数理解析研究所講究録
第 1755 巻 2011 年 76-89

A HISTORY OF THE NASH EQUILIBRIUM THEOREM IN THE FIXED POINT THEORY

Sehie Park

The National Academy of Sciences, ROK, Seoul 137–044; and
Department of Mathematical Sciences, Seoul National University,
Seoul 151–747, KOREA
shpark@math.snu.ac.kr

ABSTRACT. In 1950, John Nash [N1,2] established his celebrated equilibrium theorem by applying the Brouwer or the Kakutani fixed point theorem. Since then there have appeared several fixed point theorems from which generalizations of the Nash theorem, the Debreu theorem, and many related results can be derived. In this paper, we introduce several stages of such developments.

1. Introduction

John von Neumann's 1928 minimax theorem [V1] and 1937 intersection lemma [V2] have numerous generalizations and applications. Kakutani's 1941 fixed point theorem [K] was to give simple proofs of the above-mentioned results. In 1950, John Nash [N1,2] obtained his equilibrium theorem based on the Brouwer or Kakutani fixed point theorem. Further, in 1952, G. Debreu [De] obtained a social equilibrium existence theorem.

On the other hand, in 1952, Fan [F1] and Glicksberg [G] extended Kakutani's theorem to locally convex Hausdorff topological vector spaces, and Fan generalized the von Neumann intersection lemma by applying his own fixed point theorem. In 1961, Fan [F2] obtained his own KKM lemma and, in 1964 [F3], applied it to another intersection theorem for a finite family of sets having convex sections. This was applied in 1966 [F4] to a proof of the Nash equilibrium theorem. This is the origin of the application of the KKM theory to the Nash theorem.

Since then there have appeared many generalizations of the Nash theorem and studies on related topics. In fact, there are diverse alternative formulations of the Nash

---

2000 Mathematics Subject Classification. 47H10, 49J53, 54C60, 54H25, 90A14, 90C76, 91A13, 91A10.

Key words and phrases. Minimax theorem, von Neumann's intersection lemma, acyclic map, admissible set (in the sense of Klee), Klee approximable sets.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

equilibrium: *as a fixed point of the best response correspondence, as a fixed point of a function, as a solution of a nonlinear complementarity problem, as a solution of a stationary point problem, as a minimum of a function on a polytope, as an element of semi-algebraic set*; see, for example, [MM].

In our previous works [P17,18], we noticed that our studies on the Nash equilibrium were based on the following three methods:

(1) Fixed point method — Applications of the Kakutani theorem and its various generalizations (for example, for acyclic valued multimaps, admissible maps, or better admissible maps in the sense of Park).

(2) Continuous selection method — Applications of the fact that Fan-Browder type maps have continuous selections under certain assumptions like Hausdorffness and compactness of relevant spaces.

(3) The KKM method — As for the Sion minimax theorem [S], direct applications of the KKM theorem [KK] or its equivalents like as the Fan-Browder fixed point theorem [Br].

The history on the studies based on (2) and (3) was given recently in [P17,18].

In the present paper, we review the study based on the method (1); see [BK,D,F1,F3, G,H,IP,K,L,Lu,M,N1,2,Ni,P3,4,7-9,10,16,20,21,IP,PP,T] and others. In fact, we introduce several stages of such developments of generalizations of the Nash theorem and related results within the frame of fixed point theory. We are mainly concerned with the works of the present author.

2. From von Neumann to Nash

In order to give simple proofs of von Neumann's Lemma and the minimax theorem, Kakutani in 1941 obtained the following generalization of the Brouwer theorem to multimaps:

Theorem [K]. *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary [K]. *Theorem is also valid even if S is an arbitrary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary readily implies von Neumann's Lemma, and it is known later that those two results are directly equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

The first remarkable one of generalizations of von Neumann's minimax theorem was the Nash theorem [N1,2] on equilibrium points of non-cooperative games. The following Nash theorem is formulated by Fan [F4, Theorem 4]:

Theorem. [F4] *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\prod_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j \neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasiconcave function on X_i, then there exists a point $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) \in \prod_{i=1}^n X_i$ such that*

$$f_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) = \operatorname*{Max}_{y_i \in X_i} f_i(\hat{x}_1, \cdots, \hat{x}_{i-1}, y_i, \hat{x}_{i+1}, \cdots, \hat{x}_n) \quad (1 \leq i \leq n).$$

3. Generalizations of Debreu's work

In 1998 [P4], an acyclic version of the social equilibrium existence theorem of Debreu [De] is obtained.

A *polyhedron* is a set in $\mathbf{R}^n$ homeomorphic to a union of a finite number of compact convex sets in $\mathbf{R}^n$. The product of two polyhedra is a polyhedron [De].

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery [EM] or, more generally, to Begle [B]:

Lemma 3.1. *Let Z be an acyclic polyhedron and $T : Z \multimap Z$ an acyclic map (that is, u.s.c. with acyclic values). Then T has a fixed point $\hat{x} \in Z$; that is, $\hat{x} \in T(\hat{x})$.*

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j \quad \text{and} \quad X_{-i} = \prod_{j \in I \setminus \{i\}} X_j.$$

Any $x \in X$ can be expressed as $x = [x_{-i}, x_i]$ for any $i \in I$, where x_{-i} denotes the projection of x onto X_{-i}.

For $A \subset X$, $x_{-i} \in X_{-i}$, and $x_i \in X_i$, let

$$A(x_{-i}) := \{y_i \in X_i \mid [x_{-i}, y_i] \in A\} \quad \text{and} \quad A(x_i) := \{y_{-i} \in X_{-i} \mid [y_{-i}, x_i] \in A\}.$$

The following collectively fixed point theorem is equivalent to Lemma 3.1:

Theorem 3.2. [P20] *Let $\{X_i\}_{i\in I}$ be any family of acyclic polyhedra, and $T_i : X \multimap X_i$ an acyclic map for each $i \in I$. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

From Theorem 3.2, we have the following extension of the social equilibrium existence theorem of Debreu [De]:

Theorem 3.3. [P20] *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, $A_i : X_{-i} \multimap X_i$ closed maps, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. functions for each $i \in I$ such that*

(1) *$g_i(x) \le f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x_{-i}) = \max_{y\in A_i(x_{-i})} g_i[x_{-i}, y]$ is an l.s.c. function of $x_{-i} \in X_{-i}$; and*
(3) *for each $i \in I$ and $x_{-i} \in X_{-i}$, the set*

$$M(x_{-i}) := \{x_i \in A_i(x_{-i}) \mid f_i[x_{-i}, x_i] \ge \varphi_i(x_{-i})\}$$

is acyclic.

Then there exists an equilibrium point $\hat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\hat{a}_i \in A_i(\hat{a}_{-i}) \quad \text{and} \quad f_i(\hat{a}) = \max_{a_i \in A(\hat{a}_{-i})} g_i[\hat{a}_{-i}, a_i] \quad \text{for all} \quad i \in I.$$

This is applied in [P4] to deduce acyclic versions of theorems on saddle points and minimax theorems. The following acyclic version of the Nash equilibrium theorem is given in [P4] for a finite I and in [P20] for arbitrary I:

Corollary 3.4. *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, $X = \prod_{i\in I} X_i$, and for each $i \in I$, $f_i : X \to \overline{\mathbf{R}}$ a continuous function such that*

(0) *for each $x_{-i} \in X_{-i}$ and each $\alpha \in \overline{\mathbf{R}}$, the set*

$$\{x_i \in X_i \mid f_i[x_{-i}, x_i] \ge \alpha\}$$

is empty or acyclic.

Then there exists a point $\hat{a} \in X$ such that

$$f_i(\hat{a}) = \max_{y_i \in X_i} f_i[\hat{a}_{-i}, y_i] \quad \text{for all} \quad i \in I.$$

4. From the Idzik fixed point theorem

Let E be a real Hausdorff topological vector space (in short, a *t.v.s.*). A set $B \subset E$ is said to be *convexly totally bounded* (c.t.b.) whenever for every neighborhood V of $0 \in E$, there exist a finite subset $\{x_i \mid i \in I\} \subset E$ and a finite family of convex sets $\{C_i \mid i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i \mid i \in I\}$. See Idzik [I].

The following is a particular form of Idzik's theorem [I, Theorem 4.3]:

Theorem 4.1. [I] *Let X be a nonempty convex subset of a t.v.s. E and $T : X \multimap X$ a closed map with convex values. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Theorem 4.1 generalizes earlier results due to Zima, Rzepecki, Himmelberg, and Hadžić. For references, see [I].

As an application of the Idzik theorem, in this section, we consider a noncompact infinite optimization problem for a non-locally convex t.v.s.

From Theorem 4.1, we deduced the following:

Theorem 4.2. [PP] *Let I be an index set, and for each $i \in I$, X_i be a convex subset of a t.v.s. E_i, D_i be a nonempty compact subsets of X_i such that $D = \prod_{i\in I} D_i$ is a c.t.b. subset of $E = \prod_{i\in I} E_i$. For each $i \in I$, let $f_i : X = \prod_{i\in I} X_i \to \mathbf{R}$ be a u.s.c. function, and $S_i : X_{-i} \multimap D_i$ a closed map such that*

(1) *the function M_i defined on X^i by*

$$M_i(x_{-i}) := \sup_{y\in S_i(x_{-i})} f_i[x_{-i}, y] \quad \textit{for} \quad x_{-i} \in X_{-i}$$

is l.s.c.; and

(2) *for each $x_{-i} \in X_{-i}$, the set*

$$T_i(x_{-i}) := \{y \in S_i(x_{-i}) \mid f_i[x_{-i}, y] = M_i(x_{-i})\}$$

is convex.

Then there exists an $\overline{x} \in D$ such that for each $i \in I$,

$$\overline{x}_i \in S_i(\overline{x}_{-i}) \quad \textit{and} \quad f_i[\overline{x}_{-i}, \overline{x}_i] = M_i(\overline{x}_{-i}).$$

From Theorem 4.2, we obtain the following infinite version of the Nash equilibrium theorem:

Theorem 4.3. [PP,IP] *Let I be an index set, and for each $i \in I$, X_i be a nonempty compact convex subset of a t.v.s. E_i such that $X = \prod_{i\in I} X_i$ is a c.t.b. subset of $E = \prod_{i\in I} E_i$. For each $i \in I$, let $f_i : X \to \mathbf{R}$ be a continuous function such that for each given point $x_{-i} \in X_{-i}$, $x_i \mapsto f[x_{-i}, x_i]$ is a quasiconcave function on X_i. Then there exists an $\overline{x} \in X$ such that*

$$f_i(\overline{x}) = f_i[\overline{x}_{-i}, \overline{x}_i] = \max_{y_i\in X_i} f_i[\overline{x}_{-i}, y_i] \quad \textit{for each} \quad i \in I.$$

Remarks 1. Note that Ma already established Theorem 5 without assuming that X is c.t.b. A generalization of Ma's theorem was given by Idzik.

2. Nash's original theorem is the case E_i are Euclidean spaces and I is finite.

Moreover, in 1998 [IP], we considered two applications of the Idzik fixed point theorem [I]. First, we extended the Leray-Schauder theorem to t.v.s. which are not necessarily locally convex. As an application we derived some well-known fixed point theorems. Second, we deduced a variation of the social equilibrium existence theorem of Debreu. This was applied to results on saddle points, minimax theorems, and the Nash equilibria. These were generalizations of results of von Neumann, Kakutani, Nash, and von Neumann and Morgenstern; for the literature, see Debreu [De].

5. Fixed points of compositions of acyclic maps

From now on, a topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a t.v.s., convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible $\Longrightarrow$ ω-connected $\Longrightarrow$ acyclic $\Longrightarrow$ connected, and not conversely in each stage.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is called an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following theorems are only few examples of our previous works; for more general results, see [P14,15].

Theorem 5.1. *Let X be a nonempty convex subset of a locally convex t.v.s. E and $T \in \mathbb{V}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are ℓ^p, $L^p(0,1)$, H^p for $0 < p < 1$, and many others; see [P5,6,11,13-15] and references therein.

Theorem 5.2. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $T \in \mathbb{V}_c(X, X)$ has a fixed point.*

A *polytope* P in a subset X of a t.v.s. E is a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Examples of Klee approximable sets can be seen in [P12].

We define a class $\mathfrak{B}$ of maps from a subset X of a t.v.s. E into a topological space Y as follows [P9,11,12]:

$F \in \mathfrak{B}(X, Y) \Longleftrightarrow F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

We call $\mathfrak{B}$ the 'better' admissible class. Recently it is known that any u.s.c. map with compact values having *trivial shape* (that is, contractible in each neighborhood) belongs to $\mathfrak{B}(X, Y)$. Note that the class $\mathfrak{B}^p$ in [P11,12] should be replaced by $\mathfrak{B}$.

The following results appeared in our previous work [P12]:

Theorem 5.3. [P12, Corollary 2.3] *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

6. For admissible sets

In 2000 [P8] and 2002 [P10], we applied Theorem 5.2 to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem.

The following example of generalized forms of quasi-equilibrium theorems or social equilibrium existence theorems directly implies a generalization of the Nash-Ma type equilibrium existence theorem:

Theorem 6.1. [P10] *Let X_0 be a topological space and $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i. For each $i = 0, 1, \dots, n$, let $S_i : X_{-i} \multimap X_i$ be a closed map with compact values, and $f_i, g_i : X = \prod_{i=0}^n X_i \to \mathbf{R}$ u.s.c. real-valued functions.*

Suppose that for each i,

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*

(ii) *the function $M_i : X_{-i} \to \mathbf{R}$ defined by*

$$M_i(x_{-i}) := \max_{y_i \in S_i(x_{-i})} g_i[x_{-i}, y_i] \quad \textit{for} \quad x_{-i} \in X_{-i}$$

is l.s.c.; and

(iii) *for each $x_{-i} \in X_{-i}$, the set*

$$\{x_i \in S_i(x_{-i}) \mid f_i[x_{-i}, x_i] \geq M_i(x_{-i})\}$$

is acyclic.

If X_{-0} is admissible in $E_{-0} = \prod_{j=1}^{n} E_j$ and if all the maps S_i are compact except possibly S_n and S_n is u.s.c., then there exists an equilibrium point $\hat{x} \in X$; that is,

$$\hat{x}_i \in S_i(\hat{x}_{-i}) \quad \text{and} \quad f_i(\hat{x}) \geq \max_{y_i \in S_i(x^i)} g_i[\hat{x}_{-i}, y] \quad \text{for all} \quad i \in \mathbb{Z}_{n+1}.$$

7. For Klee approximable sets

In 2008 [P13], we deduced some collectively fixed point theorems for families of maps and, then, various von Neumann type intersection theorems.

Theorem 7.1. [P13] *Let $\{E_i\}_{i=1}^n$ be a family of t.v.s. For each i, let X_i be a subset of E_i, K_i a nonempty compact subset of X_i, and $F_i : X \multimap K_i$ a closed map with acyclic values (resp., values of trivial shape). If $K := \prod_{i=1}^n K_i$ is Klee approximable into X, then there exists an $\overline{x} = (\overline{x_i})_{i=1}^n \in X$ such that $\overline{x_i} \in F_i(\overline{x})$ for each i.*

From Theorem 7.1, we obtain the following von Neumann type intersection theorem:

Theorem 7.2. [P13] *Let $\{X_i\}_{i=1}^n$ be a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X such that $A_i(x_{-i})$ is an acyclic subset of K_i for each $x_{-i} \in X_{-i}$, where $1 \leq i \leq n$. If X is an almost convex admissible subset of E, then $\bigcap_{j=1}^n A_j \neq \emptyset$.*

Similarly, we can obtain a more general result than Theorem 7.2 as follows:

Theorem 7.2.′ [P13] *Let I be any index set, $\{X_i\}_{i\in I}$ a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X for each $i \in I$. Suppose that for each $x_{-i} \in X_{-i}$, $A_i(x_{-i})$ is a convex subset of K_i except a finite number of i's for which $A_i(x_{-i})$ is an acyclic subset of K_i. If X is an almost convex admissible subset of E, then $\bigcap_{j\in I} A_j \neq \emptyset$.*

Remark. If $I = \{1, 2\}$, E_i are Euclidean, $X_i = K_i$, and $A_i(x_{-i})$ are nonempty and convex, then Theorem 7.2 or 7.2′ reduces to the intersection lemma of von Neumann [V2].

We have another intersection theorem:

Theorem 7.3. [P13] *Let X_0 be a topological space and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For each $i = 0, 1, 2, \cdots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n and $F_i \in \mathbb{V}_c(X_{-i}, X_i)$. If K_{-0} is Klee approximable into X_{-0}, then $\bigcap_{i=0}^n \mathrm{Gr}(F_i) \neq \emptyset$.*

Remarks. 1. In case when each X_i is convex for $i \geq 1$ and X_{-0} is admissible in E_{-0}, Theorem 7.3 reduces to [P10, Theorem 4].

2. Particular forms of Theorem 7.3 were given by von Neumann, Fan, Lassonde, Chang, and Park; see [P10]. The following is one of them:

Corollary 7.4. *Let X be a topological space, Y a subset in a t.v.s. E, and $F \in \mathbb{V}_c(X,Y)$ and $G \in \mathbb{V}_c(Y,X)$. If F is compact and $F(X)$ is Klee approximable into Y, then* $\mathrm{Gr}(F) \cap \mathrm{Gr}(G) \neq \emptyset$.

From Corollary 7.4, we have the following:

Corollary 7.5. *Let X be a topological space and Y a compact subset of a t.v.s. E. Let A and B be two closed subsets of $X \times Y$ such that*

(1) *for each $x \in X$, $A(x) := \{y \in Y \mid (x,y) \in A\}$ is acyclic; and*

(2) *for each $y \in Y$, $B(y) := \{x \in X \mid (x,y) \in B\}$ is acyclic.*

If $A(X) := \bigcup\{A(x) \mid x \in X\}$ is Klee approximable into Y, then $A \cap B \neq \emptyset$.

Remarks. 1. If Y is an admissible, compact, and almost convex subset of E, then $A(X)$ is Klee approximable into Y. Especially, for the particular case when X is compact and Y is convex, Corollary 7.5 was obtained in [P8].

2. For other particular forms of Corollary 7.5, see [P8].

In [P13], from Theorem 7.3, we deduced a generalized form of the quasi-equilibrium theorem or the social equilibrium existence theorem in the sense of Debreu [De]:

Theorem 7.6. [P13] *Let X_0 be a topological space, and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For $i = 0, 1, \cdots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n, $S_i : X_{-i} \multimap K_i$ be a closed map with compact values, and $f_i, g_i : X = X_{-i} \times X_i \to \mathbf{R}$ u.s.c. real functions.*

Suppose that for each $i = 0, 1, \cdots, n$,

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*

(ii) *the real function $M_i : X_{-i} \to \mathbf{R}$ defined by*

$$M_i(x_{-i}) := \max_{y_i \in S_i(x_{-i})} g_i[x^i, y_i] \quad \text{for} \quad x_{-i} \in X_{-i}$$

is l.s.c.; and

(iii) *for each $x_{-i} \in X_{-i}$, the set*

$$\{y_i \in S_i(x_{-i}) \mid f_i[x_{-i}, y_i] \geq M_i(x_{-i})\}$$

is acyclic.

If K_{-0} is Klee approximable into X_{-0} and if S_n is u.s.c., then there exists an equilibrium point $\hat{x} \in X$; that is,

$$\hat{x}_i \in S_i(\hat{x}_{-i}) \quad \text{and} \quad f_i[\hat{x}_{-i}, \hat{x}_i] \geq \max_{y_i \in S_i(x_{-i})} g_i[x_{-i}, y_i] \quad \text{for each} \quad i \in \mathbf{Z}_{n+1}.$$

From this we deduced generalization of the Nash theorem and von Neumann type minimax theorems in [P13].

8. Existence of pure-strategy Nash equilibrium

In this section, we introduce the contents of a recent work [P21]. The following concept of generalized convex spaces is well known:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$.

We follow [Lu]. Let $I := \{1, \cdots, n\}$ be a set of players. A non-cooperative n-person game of normal form is an ordered $2n$-tuple $\Lambda := \{X_1, \cdots, X_n; u_1, \cdots, u_n\}$, where the nonempty set X_i is the ith player's pure strategy space and $u_i : X = X_i \times X_{-i} \to \mathbf{R}$ is the ith player's payoff function. A point of X_i is called a strategy of the ith player. Let us denote by x and x_{-i} an element of X and X_{-i}, resp. A strategy n-tuple $(x_1^*, \cdots, x_n^*)$ is called a *Nash equilibrium for the game* if the following inequality system holds:

$$u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*) \quad \text{for all } y_i \in X_i \quad \text{and} \quad i \in I.$$

As usual, we define an aggregate payoff function $U : X \times X \to \mathbb{R}$ as follows:

$$U(x, y) := \sum_{i=1}^{n} [u_i(y_i, x_{-i}) - u_i(x)] \quad \text{for any} \quad x = (x_i, x_{-i}), y = (y_i, y_{-i}) \in X.$$

The following is given in [Lu, Proposition 1]:

Lemma 8.1. *Let Λ be a non-cooperative game, K a nonempty subset of X, and $x^* = \{x_1^*, \ldots, x_n^*\} \in K$. Then the following are equivalent:*

(a) *x^* is a Nash equilibrium;*

(b) $\forall i \in I,\ \forall y_i \in X_i,\quad u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*)$;

(c) $\forall y \in X,\ U(x^*, y) \leq 0$.

Note that (c) implies $U(x^*, y) \leq 0$ for all $y \in D \subset X$.

Now we have our main result:

Theorem 8.2. *Let $I = \{1, \ldots, n\}$ be a set of players, K a nonempty compact subset of a Hausdorff product G-convex space $(X, D; \Gamma) = \prod_{i=1}^n (X_i, D_i; \Gamma_i)$ and Λ a non-cooperative game. Suppose that*

(i) *the function* $U : X \times X \to \mathbf{R}$ *satisfies that*

$$\{(x,y) \in X \times X \mid U(x,y) > 0\}$$

is open;

(ii) *for each* $x \in K$, $\{y \in X \mid U(x,y) > 0\}$ *is* Γ*-convex [that is,* $M \in \langle\{y \in D \mid U(x,y) > 0\}\rangle$ *implies* $\Gamma_M \subset \{y \in X \mid U(x,y) > 0\}$*];*

(iii) *for each* $y \in X$*, the set* $\{x \in K \mid U(x,y) \leq 0\}$ *is acyclic.*

Then there exists a point $x^* \in K$ *such that* x^* *is an equilibrium point for the non-cooperative game.*

Note that condition (i) can be replaced by the following:

(i)$'$ the function $U(x,y)$ is lower semicontinuous on $X \times X$.

In this case, when $X = D$ is a topological vector space, Theorem 8.2 reduces to [Lu, Theorem 1].

9. Historical remarks

In 1928, John von Neumann found his celebrated minimax theorem [V1], which is one of the fundamental theorems in the theory of games developed by himself: For the history of earlier proofs of the theorem, see von Neumann [V3] and Dantzig [D]. In 1937, the theorem was extended by himself [V2] to his intersection lemma by using a notion of integral in Euclidean spaces. The lemma was intended to establish his minimax theorem and his theorem on optimal balanced growth paths and applied to problems of mathematical economics.

In 1941, Kakutani [K] obtained a fixed point theorem for multimaps, from which von Neumann's minimax theorem and intersection lemma were easily deduced. In 1950, John Nash [N1,2] obtained his equilibrium theorem based on the Brouwer or Kakutani fixed point theorem. Further, in 1952, G. Debreu [De] obtained a social equilibrium existence theorem.

In the 1950's, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [BK] and to locally convex t.v.s. by Fan [F1] and Glicksberg [G]. These extensions were mainly used to generalize the von Neumann intersection lemma and the Nash equilibrium theorem. Further generalizations were followed by Ma [M] and others. For the literature, see [P6] and references therein.

An upper semicontinuous (u.s.c.) multimaps with nonempty compact convex values is called a *Kakutani map.* The Fan-Glicksberg theorem was extended by Himmelberg [H] in 1972 for compact Kakutani maps instead of assuming compactness of domains.

In 1988, Idzik [I] extended the Himmelberg theorem to convexly totally bounded sets instead of convex subsets in locally convex t.v.s. This result is applied in [P3,PP,IP] to various problems. In 1990, Lassonde [L] extended the Himmelberg theorem to multimaps factorizable by Kakutani maps through convex sets in Hausdorff topological vector spaces. Moreover, Lassonde applied his theorem to game theory and obtained a von Neumann type intersection theorem for finite number of sets and a Nash type equilibrium theorem comparable to Debreu's social equilibrium existence theorem [De].

On the other hand, in 1946, the Kakutani fixed point theorem was extended for acyclic maps by Eilenberg and Montgomery [EM]. Moreover, the Kakutani theorem was known to be included in the extensions, due to Eilenberg and Montgomery [EM] or Begle [B], of Lefschetz's fixed point theorem to u.s.c. multimaps of a compact *lc*-space into the family of its nonempty compact acyclic subsets. This result was applied by Park [P4] to give acyclic versions of the social equilibrium existence theorem due to Debreu [De], saddle point theorems, minimax theorems, and the Nash equilibrium theorem.

Moreover, Park [P1,2,4,10-14] obtained a sequence of fixed point theorems for various classes of multimaps (including compact compositions of acyclic maps) defined on very general subsets (including Klee approximable subsets) of t.v.s. Especially, our cyclic coincidence theorem for acyclic maps were applied to generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, the von Neumann type minimax theorems, and many other results; see [P16].

Finally, recall that there are several thousand published works on the KKM theory and fixed point theory and we can cover only a part of them. For the more historical background for the related fixed point theory and for the more involved or related results to this review, see the references of [P6,14-16,18,19] and the literature therein.

References

[B] E. G. Begle, *A fixed point theorem*, Ann. Math. **51** (1950), 544–550.

[BK] H.F. Bohnenblust and S. Karlin, *On a theorem of Ville*, Contributions to the Theory of Games, Ann. of Math. Studies **24**, 155–160, Princeton Univ. Press, 1950.

[Br] F.E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[D] G.B. Danzig, *Constructive proof of the min-max theorem*, Pacific J. Math. **6** (1956), 25–33.

[De] G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. USA **38** (1952), 886–893 [= Chap.2, Mathematical Economics: Twenty Papers of Gerald Debreu, Cambridge Univ. Press, 1983].

[EM] S. Eilenberg and D. Montgomery, *Fixed point theorems for multivalued transformations*, Amer. J. Math. **68** (1946), 214–222.

[F1] K. Fan, *Fixed point and minimax theorems in locally convex linear spaces*, Proc. Nat. Acad. Sci., U.S.A. **38** (1952), 121–126.

[F2] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[F3] K. Fan, *Sur un théorème minimax*, C.R. Acad. Sci. Paris Sér. I. Math. **259** (1964), 3925–3928.

[F4] K. Fan, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.

[G] I.L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.

[H] C.J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.

[I] A. Idzik, *Almost fixed point theorems*, Proc. Amer. Math. Soc. **104** (1988), 779–784.

[IP] A. Idzik and S. Park, *Leray-Schauder type theorems and equilibrium existence theorems*, Differential Inclusions and Optimal Control, Lect. Notes in Nonlinear Anal. **2** (1998), 191–197.

[K] S. Kakutani, *A generalization of Brouwer's fixed-point theorem*, Duke Math. J. **8** (1941), 457–459.

[KK] B. Knaster, K. Kuratowski, S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.

[L] M. Lassonde, *Fixed points of Kakutani factorizable multifunctions*, J. Math. Anal. Appl. **152** (1990), 46–60.

[Lu] H. Lu, *On the existence of pure-strategy Nash equilibrium*, Economics Letters **94** (2007), 459–462.

[M] T.-W. Ma, *On sets with convex sections*, J. Math. Anal. Appl. **27** (1969), 413–416.

[MM] R.D. McKelvey and A. McLenan, *Computation of equilibria in finite games*, Handbook of Computational Economics, Vol. 1, 87–142, Elsevier, 1966.

[N1] J.F. Nash, *Equilibrium points in N-person games*, Proc. Nat. Acad. Sci. USA **36** (1950), 48–49.

[N2] J. Nash, *Non-cooperative games*, Ann. Math. **54** (1951), 286–293.

[P1] S. Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, Ed.), 248–277, World Sci., River Edge, NJ, 1992.

[P2] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), 493–519.

[P3] S. Park, *Applications of the Idzik fixed point theorem*, Nonlinear Funct. Anal. Appl. **1** (1996), 21–56.

[P4] S. Park, *Remarks on a social equilibrium existence theorem of G. Debreu*, Appl. Math. Lett. **11**(5) (1998), 51–54.

[P5] S. Park, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829. *Corrections*, ibid. **36** (1999), 829–832.

[P6] S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 187–222.

[P7] S. Park, *Fixed points, intersection theorems, variational inequalities, and equilibrium theorems*, Inter. J. Math. Math. Sci. **24** (2000), 73–93.

[P8] S. Park, *Acyclic versions of the von Neumann and Nash equilibrium theorems*, J. Comp. Appl. Math. **113** (2000), 83–91.

[P9] S. Park, *Fixed points of better admissible maps on generalized convex spaces*, J. Korean Math. Soc. **37** (2000), 885–899.

[P10] S. Park, *Remarks on acyclic versions of generalized von Neumann and Nash equilibrium theorems*, Appl. Math. Letters **15** (2002), 641–647.

[P11] S. Park, *Fixed points of multimaps in the better admissible class*, J. Nonlinear Convex Anal. **5** (2004), 369–377.

[P12] S. Park, *Fixed point theorems for better admissible multimaps on almost convex sets*, J. Math. Anal. Appl. **329** (2007), 690–702.

[P13] S. Park, *Applications of fixed point theorems on almost convex sets*, J. Nonlinear Convex Anal. **9**(1) (2008), 45–57.

[P14] S. Park, *Generalizations of the Himmelberg fixed point theorem*, Fixed Point Theory and Its Applications (Proc. ICFPTA-2007), 123–132, Yokohama Publ., 2008..

[P15] S. Park, *Fixed point theory of multimaps in abstract convex uniform spaces*, Nonlinear Analysis **71** (2009), 2468–2480.

[P16] S. Park, *Applications of fixed point theorems for acyclic maps — A survey*, Vietnam J. Math. **37**(4) (2009), 419–441.

[P17] S. Park, *A history of the Nash equilibrium theorem in the KKM theory*, Nonlinear Analysis and Convex Analysis, RIMS Kôkyûroku, Kyoto Univ. **1685** (2010), 76–91.

[P18] S. Park, *Generalizations of the Nash equilibrium theorem in the KKM theory*, Takahashi Legacy, Fixed Point Theory and Appl., vol. 2010, Article ID 234706, 23pp, doi:10.1155/2010/234706.

[P19] S. Park, *The KKM principle in abstract convex spaces: Equivalent formulations and applications*, Nonlinear Analysis **73** (2010), 1028–1042.

[P20] S. Park, *Further extension of a social equilibrium existence theorem of G. Debreu*, to appear.

[P21] S. Park, *A variant of the Nash equilibrium theorem in generalized convex spaces*, J. Nonlinear Anal. Optim. **1** (2010), 17–22.

[PP] S. Park and J.A. Park, *The Idzik type quasivariational inequalities and noncompact optimization problems*, Colloq. Math. **71** (1996), 287–295.

[S] M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176.

[T] J.P. Torres-Martínez, *Fixed points as Nash equilibria*, Fixed Point Th. Appl. vol. 2006, Article ID 36135, 1-4.

[V1] J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), 295–320.

[V2] J. von Neumann, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergeb. Math. Kolloq. **8** (1937), 73–83.

[V3] J. von Neumann, *Communication on the Borel notes*, Econometrica **21** (1953), 124–125.