数理解析研究所講究録
第1762巻 2011年 30-38

Smoothness of hairs for some transcendental entire functions

Dedicated to Professor Shigehiro Ushiki on the occasion of his 60th birthday

Masashi KISAKA (木坂 正史)
Department of Mathematical Sciences,
Graduate School of Human and Environmental Studies,
Kyoto University, Kyoto 606-8501, Japan

Mitsuhiro SHISHIKURA (宍倉 光広)
Department of Mathematics,
Faculty of Science,
Kyoto University, Kyoto 606-8502, Japan

Abstract

We investigate the existence and smoothness of hairs for some transcendental entire functions. We show their existence and smoothness under a general setting. This is applicable for the function $P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials. This generalizes the previous results by R.L.Devaney, M.Krych and M.Viana.

1 Preliminaries

Let f be an entire function and f^n denote the n-th iterate of f, that is,

$$f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}.$$

Recall that the *Fatou set* $F(f)$ is the set of point z where $\{f^n\}_{n=1}^{\infty}$ forms a normal family in a neighborhood of z. We call the complement of $F(f)$ the *Julia set* of f and denote it by $J(f)$. By definition, $F(f)$ is open and $J(f)$ is closed in $\mathbb{C}$. Also $J(f)$ is compact if f is a polynomial, while it is non-compact if f is transcendental. This is due to the fact that ∞ is an essential singularity for a transcendental entire function.

The purpose of this paper is to construct so-called *hairs*, which is subsets of the Julia set $J(f)$, and to show their smoothness for a certain class of transcendental entire functions. Devaney and Krych first constructed hairs for exponential family $E_\lambda(z) = \lambda e^z$ ($\lambda \in \mathbb{C} \setminus \{0\}$) in 1984 ([**DK**]). Here we briefly explain their results. Define

$$B_l := \{z \mid (2l-1)\pi < \mathrm{Im}\ z + \theta < (2l+1)\pi\}, \quad \theta = \arg \lambda \in [-\pi, \pi),\ l \in \mathbb{Z}$$

then we can define itinerary $S(z) := \boldsymbol{s} = (s_0, s_1, \cdots, s_n, \cdots) \in \mathbb{Z}^{\mathbb{N}}$ for a point $z \in \mathbb{C}$ by $E_\lambda^n(z) \in B_{s_n}$.

Theorem 1.1 (Devaney-Krych, 1984). *If $\boldsymbol{s} \in \mathbb{Z}^{\mathbb{N}}$ satisfies the following "growth condition":*

$$^{\exists}x_0 \in \mathbb{R},\ ^{\forall}n,\ (2|s_n|+1)\pi + |\theta| \le g^n(x_0), \quad g(t) := |\lambda|e^t,$$

then there exists a continuous curve $h_{\boldsymbol{s}}(t) \subset J(E_\lambda)$ which satisfies the following:

(i) $E_\lambda(h_{\boldsymbol{s}}(t)) = h_{\sigma(\boldsymbol{s})}(g(t))$, where σ is the shift map on $\mathbb{Z}^{\mathbb{N}}$,

(ii) $E_\lambda^n(h_{\boldsymbol{s}}(t)) \to \infty$ $(n \to \infty)$ for every t. □

The curve $h_{\boldsymbol{s}}(t)$ is called *a hair*. Viana showed that this hair $h_{\boldsymbol{s}}(t)$ is a C^∞ curve ([**V**]).

Later, the existence of hairs was proved for some other class of functions, like $\lambda z e^z$ or the complex standard family (see [**F**]. Note that this did not mention the smoothness of hairs). In this paper we consider the existence and smoothness of hairs under a general setting. In particular we generalize this result for the exponential functions to $f(z) := P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials. We state our detailed setting and the results of existence in §2. In §3 and §4 we explain the smoothness of hairs. In §5 we state the result for $f(z) = P(z)e^{Q(z)}$ as an application of our general results.

2 C^0 a priori estimates — existence of a hair $h(t)$ —

Definition 2.1. Let $\rho : [\tau_*, \infty) \to \mathbb{R}_+$ be a positive function (called *weight function*). Define for a function $\psi : [\tau_*, \infty) \to \mathbb{C}$,

$$||\psi||_{\rho,\tau} = \sup_{t \ge \tau} |\psi(t)|\rho(t).$$

The set of continuous functions ψ with $||\psi||_{\rho,\tau} < \infty$ forms a Banach space $X_{\rho,\tau}$.

Our setting is as follows:

A: Let $f_n : U_n \to V_n$ $(n = 0, 1, 2, \dots)$ be holomorphic diffeomorphisms between unbounded domains U_n and V_n in $\mathbb{C}$. The reference mapping $g : [\tau_*, \infty) \to \mathbb{R}$ is an increasing C^∞ function such that $g(t) > t$ for $t \ge \tau_*$. (Hence $g^n(t) \to \infty$ $(n \to \infty)$.) Denote $\tau_n = g^n(\tau_*)$ $(n = 0, 1, 2, \dots)$.

For the application in §5, we will take f_n as a restriction of a single function f to some domains U_n, that is, $f_n := f|_{U_n}$, but in general we do not need this.

Our goal is to construct functions $h_n : [\tau_n, \infty) \to U_n$ $(n = 0, 1, 2, \dots)$ satisfying

$$f_n \circ h_n(t) = h_{n+1} \circ g(t) \quad \text{for } t \in [\tau_n, \infty), \tag{1}$$

and show their smoothness. In order to construct such functions, we start with a function $h_{l,l} : [\tau_l, \infty) \to \mathbb{C}$, then define $h_{l,n} : [\tau_n, \infty) \to \mathbb{C}$ $(0 \le n < l)$ by "lifting" it successively so that for $n = l-1, l-2, \dots, 1, 0$,

$$f_n \circ h_{l,n}(t) = h_{l,n+1} \circ g(t) \quad (t \in [\tau_n, \infty)). \tag{2}$$

See Figure 1 and Diagram 1 below. Once $h_{l,n}(t)$ $(l = 0, 1, 2 \dots,\ 0 \le n \le l)$ are defined, our $h_n(t)$ will be obtained as $\lim_{l \to \infty} h_{l,n}(t)$. To ensure the convergence and the smoothness,

we need to impose various conditions on f_n and $h_{n,n}$ together with auxiliary functions $R(t)$, $\rho_k(t)$ and $\sigma_k(t)$, which are defined below. In particular, the initial curves should be chosen so that $h_{n+1,n} - h_{n,n}$ is small (or $f_n \circ h_{n,n} - h_{n+1,n+1} \circ g$ not too big). So we assume the following:

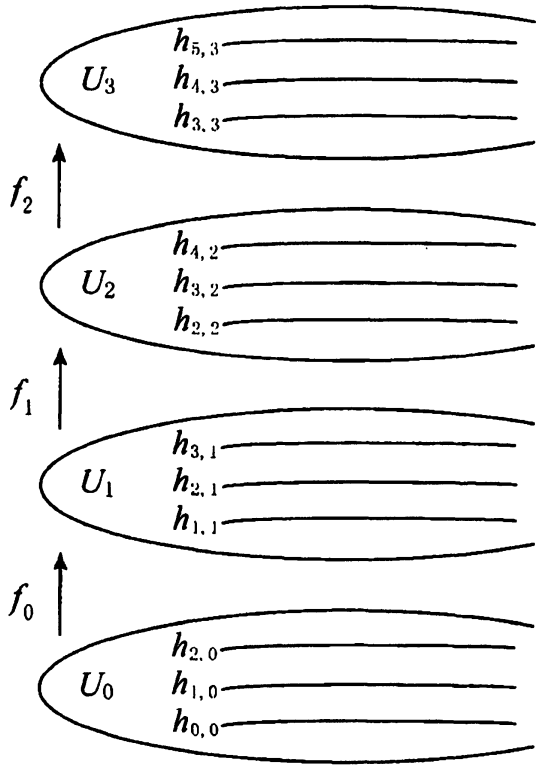

Figure 1. Construction of $h_{l,n}$.

$$
\begin{array}{lccccccccc}
 & h_{0,0} & & & & & & & & \\
 & h_{1,0} & h_{1,1} & & & & & & & \\
 & h_{2,0} & h_{2,1} & h_{2,2} & & & & & & \\
 & h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3} & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\
 & h_{n,0} & h_{n,1} & h_{n,2} & h_{n,3} & \cdots & h_{n,n} & & & \\
 & h_{n+1,0} & h_{n+1,1} & h_{n+1,2} & h_{n+1,3} & \cdots & h_{n+1,n} & h_{n+1,n+1} & & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
 & h_{l,0} & h_{l,1} & h_{l,2} & h_{l,3} & \cdots & h_{l,n} & h_{l,n+1} & \cdots & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \\
(l \to \infty) & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \ddots & \\
 & h_0 & h_1 & h_2 & h_3 & \cdots & h_n & h_{n+1} & \cdots &
\end{array}
$$

Diagram 1.

B: (**Initial curves**) Suppose that continuous functions $h_{n,n}, h_{n+1,n} : [\tau_n, \infty) \to U_n$ $(n = 0, 1, 2, \dots)$, $R : [\tau_*, \infty) \to \mathbb{R}_+$ and a constant $0 < \kappa < 1$ satisfy for $t \in [\tau_n, \infty)$:

- $f_n \circ h_{n+1,n}(t) = h_{n+1,n+1} \circ g(t)$; (3)
- $|h_{n+1,n}(t) - h_{n,n}(t)| \leq (1 - \kappa)R(t)$; (4)
- There exists an open set $B_n(t) \subset U_n$ with $\overline{B_n(t)} \subset U_n$ such that
$$f_n : B_n(t) \to \mathbb{D}(h_{n+1,n+1}(g(t)), R(g(t)))$$
is bijective. In particular, $\mathbb{D}(h_{n+1,n+1}(g(t)), R(g(t))) \subset V_n$; (5)
- For $z \in B_n(t)$, $|f_n'(z)|\dfrac{R(t)}{R(g(t))} \geq \dfrac{1}{\kappa}$. (6)

We have a sufficient condition for **B**.

Lemma 2.2. *Suppose that continuous functions $h_{n,n} : [\tau_n, \infty) \to U_n$ $(n = 0, 1, 2, \dots)$ and constants $\tilde{R} > 0$, $0 < \kappa < \frac{1}{2}$ satisfy for $t \in [\tau_n, \infty)$:*

$$\mathbb{D}(h_{n+1,n+1}(t'), \tilde{R}) \subset V_n, \text{ where } t' = g(t) \in [\tau_{n+1}, \infty); \tag{7}$$
$$|f_n \circ h_{n,n}(t) - h_{n+1,n+1} \circ g(t)| \leq \tilde{R}/3; \tag{8}$$
$$|f_n'(h_{n,n}(t))| \geq 16/\kappa. \tag{9}$$

If we choose $R(t)$ so that

$$\frac{4\tilde{R}}{3(1-\kappa)|f_n'(h_{n,n}(t))|} \leq R(t) \leq \frac{\kappa\tilde{R}}{12(1-\kappa)}, \tag{10}$$

(which is possible by (9)), then there exist $h_{n+1,n} : [\tau_n, \infty) \to U_n$ $(n = 0, 1, 2, \dots)$ satisfying **B**. □

Let us denote $\rho_*(t) = 1/R(t)$. Using the norm $||\cdot||_{\rho,\tau}$ defined in the beginning of this section, the above condition (4) can be expressed as $||h_{n+1,n} - h_{n,n}||_{\rho_*,\tau_n} \leq 1 - \kappa$.

Under the above setting, we can show the existence of a hair $h_n(t)$ $(n = 0, 1, \cdots)$.

Lemma 2.3. *Under the assumptions* **A***and* **B**, *there exist continuous functions $h_{l,n} : [\tau_n, \infty) \to \mathbb{C}$ $(l = 0, 1, 2, \dots,\ 0 \leq n \leq l)$ such that*

$$f_n \circ h_{l,n}(t) = h_{l,n+1} \circ g(t) \quad \text{for } t \in [\tau_n, \infty),\ n < l; \tag{11}$$
$$||h_{l+1,n} - h_{l,n}||_{\rho_*,\tau_n} \leq (1 - \kappa)\kappa^{l-n}; \tag{12}$$
$$||h_{l,n} - h_{n,n}||_{\rho_*,\tau_n} \leq 1 - \kappa^{l-n}. \tag{13}$$

Therefore there exists continuous functions $h_n(t) = \lim_{l\to\infty} h_{l,n}(t)$ satisfying

$$f_n \circ h_n(t) = h_{n+1} \circ g(t) \quad \text{for } t \in [\tau_n, \infty) \quad \text{and} \quad |h_n(t) - h_{n,n}(t)| \leq R(t). \tag{14}$$

□

3 C^1 estimates

We are now going to show that h_n are C^1 under additional assumptions. If we know that $h_{l,n}$ are C^1, then the differentiation of (11) gives

$$\log h'_{l,n} = \log h'_{l,n+1} \circ g + \log g' - \log f'_n \circ h_{l,n}. \tag{15}$$

Fix an l and denote

$$\psi_n(t) = \log h'_{l,n}(t) \quad \text{and} \quad \hat{\psi}_n(t) = \log h'_{l+1,n}(t) \quad (n = 0, \dots, l), \tag{16}$$

where an appropriate branch of log should be taken along the hairs so that $\psi_n(t) - \hat{\psi}_n(t) \to 0$ $(t \to \infty)$. Then it follows from (15) that for $n = 1, 2, \dots$

$$\psi_n - \hat{\psi}_n = (\psi_{n+1} - \hat{\psi}_{n+1}) \circ g - (\log f'_n \circ h_{l,n} - \log f'_n \circ h_{l+1,n}). \tag{17}$$

Our goal is to derive a geometric estimate of the form

$$|\psi_n(t) - \hat{\psi}_n(t)| \le const\, \kappa_0^{l-n}$$

with $0 < \kappa_0 < 1$. Note that Theorem in the previous section gives an estimate for the second term of (17) by $const\, \kappa^{l-n}$. In order to give recursive estimates on $\psi_n - \hat{\psi}_n$ from $n = l$ down to $n = 0$, observe the following fact: if $\psi_{n+1} - \hat{\psi}_{n+1}$ goes to 0 as $t \to \infty$, by composing g, $(\psi_{n+1} - \hat{\psi}_{n+1}) \circ g$ may go to 0 faster. This can be formulated in terms of the norm $\|\cdot\|_{\rho_0,\tau_*}$ with an appropriate weight function $\rho_0 : [\tau_*, \infty) \to \mathbb{R}^+$. If fact for a function $\psi : [\tau_*, \infty) \to \mathbb{C}$, we have

$$\begin{aligned} \|\psi \circ g\|_{\rho_0,\tau} &= \sup_{t \ge \tau} |\psi(g(t))| \rho_0(t) = \sup_{t \ge \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \cdot |\psi(g(t))| \rho_0(g(t)) \\ &\le \left(\sup_{t \ge \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \cdot \left(\sup_{t' \ge g(\tau)} |\psi(t')| \rho_0(t') \right) = \left(\sup_{t \ge \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \|\psi\|_{\rho_0, g(\tau)}. \end{aligned} \tag{18}$$

So if $\sup_{t \ge \tau} \dfrac{\rho_0(t)}{\rho_0(g(t))} < 1$, then $\|\cdot\|_{\rho_0,\tau}$-norm is contracted by composing g. This gives a possibility to prove the geometric estimate on $\psi_n - \hat{\psi}_n$.

For further estimates (C^k, $k = 1, 2, \dots$), we need to prepare the following.

Definition 3.1. In what follows, we shall introduce weight functions $\rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+$ to measure the norm $\|\cdot\|_{\rho_k,\tau}$ of $\psi_{n+1}^{(k)} - \psi_n^{(k)}$ and the norm $\|\cdot\|_{\sigma_k,\tau}$ of $\psi_n^{(k)}$ for $k = 0, 1, 2, \dots$, with $\sigma_k(t) \le \rho_k(t)$. Given those weight functions, define

$$\alpha_k(t) = \frac{\rho_k(t) |g'(t)|^k}{\rho_k(g(t))} \quad \text{and} \quad \bar{\alpha}_k(\tau) = \sup_{t \ge \tau} \alpha_k(t)$$

for $k = 0, 1, 2, \dots$ and $t, \tau \ge \tau_*$. We also need

$$D_{n,k}(t) = \sup_{z \in B_n(t)} \left| (\log f'_n)^{(k)}(z) \right|,$$

$$\text{where} \quad B_n(t) := \{ z \in U_n : |f_n(z) - h_{n+1,n+1}(g(t))| \le R(g(t))) | \}.$$

Suppose there exist weight functions $\rho_0, \sigma_0 : [\tau_*.\infty) \to \mathbb{R}_+$ satisfying $\mathbf{C}_0$, $\mathbf{D}_0$ and $\mathbf{F}_0$.

$\mathbf{C}_0$: $h_{l,l}$, $h_{l+1,l}$ are C^1 with $h'_{l,l}(t), h'_{l+1,l}(t) \neq 0$ and $\psi_{l,l}(t) = \log h'_{l,l}(t)$, $\psi_{l+1,l}(t) = \log h'_{l+1,l}(t)$ satisfy

$$||\psi_{l+1,l} - \psi_{l,l}||_{\rho_0,\tau_l} < \infty \quad \text{and} \quad ||\psi_{l,l}||_{\sigma_0,\tau_l} < \infty.$$

$\mathbf{D}_0$: $\lim_{\tau\to\infty} \bar{\alpha}_0(\tau) = \limsup_{t\to\infty} \dfrac{\rho_0(t)}{\rho_0(g(t))} < 1.$

$\mathbf{F}_0$: $K_0 := \sup_{n\geq 0} \sup_{t\geq\tau_n} D_{n,1}(t)R(t)\rho_0(t) < \infty.$

Lemma 3.2. *Suppose* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_0$, $\mathbf{D}_0$ *and* $\mathbf{F}_0$ *are satisfied. Then* $h_{l,n}$ *are* C^1 $(l = 0, 1, 2, \dots,\ 0 \leq n \leq l)$ *and there exists* $\kappa_0 < 1$ *and* C_0 *such that* $\psi_{l,n}(t) = \log h'_{l,n}(t)$ *satisfy*

$$||\psi_{l+1,n} - \psi_{l,n}||_{\rho_0,\tau_n} \leq C_0\kappa_0^{l-n} \quad (l = 0, 1, 2, \dots,\ 0 \leq n \leq l). \tag{19}$$

Therefore the limits $h_n(t)$ *are also* C^1 *and* $\psi_n(t) = \log h'_n(t)$ *satisfies*

$$||\psi_n - \psi_{n,n}||_{\rho_0,\tau_n} \leq C_0/(1-\kappa_0) \quad \textit{and} \quad ||\psi_n||_{\sigma_0,\tau_n} \leq C_0/(1-\kappa_0) + ||\psi_{n,n}||_{\sigma_0,\tau_n} < \infty.$$

□

4 Higher order derivatives — estimate for $\psi_n^{(k)}(k = 1, 2, \dots)$ —

We now try to apply similar estimates as in the previous section to $\psi_{l,n}^{(k)}$ $(k = 1, 2, \dots)$, but the estimates must involve with more terms. Differentiating (15) and using $h'_{l,n} = e^{\psi_{l,n}}$, we have

$$\psi'_{l,n} = \left(\psi'_{l,n+1} \circ g\right) \cdot g' + (\log g')' - \left((\log f'_n)' \circ h_{l,n}\right) e^{\psi_{l,n}}, \tag{20}$$

$$\begin{aligned}\psi''_{l,n} = {} & \left(\psi''_{l,n+1} \circ g\right) \cdot (g')^2 + \left(\psi'_{l,n+1} \circ g\right) \cdot g'' + (\log g')'' \\ & - \left((\log f'_n)'' \circ h_{l,n}\right) e^{2\psi_{l,n}} - \left((\log f'_n)' \circ h_{l,n}\right) e^{\psi_{l,n}}\psi'_{l,n}.\end{aligned} \tag{21}$$

More generally, it is easy to check the following by the induction:

Lemma 4.1. *For* $k = 1, 2, \dots,$ *we have*

$$\begin{aligned}\psi_{l,n}^{(k)} = {} & \left(\psi_{l,n+1}^{(k)} \circ g\right)(g')^k + \sum_{\substack{1\leq\ell<k \\ j_1\geq\cdots\geq j_\ell\geq 1 \\ j_1+\cdots+j_\ell=k}} const \left(\psi_{l,n+1}^{(\ell)} \circ g\right) g^{(j_1)} \dots g^{(j_\ell)} + (\log g')^{(k)} \\ & - \sum_{\substack{1\leq\ell\leq k,\, 0\leq\nu \\ j_1\geq\cdots\geq j_\nu\geq 1 \\ \ell+j_1+\cdots+j_\nu=k}} const \left((\log f'_n)^{(\ell)} \circ h_{l,n}\right) e^{\ell\psi_{l,n}} \psi_{l,n}^{(j_1)} \dots \psi_{l,n}^{(j_\nu)},\end{aligned} \tag{22}$$

where the coefficients "const" are some constants depending the indices $\ell, j_1, j_2, \dots$. □

Remark 4.2. (1) Note that in the right hand side of (22), only the first term contains k-th derivative of ψ_n and all other terms involve lower order derivatives of ψ_n (or none). Therefore if lower order derivatives are "under control," it is expected that we can proceed as in the previous section.

(2) For the exponential map $f(z) = \lambda e^z$ and $g(t) = |\lambda| e^t$, we have $(\log f')' \equiv 1$ and $(\log f')^{(\ell)} \equiv 0$ $(\ell > 1)$. So the formula (22) simplifies substantially. Moreover $g^{(j_1)} \dots g^{(j_\ell)}$ is a constant multiple of g^ℓ which also simplifies the expression.

Suppose weight functions $\rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+$ are given. We require the following:

$\mathbf{C}_k$: $h_{l,l}$, $h_{l+1,l}$ are C^{k+1} and $\psi_{l,l} = \log h'_{l,l}$ and $\psi_{l+1,l} = \log h'_{l+1,l}$ satisfy

$$||\psi^{(k)}_{l+1,l} - \psi^{(k)}_{l,l}||_{\rho_k,\tau_l} < \infty \quad \text{and} \quad ||\psi^{(k)}_{l,l}||_{\sigma_k,\tau_l} < \infty.$$

$\mathbf{D}_k$: $\lim_{\tau\to\infty} \bar{\alpha}_k(\tau) < 1$.

$\mathbf{E}_k$: For $1 \le \ell < k$ and $j_1, \dots, j_\ell \ge 1$ with $j_1 + \cdots + j_\ell = k$,

$$\sup_{t\ge\tau_*} \frac{\rho_k(t)|g^{(j_1)}(t)\cdots g^{(j_\ell)}(t)|}{\rho_\ell(g(t))} < \infty.$$

$\mathbf{F}_k$: For $1 \le \ell \le k$, $\nu \ge 0$, $j_1, \dots, j_\nu \ge 1$ with $\ell + j_1 + \cdots + j_\nu = k$,

$$\sup_{n\ge0}\sup_{t\ge\tau_*} D_{n,\ell+1}(t)R(t)\frac{\rho_k(t)}{\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t)} < \infty;$$

$$\sup_{n\ge0}\sup_{t\ge\tau_*} D_{n,\ell}(t)\frac{\rho_k(t)}{\rho_0(t)\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t)} < \infty;$$

$$\text{if } \nu \ge 1, \text{ for } 1 \le i \le \nu, \quad \sup_{n\ge0}\sup_{t\ge\tau_*} D_{n,\ell}(t)\frac{\rho_k(t)}{\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t)}\frac{\sigma_{j_i}(t)}{\rho_{j_i}(t)} < \infty.$$

Here if $\nu = 0$, set $\sigma_{j_1}(t)\cdots\sigma_{j_\nu}(t) = 1$. Note that the last condition should be satisfied only when $\nu \ge 1$.

Under these assumptions, we can show the following:

Lemma 4.3. *Let $k \ge 1$. Suppose* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_j$ *$(0 \le j \le k)$*, $\mathbf{D}_j$ *$(0 \le j \le k)$*, $\mathbf{E}_j$ *$(1 \le j \le k)$ and* $\mathbf{F}_j$ *$(0 \le j \le k)$ are satisfied. Then h_n are C^{k+1} $(n = 2, 3, \dots)$ and there exist constants $0 < \kappa_k < 1$ and C_k such that*

$$||\psi^{(k)}_{l+1,n} - \psi^{(k)}_{l,n}||_{\rho_k,\tau_n} \le C_k\kappa_k^n \quad (n = 0, 1, 2, \dots). \tag{23}$$

Therefore the limits $h(t)$ are also C^{k+1} and $\psi_n = \log h'_n$ satisfies

$$||\psi^{(k)}_n - \psi^{(k)}_{n,n}||_{\rho_k,\tau_n} \le C_k/(1-\kappa_k) \quad \text{and} \quad ||\psi^{(k)}_{l,n}||_{\sigma_k,\tau_n}, ||\psi^{(k)}_n||_{\sigma_k,\tau_n} \le C'_k.$$

□

5 Examples

As an application of our results, we consider the following function:

$$f(z) = P(z)e^{Q(z)}, \quad P(z) = b_m z^m + \cdots + b_0, \quad Q(z) = a_d z^d + \cdots + a_1 z + a_0$$
$$m = \deg P \geq 0, \ d = \deg Q \geq 1, \ (a_d \neq 0, \ b_m \neq 0).$$

By a linear change of coordinate and multiplying P by e^{a_0}, we may assume that $a_d = 1$ and $a_0 = 0$. Since the function $f(z) = P(z)e^{Q(z)}$ is structurally finite, we can define the itinerary $\boldsymbol{s} \in (\{0, 1, \cdots, d-1\} \times \mathbb{Z})^{\mathbb{N}}$, where $d = \deg Q$. See Figure 2. For the details, see [**Ki**]. So by taking $f_n : U_n \to V_n$ to be the restriction of f to a suitable domain U_n according to $\boldsymbol{s}$, we can apply our results for general setting and obtain the smooth hair $h_{\boldsymbol{s}}(t)$ corresponding to $\boldsymbol{s}$. Here for simplicity, we consider only a fixed itinerary $\boldsymbol{s}$, that is, a constant sequence of a single symbol. So the hair $h_{\boldsymbol{s}}(t)$ is invariant as a set for this $\boldsymbol{s}$. Also $f_0 = f_1 = \cdots = f_n = \cdots$ and this is a restriction of f to a suitable domain $U_0 = U_1 = \cdots = U_n = \cdots$.

Let $g(t) = t^m e^{t^d}$ be the "reference function" to compare.

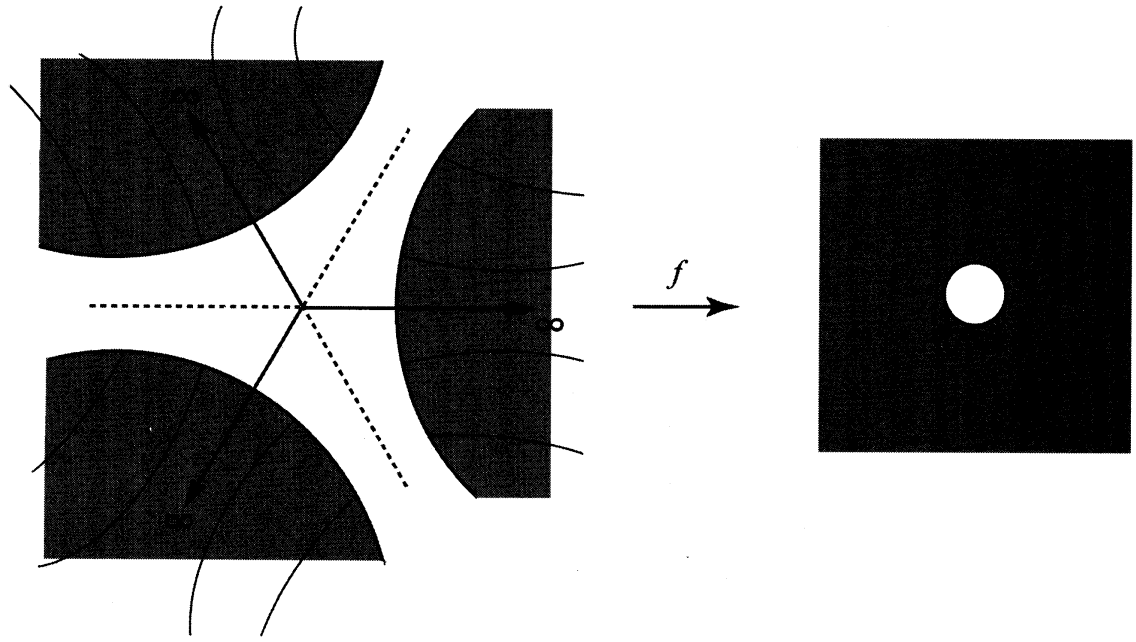

Figure 2. The case of $d = 3$

Lemma 5.1. *For any $\varepsilon > 0$, there exists $R > 0$ such that for $t \in \mathbb{C}$ with $|t| \geq R$, there exists a unique $w = w(t)$ such that $|w| < \varepsilon$, $P(t(1+w))e^{Q(t(1+w))} = t^m e^{t^d}$ and $|tw| \leq C$, where C is a constant.* □

To apply the previous result, we change the notation as follows: We set

$$h_{0,0}(t) = h_{1,1}(t) = \cdots = h_{n,n}(t) = \cdots$$

and denote this by $h_0(t)$. Also we set $h_n(t) := h_{n,0}(t)$. Then by using the function $w(t)$ in Lemma 5.1, we define $h_0(t)$ and start constructing $h_n(t)$.

Proposition 5.2. *There exist* $\tau_* > 0$ *and* C^∞*-function* $h_0 : [\tau*, \infty) \to \mathbb{C}$ *such that* $h_0'(t) \neq 0$ *and*

$$f \circ h_0(t) = g(t) \ (= t^m e^{t^d}) \tag{24}$$

$$h_0(t) := t(1 + w(t)) = t + O(1) \quad (as\ t \to \infty) \tag{25}$$

$$(\log h_0'(t))^{(k)} = O\left(\frac{1}{t^{k+2}}\right) \quad (k = 0, 1, 2, \dots). \tag{26}$$

Moreover h_0, $h_1 := f^{-1}(h_0 \circ g)$ *satisfies* **A** *and* **B** *with* $R(t) = \dfrac{const}{t^{d-1}g(t)}$. □

Proposition 5.3. *Let* $\sigma_k(t) = t^{k+2}$ $(k = 0, 1, 2, \dots)$. *Suppose that* $\rho_k(t)$ $(k = 0, 1, 2, \dots)$ *satisfy*

$$\sigma_k(t) \leq \rho_k(t) \tag{27}$$

$$\limsup_{t\to\infty} \frac{\rho_k(t)t^{k(d-1)}(g(t))^k}{\rho_k(g(t))} < 1 \tag{28}$$

$$\rho_k(t) \leq const\, \frac{\rho_\ell(g(t))}{t^{k(d-1)}(g(t))^\ell} \quad (1 \leq \ell < k) \tag{29}$$

$$\rho_k(t) \leq const \cdot t^k g(t) \tag{30}$$

$$\rho_k(t) \leq const\, \frac{\rho_0(t)}{t^{d-k}} \quad (k \geq 1) \tag{31}$$

$$\rho_k(t) \leq const\, \frac{\rho_j(t)}{t^{d+j-1}} \quad (1 \leq j < k). \tag{32}$$

Then $\mathbf{C}_j$ $(0 \leq j \leq k)$, $\mathbf{D}_j$ $(0 \leq j \leq k)$, $\mathbf{E}_j$ $(1 \leq j \leq k)$ *and* $\mathbf{F}_j$ $(0 \leq j \leq k)$ *are satisfied.* □

Corollary 5.4. *For a suitable choice of const and* $\mu_k > 0$, $\rho_k(t) = const\dfrac{e^{\varepsilon t}}{t^{\mu_k}}$ *satisfies the hypothesis.* □

References

[**DK**] R. Devaney, M. Krych, *Dynamics of Exp(z)*, Ergod. Th. & Dynam. Sys. **4** No.1 (1984), 35–52.

[**F**] N. Fagella, *Limiting dynamics of the complex standard family*, Int. J. of Bif. and Chaos **3** (1995), 673–700.

[**Ki**] M. Kisaka, *Dynamics of structurally finite transcendental entire functions.*, Studies on complex dynamics and related topics (Kyoto, 2000), Sūrikaisekikenkyū-sho Kōkyūroku, No.1220 (2001), 17–25.

[**V**] M. Viana da Silva, *The differentiability of the hairs of exp(Z)*, Procs. Amer. Math. Soc. **103** No. 4 (1988), 1179-1184.