call such a mapping an (α, β)-generalized hybrid mapping. We note that an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We first prove a fixed point theorem for generalized hybrid mappings in a Banach space. For proving this, we need the following lemma; see, for instance, [33] and [28].

Lemma 5.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, let $\{x_n\}$ be a bounded sequence in E and let μ be a mean on l^∞. If $g : E \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in E,$$

then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Using Lemma 5.1, we can prove the following theorem [13].

Theorem 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a mapping of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. If*

$$\mu_n \|x_n - Ty\|^2 \le \mu_n \|x_n - y\|^2$$

for all $y \in C$, then T has a fixed point in C.

Using Theorem 5.2 and properties of Banach limit, we prove a fixed point theorem [13] for generalized hybrid mappings in a Banach space.

Theorem 5.3. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a generalized hybrid mapping. Then the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

On the other hand, Kocourek, Takahashi and Yao [18] extended a generalized hybrid mapping in a Hilbert space to Banach spaces as follows: Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to E$ is called generalized nonspreading [18] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\quad \le \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{5.2}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for $x, y \in E$. So, we obtain the following:

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ &\quad \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned}$$

for all $x, y \in C$. This implies that

$$\begin{aligned} &(\alpha+\gamma)\|Tx - Ty\|^2 + \{1 - (\alpha+\gamma)\}\|x - Ty\|^2 \\ &\quad \le (\beta+\delta)\|Tx - y\|^2 + \{1 - (\beta+\delta)\}\|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. That is, T is a generalized hybrid mapping in a Hilbert space. The following is Kocourek, Takahashi and Yao's fixed point theorem [18].

Theorem 5.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Let T be a generalized nonspreading mapping of C into itself. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself. Define a mapping T^* as follows:

$$T^* x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the duality mapping of T; see [37] and [12]. It is easy to show that T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^* x^* = JTJ^{-1}x^* \in JC.$$

Then, T^* is a mapping of JC into itself. Furthermore, we define the duality mapping T^{**} of T^* as follows:

$$T^{**}x = J^{-1}T^*Jx, \quad \forall x \in C.$$

It is easy to show that T^{**} is a mapping of C into itself. In fact, for $x \in C$, we have

$$T^{**}x = J^{-1}T^*Jx = J^{-1}JTJ^{-1}Jx = Tx \in C.$$

So, T^{**} is a mapping of C into itself. We know the following result in a Banach space; see [9] and [37].

Lemma 5.5. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into itself and let T^* be the duality mapping of JC into itself. Then, the following hold:*

(1) $JF(T) = F(T^*)$;
(2) $\|T^n x\| = \|(T^*)^n Jx\|$ for each $x \in C$ and $n \in \mathbb{N}$.

Let E be a smooth Banach space, let J be the duality mapping from E into E^* and let C be a nonempty subset of E. A mapping $T : C \to E$ is called skew-generalized nonspreading if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ &\quad \leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\} \end{aligned} \tag{5.3}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspreading mapping. Let T be an $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(Ty, u) \leq \phi(y, u)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (5.3), we obtain

$$\phi(Ty, u) + \gamma\{\phi(u, Ty) - \phi(u, Ty)\} \leq \phi(y, u) + \delta\{\phi(u, y) - \phi(u, y)\}.$$

So, we have that

$$\phi(Ty, u) \leq \phi(y, u) \tag{5.4}$$

for all $u \in F(T)$ and $y \in C$. Now, we can prove a fixed point theorem [13] for skew-generalized nonspreading mappings in a Banach space.

Theorem 5.6. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let T be a skew-generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

6 Convergence Theorems in Banach Spaces

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to E$ be a generalized nonspreading mapping. Then, we have that for any $u \in F(T)$ and $x \in C$, $\phi(u, Tx) \leq \phi(u, x)$. This property can be revealed by putting $x = u \in F(T)$ in (5.2). Similarly, putting $y = u \in F(T)$ in (5.2), we obtain that for $x \in C$,

$$\begin{aligned}\alpha\phi(Tx, u) + (1-\alpha)\phi(x, u) + \gamma\{\phi(u, Tx) - \phi(u, x)\} \\ \leq \beta\phi(Tx, u) + (1-\beta)\phi(x, u) + \delta\{\phi(u, Tx) - \phi(u, x)\}\end{aligned}$$

and hence

$$(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0. \tag{6.1}$$

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u).$$

Now, we can prove the following nonlinear ergodic theorem [18] for generalized nonspreading mappings in a Banach space.

Theorem 6.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Using Theorem 6.1, we obtain the following theorem.

Theorem 6.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Using Theorem 6.1, we can also prove Kocourek, Takahashi and Yao's nonlinear ergodic theorem (Theorem 1.4) in Introduction.

Remark We do not know whether a nonlinear ergodic theorem of Baillon's type for non-spreading mappings holds or not.

Next, we prove a weak convergence theorem of Mann's iteration [21] for generalized non-spreading mappings in a Banach space. For proving it, we need the following lemma obtained by Takahashi and Yao [36].

Lemma 6.3. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = R_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where R_C is a sunny generalized nonexpansive retraction of E onto C. Then $\{R_{F(T)}x_n\}$ converges strongly to an element z of $F(T)$, where $R_{F(T)}$ is a sunny generalized nonexpansive retraction of C onto $F(T)$.

Using Lemma 6.3 and the technique developed by [14], we can prove the following weak convergence theorem.

Theorem 6.4. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Using Theorem 6.4, we can prove the following theorems.

Theorem 6.5. *Let E be a uniformly convex and uniformly smooth Banach space. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Theorem 6.6 (Kocourek, Takahashi and Yao [17]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$.

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