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n-means on a metric space

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1. Introduction

In [1], Ando, Li and Mathias constructed a geometric mean of k-positive operators on a Hilbert space and showed that it has many required properties on the geometric mean. In [6], Lawson and Lim showed that the basic approach due to Horwitz [4] and Ando-Li-Mathias [1] can be generalized to means on metric spaces, and developed the theory of the extensions in this context. Among others, they showed that every nonexpansive and coordinatewise contractive mean has extensions of higher orders. The principle is to "extend" a k-mean on X to a $k+1$-mean. So, to obtain k-means on X is quite difficult. In [3], Yamazaki *et al.* proposed a new construction of the geometric mean of n-positive operators. The idea due to Yamazaki is to "extend" the geometric mean of two positive operators to the geometric mean of k-positive operators.

In this paper, based on the method due to Lawson-Lim, and the construction due to Yamazaki [3], we consider a method of extending means to higher orders. We show that a symmetric convex 2-mean on a complete metric space admits extensions to all higher orders.

2. Extending means

Let (X, d) be a metric space. A k-ary operation $\nu : X^k \mapsto X$ is said to be a k-mean on X if ν satisfies a generalized idempotency law: $\nu(x, x, \cdots, x) = x$ for all $x \in X$. An operation is said to be a mean if it is a k-mean for some $k \geq 2$. A mean is symmetric if it is invariant under permutations:

$$\nu(x_{\pi(1)}, \cdots, x_{\pi(k)}) = \nu(x_1, \cdots, x_k) \quad \text{for any permuation } \pi \text{ on } \{1, 2, \cdots, k\}.$$

The following definition is based on an idea due to Yamazaki [3]:

Definition 2.1. *For a 2-mean $\mu : X^2 \mapsto X$, a shift operator $\beta = \beta_\mu : X^k \mapsto X^k$ $(k \geq 3)$ is defined by*

$$\beta(\mathbf{x}) := (\mu(x_1, x_2), \mu(x_2, x_3), \cdots, \mu(x_k, x_1))$$

for every $\mathbf{x} = (x_1, \cdots, x_k) \in X^k$. The shift map β is said to be power convergent if for each $\mathbf{x} \in X^k$, there exist some $x^* \in X$ such that $\lim_n \beta^n(\mathbf{x}) = (x^*, \cdots, x^*)$.

Definition 2.2. *A k-mean ν is a β-invariant extension of a 2-mean μ if $\nu \circ \beta_\mu = \nu$, that is,*

$$\nu(\mathbf{x}) = \nu(\mu(x_1, x_2), \cdots, \mu(x_k, x_1)) \quad \textit{for all } \mathbf{x} = (x_1, \cdots, x_k) \in X^k$$

Proposition 2.3. *Assume that μ is a 2-mean and the corresponding shift operator $\beta = \beta_\mu$ is power convergent. Define $\mu^{(k)} : X^k \mapsto X$ by $\mu^{(k)}(\mathbf{x}) = x^*$ where $\lim_n \beta^n(\mathbf{x}) = (x^*.\cdots, x^*)$. Then*
(i) *$\mu^{(k)}$ is a k-mean on X that is a β-invariant extension of μ.*
(ii) *Any continuous k-mean on X that is a β-invariant extension of μ must equal $\mu^{(k)}$.*

Proof. (i) For $x \in X$, put $\mathbf{x} = (x, \cdots, x) \in X^k$. Since

$$\beta(\mathbf{x}) = (\mu(x,x), \cdots, \mu(x,x)) = (x, \cdots, x) = \mathbf{x},$$

it follows that $\lim_n \beta^n(\mathbf{x}) = \mathbf{x}$ and hence $\mu^{(k)}(\mathbf{x}) = x$, i.e., $\mu^{(k)}$ is a k-mean. Furthermore, since

$$\mu^{(k)}(\beta(\mathbf{x})) = \pi_1(\lim_n \beta^n(\beta(\mathbf{x}))) = \pi_1(\lim_n \beta^{n+1}(\mathbf{x})) = \mu^{(k)}(\mathbf{x}),$$

where π_1 is the projection into the first coordinate, we have $\mu^{(k)} \circ \beta = \mu^{(k)}$, i.e., $\mu^{(k)}$ is a β-invarinat extension of μ.
(ii) Suppose that ν is a continuous k-mean on X that is a β-invariant extension of μ. Since $\nu = \nu \circ \beta = \nu \circ \beta^n$, for each $\mathbf{x} \in X^k$

$$\nu(\mathbf{x}) = \nu(\beta(\mathbf{x})) = \nu(\beta^n(\mathbf{x})) = \nu(x^*, \cdots, x^*) = x^* = \mu^{(k)}(\mathbf{x})$$

where $\lim_n \beta^n(\mathbf{x}) = (x^*, \cdots, x^*)$. Hence $\nu = \mu^{(k)}$. □

Definition 2.4. *A k-mean ν is a β-extension of a 2-mean μ if for each $\mathbf{x} \in X^k$, $\lim_n \beta^n(\mathbf{x}) = (\nu(\mathbf{x}), \cdots, \nu(\mathbf{x}))$. In this case we say that β power converges to ν, denoted by $\beta_\mu^n \mapsto \nu$ as $n \to \infty$.*

Corollary 2.5. *If μ is a 2-mean and the corresponding shift operator $\beta = \beta_\mu$ is power convergent, then β power converges to a k-mean $\mu^{(k)}$, which by definition is a β-extension of μ. Furthermore, if $\mu^{(k)}$ is continuous, then it is the unique β-invariant extension of μ.*

3. k-MEANS

For a k-mean ν on a metric space (X, d), a subset $C \subset X$ is ν-convex if $\nu(x_1, \cdots, x_k) \in C$ whenever $x_1, \cdots, x_k \in C$.

Lemma 3.1. *If a k-mean ν is a β-extension of a 2-mean μ, then any closed set that is convex with respect to the mean μ is convex with respect to the extension ν.*

Proof. Let A be a closed μ-convex set and $x_1, \cdots, x_k \in A$. Put $\mathbf{x} = (x_1, \cdots, x_k)$. Then by convexity each coordinate of $\beta(\mathbf{x})$ is in A and by induction each coordinate of $\beta^n(\mathbf{x})$ is in A. Since A is closed, it follows that the coordinate limits, which are all $\nu(\mathbf{x})$, belong to A. Hence A is ν-convex. □

If C is ν-convex and ν is continuous, then it follows that the closure of C is ν-convex. We recall that the ν-convex hull of a subset C is the smallest ν-convex subset containing $C \subset X$, and can be obtained by intersecting all ν-convex sets containing C. In a similar way, if ν is continuous, then the closed ν-convex hull of C is can be obtained by intersecting all closed ν-convex sets containing C and coincides with the closure of the ν-convex hull of C.

Definition 3.2. *Let (X, d) be a metric space endowed with a continuous k-mean ν. X is locally convex if there exists at each point a basis of not necessary open neighborhoods that are ν-convex. X is uniformly locally convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that the diameter of the ν-convex hull of A is less than ε whenever the diameter of A is less than δ. X is closed ball convex if all closed balls $B_\varepsilon(x) = \{y \in X : d(x, y) \le \varepsilon\}$ are ν-convex for all $x \in X$.*

By definition, given a continuous mean on a metric space, closed ball convexity implies uniformly local convexity, which in turn implies local convexity, also see [6].

Lemma 3.3. *If ν is a β-extension of the continuous 2-mean μ and if (X, d) is locally convex, then ν is continuous.*

Proof. Let $\mathbf{x} = (x_1, \cdots, x_k) \in X^k$ and $x^* = \nu(\mathbf{x})$ and let U be an open set containing x^*. Take a closed ν-convex neighborhood V of x^* such that $V \subset U$. Since by hypothesis the sequences $\beta^n(\mathbf{x})$ power converges to the k-string with entries x^*, we have $\beta^n(\mathbf{x}) \in V^k$ for some n large enough.By continuity of μ and hence of β^n, there exists W open in X^k containing $\mathbf{x}$ such that $\beta^n(W) \subset V^k$. For any $\mathbf{y} \in W$, we have $\beta^n(\mathbf{y}) \in V^k$, and hence $\beta^m(\mathbf{y}) \in V^k$ for all $m > n$ since V is ν-convex. Since V is closed it follows that $\nu(\mathbf{y}) \in V$. Thus ν is continuous. □

Definition 3.4. *Let $\mu : X^2 \mapsto X$ be a 2-mean on a metric space (X, d). For $\mathbf{x} = (x_1, \cdots, x_k) \in X^k$, put $|\mathbf{x}| = \{x_1, \cdots, x_k\}$, the underlying set of the k-tuple, and define the diameter $\Delta(\mathbf{x})$ of $\mathbf{x}$ by*

$$\Delta(\mathbf{x}) = \mathrm{diam}|\mathbf{x}| = \max\{d(x_i, x_j) : 1 \le i, j \le k\}.$$

A mean μ is weakly β-contractive if $\lim_n \Delta(\beta^n(\mathbf{x})) = 0$ for each $\mathbf{x} \in X^k$. A 2-mean μ is convex if

$$d(\mu(x_1, x_2), \mu(y_1, y_2)) \le \frac{1}{2} d(x_1, y_1) + \frac{1}{2} d(x_2, y_2)$$

for every $x_1, x_2, y_1, y_2 \in X$. Generally, a k-mean ν is convex if

$$d(\nu(x_1, \cdots, x_k), \nu(y_1, \cdots, y_k)) \le \frac{1}{k} \sum_{i=1}^{k} d(x_i, y_i)$$

for every $(x_1, \cdots, x_k), (y_1, \cdots, y_k) \in X^k$. It follows that if a mean μ is convex, then μ is continuous.

Example 3.5.

(1) Let X be a Banach space with the metric $d(x, y) = \| x - y \|$. If μ is defined by $\mu(x, y) = \frac{1}{2}(x + y)$ for all $x, y \in X$, then μ is a symmetric 2-mean and convex. But, if ρ is defined by $\rho(x, y) = 2x - y$ for all $x, y \in X$, then ρ is a non-symmetric 2-mean and not convex.

(2) Let $\mathcal{A}^+$ (resp. $\mathcal{A}^h$) be the set of positive invertible elements (resp. selfadjoint elements) in a unital C*-algebra $\mathcal{A}$. If the manifold $\mathcal{A}^+$ has a metric L: $L_2(X; P) = \| P^{-1} X P^{-1} \|$ on the tangent space $\mathcal{A}^h$, then the geodesic and the distance from x to y for $x, y \in \mathcal{A}^+$ are given by

$$x \,!\, ty = ((1-t)x^{-1} + ty^{-1})^{-1} \qquad \text{and} \qquad d(x, y) = \| x^{-1} - y^{-1} \|$$

for $t \in [0, 1]$, also see [2]. If we define the symmetric 2-mean $\mu(x, y) = 2(x^{-1} + y^{-1})^{-1}$, then μ is convex.

(3) Let λ, μ be symmetric convex 2-mean on a metric space X. Define two symmetric 2-mean

$$\lambda'(x,y) = \lambda(\lambda(x,y), \mu(x,y)) \quad \text{and} \quad \mu'(x,y) = \mu(\lambda(x,y), \mu(x,y)).$$

Then λ' and μ' are convex.

Lemma 3.6. *If $\mu : X^2 \mapsto X$ is a convex symmetric 2-mean, then it is weakly β-contractive.*

Proof. For $\mathbf{x} = (x_1, \cdots, x_k) \in X^k$ and $n \in \mathbb{N}$, put $x_i^{(0)} = x_i$ and $x_i^{(n)} = \mu(x_i^{(n-1)}, x_{i+1}^{(n-1)})$ for $i = 1, \cdots, k-1$ and $x_k^{(n)} = \mu(x_k^{(n-1)}, x_1^{(n-1)})$. Moreover, put $\Delta_l^{(n)} = \max\{d(x_i^{(n)}, x_j^{(n)}) : |i-j| = l\}$ for $l = 1, \cdots, k-1$. For $n \in \mathbb{N}$ and $|i-j| = l$, it follows form the convexity of μ that

$$\begin{aligned} d(x_i^{(n)}, x_j^{(n)}) &= d(\mu(x_i^{(n-1)}, x_{i+1}^{(n-1)}), \mu(x_j^{(n-1)}, x_{j+1}^{(n-1)})) \\ &\le \frac{1}{2} d(x_i^{(n-1)}, x_j^{(n-1)}) + \frac{1}{2} d(x_{i+1}^{(n-1)}, x_{j+1}^{(n-1)}) \\ &\le \Delta_l^{(n-1)} \end{aligned}$$

and hence we have $0 \le \Delta_l^{(n)} \le \Delta_l^{(n-1)}$. Since $\{\Delta_l^{(n)}\}$ is bounded below and monotone decreasing, put $\Delta_l = \lim_n \Delta_l^{(n)}$ for $l = 1, \cdots, k-1$.

Since μ is symmetric, it follows that

$$\begin{aligned} d(x_i^{(n)}, x_{i+1}^{(n)}) &= d(\mu(x_i^{(n-1)}, x_{i+1}^{(n-1)}), \mu(x_{i+1}^{(n-1)}, x_{i+2}^{(n-1)})) \\ &= d(\mu(x_i^{(n-1)}, x_{i+1}^{(n-1)}), \mu(x_{i+2}^{(n-1)}, x_{i+1}^{(n-1)})) \\ &\le \frac{1}{2} d(x_i^{(n-1)}, x_{i+2}^{(n-1)}) \le \frac{1}{2} \Delta_2^{(n-1)} \end{aligned}$$

and hence $\Delta_1^{(n)} \le \frac{1}{2}\Delta_2^{(n-1)}$. For $|i-j| = l$ and $l = 2, \cdots, k-2$, we have

$$\begin{aligned} d(x_i^{(n)}, x_j^{(n)}) &\le \frac{1}{2} d(x_{i+1}^{(n-1)}, x_j^{(n-1)}) + \frac{1}{2} d(x_i^{(n-1)}, x_{j+1}^{(n-1)}) \\ &\le \frac{1}{2}\Delta_{l-1}^{(n-1)} + \frac{1}{2}\Delta_{l+1}^{(n-1)} \end{aligned}$$

and hence $\Delta_l^{(n)} \le \frac{1}{2}\Delta_{l-1}^{(n-1)} + \frac{1}{2}\Delta_{l+1}^{(n-1)}$. In the case of $l = k-1$, we have

$$\begin{aligned} d(x_1^{(n)}, x_k^{(n)}) &= d(\mu(x_1^{(n-1)}, x_2^{(n-1)}), \mu(x_k^{(n-1)}, x_1^{(n-1)})) \\ &\le \frac{1}{2} d(x_2^{(n-1)}, x_k^{(n-1)}) \end{aligned}$$

and hence

$$\Delta_{k-1}^{(n)} \le \frac{1}{2}\Delta_{k-2}^{(n-1)}.$$

As $n \to \infty$, it follows that $\Delta_1 \le \frac{1}{2}\Delta_2$ and $\Delta_l \le \frac{1}{2}\Delta_{l-1} + \frac{1}{2}\Delta_{l+1}$ for $l = 2, \cdots, k-2$, and $\Delta_{k-1} \le \frac{1}{2}\Delta_{k-2}$. This in turn implies $\Delta_{k-1} \le \frac{1}{2}\Delta_{k-2}, \Delta_{k-2} \le \frac{2}{3}\Delta_{k-3}, \cdots, \Delta_3 \le \frac{k-3}{k-2}\Delta_2, \Delta_2 \le \frac{k-2}{k-1}\Delta_1$ and we have

$$0 \le \Delta_1 \le \frac{k-2}{2k-2}\Delta_1 \le \Delta_1.$$

Hence we have $\Delta_l = 0$ for all $l = 1, \cdots, k-1$. Since $\Delta(\beta^n(\mathbf{x})) = \max\{\Delta_1^{(n)}, \cdots, \Delta_{k-1}^{(n)}\}$, in conclusion, we have $\lim_n \Delta(\beta^n(\mathbf{x})) = \max\{\Delta_1, \cdots, \Delta_{k-1}\} = 0$. □

Note that if β_μ is power convergent, then μ must be weakly β-contractive. The following lemma provides a converse:

Lemma 3.7. *Let X be a complete metric space endowed with a weakly β-contractive continuous 2-mean μ. If X is uniformly locally convex, then β is power convergent, so that there exists a β-extension of μ.*

Proof. For $\mathbf{x} \in X$, set $C_n(\mathbf{x})$ equal to the closed μ-convex hull of $|\beta^n(\mathbf{x})|$. By hypothesis $\Delta(\beta^n(\mathbf{x})) = \mathrm{diam}|\beta^n(\mathbf{x})| \to 0$ and then by uniform local convexity $\mathrm{diam}C_n(\mathbf{x}) \to 0$. Note that since $C_n(\mathbf{x})$ is μ-convex, it contains $|\beta^m(\mathbf{x})|$ for all $m > n$, and hence contains $C_m(\mathbf{x})$. Then the collection $\{C_n(\mathbf{x})\}$ is a decreasing sequence of closed μ-convex sets whose diameters converge to 0. Since X is a complete metric space the intersection consists of a single point $\{x^*\}$, and it is now easy to show that $\beta^n(\mathbf{x})$ converges to the k-tuple with all entries x^*. □

Definition 3.8. *Let (X, d) be a metric space. The sup metric d_k on X^k is defined by*

$$d_k(\mathbf{x}, \mathbf{y}) = \max\{d(x_i, y_i) : 1 \leq i \leq k\}$$

for all $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k) \in X^k$. A map γ of a metric space (X^k, d_k) into a metric space (X^m, d_m) is said to be nonexpansive if

$$d_m(\gamma(\mathbf{x}), \gamma(\mathbf{y})) \leq d_k(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in X^k$.

Lemma 3.9. *Let (X, d) be a metric space endowed with a continuous 2-mean μ. Then the following conditions are equivalent:*
(i) *The 2-mean $\mu : X^2 \mapsto X$ is nonexpansive.*
(ii) *The shift operator $\beta = \beta_\mu : X^k \mapsto X^k$ is nonexpansive.*
These conditions imply
(iii) *X is closed ball convex.*

Proof. (i) $\Longrightarrow$ (ii): Since μ is nonexpansive, it follows that

$$\begin{aligned}
d_k(\beta(\mathbf{x}), \beta(\mathbf{y})) &= d_k((\mu(x_1, x_2), \cdots, \mu(x_k, x_1)), (\mu(y_1, y_2), \cdots, \mu(y_k, y_1)) \\
&= \max\{d(\mu(x_1, x_2), \mu(y_1, y_2)), \cdots, d(\mu(x_k, x_1), \mu(y_k, y_1))\} \\
&\leq \max\{d_2((x_1, x_2), (y_1, y_2)), \cdots, d_2((x_k, x_1), (y_k, y_1))\} \\
&= \max\{\max\{d(x_1, y_1), d(x_2, y_2)\}, \cdots, \max\{d(x_k, y_k), d(x_1, y_1)\}\} \\
&= \max\{d(x_1, y_1), \cdots, d(x_k, y_k)\} \\
&= d_k(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

(ii) $\Longrightarrow$ (i): For $\mathbf{x} = (x_1, x_2) \in X^2$ and some $z \in X$, we put $\tilde{\mathbf{x}} = (x_1, x_2, z) \in X^3$. Then we have $\mu(\mathbf{x}) = \pi_1(\beta(\tilde{\mathbf{x}}))$ where π_1 is the projection into the first coordinate. Hence

$$\begin{aligned}
d(\mu(\mathbf{x}), \mu(\mathbf{y})) &= d(\pi_1(\beta(\tilde{\mathbf{x}})), \pi_1(\beta(\tilde{\mathbf{y}}))) \\
&\leq d_3(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = d_2(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

(i) $\Longrightarrow$ (iii): For $\varepsilon > 0$ and $x \in X$, $y_1, y_2 \in B_\varepsilon(x)$ imply

$$d(x, \mu(y_1, y_2)) = d(\mu(x, x), \mu(y_1, y_2)) \leq d_2((x, x), (y_1, y_2)) = \max\{d(x, y_1), d(x, y_2)\} \leq \varepsilon$$

and we have $\mu(y_1, y_2) \in B_\varepsilon(x)$. Hence $B_\varepsilon(x)$ is μ-convex for all $x \in X$ and X is closed ball convex. $\square$

Lemma 3.10. *If μ is a nonexpansive 2-mean on a metric space X and if μ has a β-extension $\mu^{(k)}$, then $\mu^{(k)}$ is a nonexpansive.*

Proof. Let $\pi_1 : X^k \mapsto X$ denote the projection into the first coordinate. For $\mathbf{x} \in X^k$,

$$\mu^{(k)}(\mathbf{x}) = \pi_1(\lim_n \beta^n(\mathbf{x})) = \lim_n (\pi_1 \circ \beta^n)(\mathbf{x}).$$

Here, $\pi_1 \circ \beta^n$ is nonexpansive, so is $\mu^{(k)}$. $\square$

Theorem 3.11. *Let X be a complete metric space equipped with a symmetric convex 2-mean $\mu : X^2 \mapsto X$. Then the shift operator β is power convergent, and hence there exists a unique convex k-mean $\mu^{(k)} : X^k \mapsto X$ that β-extends μ.*

Proof. By Lemma 3.6, it follows that μ is weakly β-contractive. Since μ is convex, μ is nonexpansive and it follows from Lemma 3.9 that X is closed ball convex. Therefore, X is uniformly locally convex. By Lemma 3.7, β is power convergent and we have β-extension $\mu^{(k)}$ of μ. Since $\mu^{(k)}$ is nonexpansive, $\mu^{(k)}$ is continuous. Therefore, $\mu^{(k)}$ is a unique continuous k-mean that is a β-extension of μ. Finally we show $\mu^{(k)}$ is convex:

$$d(\mu^{(k)}(\mathbf{x}), \mu^{(k)}(\mathbf{y})) \le \frac{1}{k}\sum_{i=1}^{k} d(x_i, y_i)$$

for all $\mathbf{x} = (x_1, \cdots, x_k)$ and $\mathbf{y} = (y_1, \cdots, y_k)$.

Put $\beta^n(\mathbf{x}) = (x_1^{(n)}, \cdots, x_k^{(n)})$ and then $x_l^{(n)} = \mu(x_l^{(n-1)}, x_{l+1}^{(n-1)})$ for $l = 1, \cdots, k-1$ and $x_k^{(n)} = \mu(x_k^{(n-1)}, x_1^{(n-1)})$. Moreover, put for each $m = 1, \cdots, k$

$$\begin{aligned}
&d(x_m^{(n+k)}, y_m^{(n+k)}) \\
&\le a_{10} d(x_1^{(n)}, y_1^{(n)}) + a_{20} d(x_2^{(n)}, y_2^{(n)}) + \cdots + a_{k0} d(x_k^{(n)}, y_k^{(n)}) \\
&\le \cdots \\
&\le a_{1n} d(x_1, y_1) + a_{2n} d(x_2, y_2) + \cdots + a_{kn} d(x_k, y_k)
\end{aligned}$$

for positive real numbers a_{ij} such that

$$a_{1i} + a_{2i} + \cdots + a_{ki} = 1 \quad \text{for all } i = 0, \cdots, n$$

and

$$a_{li} = \frac{1}{2} a_{li-1} + \frac{1}{2} a_{l-1i-1} \quad \text{for all } l = 1, \cdots, k.$$

Put an irreducible probability matrix $A = \frac{1}{2}(I_k + S_k)$ where I_k is the identity matrix and S_k is the shift unitary matrix, then we have

$$\begin{pmatrix} a_{1n} \\ \vdots \\ a_{kn} \end{pmatrix} = A^n \begin{pmatrix} a_{10} \\ \vdots \\ a_{k0} \end{pmatrix} \qquad \text{and} \qquad A^n \to \frac{1}{k}\begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \qquad \text{as } n \to \infty,$$

since A has the stationary probability vector $\frac{1}{k}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$. Hence

$$\begin{aligned} d(\mu^{(k)}(\mathbf{x}), \mu^{(k)}(\mathbf{y})) &= \lim_n d(\beta^{(n)}(\mathbf{x}), \beta^{(n)}(\mathbf{y})) \\ &\le \lim_n \max\{d(x_1^{(n)}, y_1^{(n)}), \cdots, d(x_k^{(n)}, y_k^{(n)})\} \\ &\le \frac{1}{k}\sum_{i=1}^{k} d(x_i, y_i). \end{aligned}$$

□

Theorem 3.12. *Let X be a complete metric space equipped with a symmetric convex 2-mean $\mu : X^2 \mapsto X$. Then for each $k \ge 3$, $\mu^{(k)} : X^k \mapsto X$ is uniquely determined in a family of convex means which is a β-invariant extension of μ.*

4. Concluding remarks

Let X be a complete metric space equipped with a symmetric convex 2-mean $\mu : X^2 \mapsto X$ and $\mu^{(k)}$ the convex k-mean obtained by μ. We further assume that X is equipped a closed partial order $\le$, that is, $x_n \le y_n$ for all n implies $\lim_n x_n \le \lim_n y_n$. Let $\le_k$ be the product order on X^k defined by

$$(x_1, \cdots, x_k) \le_k (y_1, \cdots, y_k) \quad \text{if and only if} \quad x_j \le y_j \quad 1 \le j \le k.$$

Definition 4.1. *A k-mean ν on X is said to be monotone for the partial order $\le$ if $\nu(\mathbf{x}) \le \nu(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X^k$ with $\mathbf{x} \le_k \mathbf{y}$.*

Theorem 4.2. *If a symmetric convex 2-mean μ is monotone for the closed partial order $\le$, then $\mu^{(k)}$ is monotone for $k \ge 3$:*

$$\mathbf{x} \le_k \mathbf{y} \qquad \Longrightarrow \qquad \mu^{(k)}(\mathbf{x}) \le \mu^{(k)}(\mathbf{y}).$$

Proof. Let $\mathbf{x}, \mathbf{y} \in X^k$ with $\mathbf{x} \le_k \mathbf{y}$. Then by assumption we have $\mu(x_i, x_{i+1}) \le \mu(y_i, y_{i+1})$ for $i = 1, \cdots, k$ and hence $\beta^n(\mathbf{x}) \le_k \beta^n(\mathbf{y})$, $n = 1, 2, \cdots$. By the closeness of the order, $\lim_n \beta^n(\mathbf{x}) \le_k \lim_n \beta^n(\mathbf{y})$ and we have $\mu^{(k)}(\mathbf{x}) \le_k \mu^{(k)}(\mathbf{y})$. □

Finally, we shall present Yamazaki's geometric mean of k-positive operators on a Hilbert space in [3]. We use the following Thompson metric on the convex cone Ω of positive invertible operators:

$$d(A, B) = \max\{\log M(A/B), \log M(B/A)\}$$

where $M(A/B) = \inf\{\lambda > 0 : A \le \lambda B\}$, see [7]. We remark that Ω is a complete metric space with respect to this metric and the corresponding metric topology on Ω agrees with the relative norm topology. For positive invertible operators A and B on a Hilbert space, the operator geometric mean $A \sharp B$ of A and B is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}},$$

also see [5]. Then the geometric mean $A \sharp B$ is a symmetric convex 2-mean on Ω endowed with the respect to the Thompson metric:

$$d(A \sharp C, B \sharp D) \leq \frac{1}{2}\, d(A, B) + \frac{1}{2}\, d(C, D).$$

For $A_1, \cdots, A_k$ of any k-tuple of positive invertible operators on a Hilbert space, the shift operator β is defined by

$$\beta(A_1, \cdots, A_k) = (A_1 \sharp A_2, \cdots, A_k \sharp A_1).$$

By Theorem 3.12 there exists $\lim_n \beta^n(A_1, \cdots, A_k)$ uniformly. Hence Yamazaki's geometric mean of k-positive operators is defined by

$$Y(A_1, \cdots, A_k) = \lim_n \pi_1 \circ \beta^n(A_1, \cdots, A_k).$$

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