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NATURAL TRANSFORMATIONS ASSOCIATED TO ADDITIVE HOMOLOGY CLASSES

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1. INTRODUCTION

For a topological space X a homology class $\alpha(X)$ shall be called *additive* if we have that $\alpha(X \sqcup Y) = \alpha(X) + \alpha(Y)$. Almost all invariants, for example, Euler–Poincaré characteristic, signature, all the classical characteristic cohomology classes of manifolds, etc. are additive. When it comes to the case of singular spaces, characteristic classes such as Chern–Schwartz–MacPherson class [23], Baum–Fulton–MacPherson's Todd class [9], Goresky–MacPherson's L-class [20], Cappell–Shaneson's L-class [15] are also additive. In fact, these characteristic (co)homology classes are all formulated as *natural transformations from suitable (contravariant) covariant functors to the (co)homology theory*. This is an important or key aspect of characteristic (co)homology classes.

Besides these characteristic classes formulated as natural transformations, there are several important homology classes which are *usually not formulated as such natural transformations*; for example,

- Chern-Mather class $c_*^M(X)$ (e.g., [23]),
- Segre–Mather class $s_*^M(X)$ (e.g., [38]),
- Fulton's canonical Chern class $c_*^F(X)$ ([17]),
- Fulton–Johnson's Chern class $c_*^{FJ}(X)$ ([18]),
- Milnor class $\mathcal{M}(X)$ (e.g., [1], [11], [25], [40], etc.),
- Aluffi class α_X ([2], [10]), etc.

In [43] we captured Fulton–Johnson's Chern class as a natural transformation and also captured the Milnor class $\mathcal{M}(X)$ as a natural transformation, which is a special case of the Hirzebruch–Milnor class (also see [14]), using the motivic Hirzebruch class [12].

Motivated by the construction or approach in [43], in [47] we generalize the results of [43] in more general situations and also we consider very abstract situations in category-functor.

In this paper we give a survey of our results of [47] and finally we make a remark on the recent theory of *Intersection Spaces* due to Markus Banagl [5] (see also [4]).

2. SOME BACKGROUNDS

Theories of characteristic classes of singular spaces which have been developed so far are all formulated as *natural transformations from certain covariant functors $\mathcal{F}$ to the homology theory H_*, satisfying a normalization condition that for a smooth variety X the value of a distinguished element Δ_X of $\mathcal{F}(X)$ is equal to the Poincaré dual of the corresponding characteristic cohomology class of the tangent bundle*:

$$\tau_{c\ell} : \mathcal{F}(-) \to H_*(-) \quad \text{such that for } X \text{ smooth} \quad \tau_{c\ell}(\Delta_X) = c\ell(TX) \cap [X].$$

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Here are the three well-known and well-studied ones:

(1) MacPherson's Chern class [23] is the unique natural transformation

$$c_*^{Mac} : F(X) \to H_*(X)$$

satisfying the normalization condition that for a smooth variety X the value of the characteristic function is the Poincaré dual of the total Chern class of the tangent bundle: $c_*^{Mac}(1\!\!1_X) = c(TX) \cap [X]$.

(2) Baum–Fulton–MacPherson's Todd class [9] is the unique natural transformation

$$td_*^{BFM} : G_0(X) \to H_*(X) \otimes \mathbb{Q}$$

satisfying the normalization condition that for a smooth variety X the value of the sturcture sheaf is the Poincaré dual of the total Todd class of the tangent bundle: $td_*^{BFM}(\mathcal{O}_X) = td(TX) \cap [X]$.

(3) Goresky– MacPherson's homology L-class [20], which is extended as a natural transformation by Sylvain Cappell and Julius Shaneson [15] (also see [39]), is the unique natural transformation

$$L_*^{CS} : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$$

satisfying the normalization condition that for a smooth variety X the value of the shifted constant sheaf is the Poincaré dual of the total Hirzebruch–Thom's L-class of the tangent bundle: $L_*^{CS}(\mathbb{Q}_X[\dim X]) = L(TX) \cap [X]$.

The *motivic Hirzeruch class* constructed in [12] (see also [29], [28] and [44]) in a sense unifies these three theories c_*^{Mac}, td_*^{BFM} and L_*^{CS}.

Let $\mathcal{C}$ be a category of topological spaces with some additional structures, such as the category of complex algebraic varieties, etc. An additive function on objects $Obj(\mathcal{C})$ with values in R-homology classes is a function α such that

- $\alpha(X) \in H_*(X; R)$
- $\alpha(X \sqcup Y) = \alpha(X) + \alpha(Y)$. More precisely,

$$\alpha(X \sqcup Y) = (\iota_X)_*\alpha(X) + (\iota_Y)_*\alpha(Y)$$

with $\iota_X : X \to X \sqcup Y$, $\iota_Y : Y \to X \sqcup Y$ being the inclusions.

A categorification of the additive function α is meant to be *an associated natural transformation from a certain covariant functor* $\diamondsuit(-)$ *to the homology theory* $H_*(-; R)$

$$\tau_\alpha : \diamondsuit(-) \to H_*(-; R)$$

such that for some distinguished element $\delta_X \in \diamondsuit(X)$ *of a special space* X

$$\tau_\alpha(\delta_X) = \alpha(X).$$

To construct such a covariant functor $\diamondsuit(-)$, we introduce *generalized relative Grothendieck groups,* using comma categories in a more abstract category-functorial situation. The construction of such a covariant functor is hinted by the definition of the relative Grothendieck group $K_0(\mathcal{V}_{\mathbb{C}}/X)$ and more clearly by the description of the oriented bordism group $\Omega_*(X)$. This bordism group $\Omega_m(X)$ of a topological space X is defined to be the free abelian group generated by the isomorphism classes $[M \xrightarrow{h} X]$ of *continuous maps* $M \xrightarrow{h} X$ from *closed oriented smooth manifolds* M of dimension m to the given *topological space* X, modulo the following relations

(1) $[M \xrightarrow{h} X] + [M' \xrightarrow{h'} X] = [M \sqcup M' \xrightarrow{h+h'} X]$,

(2) $0 = [\emptyset \to X]$,

(3) if $M \xrightarrow{h} X$ and $M' \xrightarrow{h'} X$ are *bordant*, then $[M \xrightarrow{h} X] = [M' \xrightarrow{h'} X]$.

In the definition of the bordism group two categories are involved:

- the category $co\mathcal{C}^\infty$ of closed oriented smooth manifolds,
- the category $\mathcal{TOP}$ of topological spaces

Here we emphasize that *even though we consider a finer category $co\mathcal{C}^\infty$ for a source space M the map $h : M \to X$ of course has to be considered in the cruder category $\mathcal{TOP}$.*

The bordism group $\Omega_*(-)$ is a covariant functor

$$\Omega_* : \mathcal{TOP} \to \mathcal{AB},$$

where $\mathcal{AB}$ is the category of abelian groups. We can consider this covariant functor on a different category finer than the category $\mathcal{TOP}$ of topological spaces, e.g., the category $\mathcal{V}_{\mathbb{C}}$ of complex algebraic varieties. Namely we consider continuous maps $h : M \to V$ from closed oriented manifolds M to a complex algebraic variety V, and we get a covariant functor

$$\Omega_* : \mathcal{V}_{\mathbb{C}} \to \mathcal{AB}.$$

In this set-up *three different categories $co\mathcal{C}^\infty$, $\mathcal{TOP}$* and $\mathcal{V}_{\mathbb{C}}$ are involved, i.e., we have the following forgetful functors

$$co\mathcal{C}^\infty \xrightarrow{\mathfrak{f}_s} \mathcal{TOP} \xleftarrow{\mathfrak{f}_t} \mathcal{V}_{\mathbb{C}}$$

where "s" and "t" mean "source object" and "target object".

A commutative triangle

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "h"'] & & M' \arrow[dl, "h'"] \\ & V & \end{tikzcd}$$

really means a commutative triangle in the base category $\mathcal{TOP}$:

$$\begin{tikzcd} \mathfrak{f}_s(M) \arrow[rr, "\mathfrak{f}_s(\phi)"] \arrow[dr, "h"'] & & \mathfrak{f}_s(M') \arrow[dl, "h'"] \\ & \mathfrak{f}_t(V). & \end{tikzcd}$$

More generally we can deal with a cospan $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ of categories $\mathcal{C}_s, \mathcal{C}_t, \mathcal{B}$ equipped with coproduct structures:

From this *cospan* $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ we get *the canonical generalized $(\mathfrak{S}, \mathfrak{T})$-relative Grothendeick groups* $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-))$ and also from the following commutative diagrams of categories and functors

$$\begin{tikzcd} \mathcal{C}_s \arrow[r, "\mathfrak{S}"] \arrow[dr, "\mathfrak{S}'"'] & \mathcal{B} \arrow[d, "\Phi"] & \mathcal{C}_t \arrow[l, "\mathfrak{T}"'] \arrow[dl, "\mathfrak{T}'"] \\ & \mathcal{B}' & \end{tikzcd}$$

we obtain *a categorification of an additive function $\alpha(X)$ on objects $Obj(\mathcal{C}_s)$ with values $\alpha(X) \in \mathfrak{T}'(X)$*:

$$\tau_\alpha : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-)) \to \mathfrak{T}'(-).$$

In particular, for the following commutative diagram

$$\begin{tikzcd} \mathcal{C}_s \arrow[r, tail, "\mathfrak{S}"] \arrow[dr, "\mathfrak{S}'"'] & \mathcal{B} \arrow[d, "\Phi"] & \mathcal{C}_s \arrow[l, tail, "\mathfrak{S}"'] \arrow[dl, "\mathfrak{S}'"] \\ & \mathcal{B}' & \end{tikzcd}$$

with $\mathfrak{S} : \mathcal{C} \to \mathcal{B}$ being a full functor, then the natural transformation $\tau_\alpha : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-)) \to \mathfrak{S}'(-)$ satisfying the condition that $\tau_\alpha([(V, V, id_V)]) = \alpha(V) \in \mathfrak{S}'(V)$ for $V \in Obj(\mathcal{C}_s)$ is unique.

We apply these to geometric situations and in particular all additive homology classes such as characterisitic classes cited above are captured as natural transformations (cf. [41]).

3. Generalized relative Grothendieck groups

Definition 3.1. Let $\mathcal{C}$ be a bimonoidal category equipped with two monoidal structures $\oplus$ with unit $\emptyset$ and $\otimes$ with unit 1. The Grothendieck group $K(\mathcal{C})$ is defined to be the free abelian group generated by the isomophism classes $[X]$ of objects $X \in Obj(\mathcal{C})$ modulo the relations

$$[X] + [Y] = [X \oplus Y], \quad 0 = [\emptyset].$$

If we furthermore define

$$[X] \times [Y] := [X \otimes Y],$$

then the Grothedieck group $K(\mathcal{C})$ becomes a ring, called the Grothendieck ring of the bimonoidal category.

Example 3.2. The category of sets, the category of topological spaces, the category of manifolds, etc. are bimonoidal categories with the disjoint sum and the Cartesian product.

A functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ of two monoidal categories is a functor which preserves $\oplus$ and $\otimes$ in the relaxed sense that there are natural transformations:

$$\Phi(A) \oplus_{\mathcal{C}_2} \Phi(B) \to \Phi(A \oplus_{\mathcal{C}_1} B),$$

$$\Phi(A) \otimes_{\mathcal{C}_2} \Phi(B) \to \Phi(A \otimes_{\mathcal{C}_1} B).$$

In some usage it requires both isomorphisms

$$\Phi(A) \oplus_{\mathcal{C}_2} \Phi(B) \cong \Phi(A \oplus_{\mathcal{C}_1} B)$$

$$\Phi(A) \otimes_{\mathcal{C}_2} \Phi(B) \cong \Phi(A \otimes_{\mathcal{C}_1} B),$$

in which case it is sometimes called a *strong* monoidal functor. However, the cases with which we deal satisfy that as to the monidal structure $\oplus$ we have the isomorphism $\Phi(A) \oplus_{\mathcal{C}_2} \Phi(B) \cong \Phi(A \oplus_{\mathcal{C}_1} B)$, but possibly we have $\Phi(A) \otimes_{\mathcal{C}_2} \Phi(B) \ncong \Phi(A \otimes_{\mathcal{C}_1} B)$, as given in the following example.

Example 3.3. Let $H_*(-) : \mathcal{TOP} \to \mathcal{AB}$ be the integral homology functor. Then we have

$$H_*(X \sqcup Y) \cong H_*(X) \oplus H_*(Y),$$

but in general we have

$$H_*(X \times Y) \ncong H_*(X) \otimes H_*(Y)$$

and we have just a cross product homomorphism

$$\times : H_*(X) \otimes H_*(Y) \to H_*(X \times Y).$$

However, for a field k, the k-coefficient homology functor $H_*(-;k) = H_*(-) \otimes k : \mathcal{TOP} \to \mathcal{AB}$ is a strong monoidal functor, i.e., we do have the isomorphism

$$H_*(X;k) \otimes H_*(Y;k) \cong H_*(X \times Y;k),$$

which is the *Künneth Theorem.*

Lemma 3.4. (1) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories equipped with coproduct structures $\sqcup$ and let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor preserving the coproduct structure strongly, i.e., $\Phi(A \sqcup B) = \Phi(A) \sqcup \Phi(B)$ for any objects A, B in $\mathcal{C}_1$. Then the map*

$$\Phi_* : K(\mathcal{C}_1) \to K(\mathcal{C}_2), \quad \Phi_*([X]) := [\Phi(X)]$$

is well-defined and a group homomorphism. Namely, the Grothendieck group K is a covariant functor from the category of such categories and functors to the category of abelian groups.

(2) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two bimonoidal categories equipped with coproduct structures and product structures and let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a strong monoidal functor. Then the map $\Phi_* : K(\mathcal{C}_1) \to K(\mathcal{C}_2)$ is a ring homomorphism.*

Definition 3.5. Let

$$\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$$

be two functors among the three categories $\mathcal{C}_s$, $\mathcal{C}_t$ and $\mathcal{B}$. This shall be called *a cospan of categories*. The *comma category* $(\mathfrak{S} \downarrow \mathfrak{T})$ (e.g., see [22]) is defined by

- $Obj((\mathfrak{S} \downarrow \mathfrak{T}))$ consists of triples (V, X, h) with

$$V \in Obj(\mathcal{C}_s),\ X \in Obj(\mathcal{C}_t),\ h \in Hom_{\mathcal{B}}(\mathfrak{S}(V), \mathfrak{T}(X))$$

- $Hom_{(S \downarrow T)}((V, X, h), (V', X', h'))$ consists of the pairs (g_s, g_t) where

$$g_s : V \to V' \in Hom_{\mathcal{C}_s}(V, V'),\ g_t : X \to X' \in Hom_{\mathcal{C}_t}(X, X')$$

such that the following diagram commutes in the base category $\mathcal{B}$:

$$\begin{tikzcd} \mathfrak{S}(V) \arrow[r, "\mathfrak{S}(g_s)"] \arrow[d, "h"'] & \mathfrak{S}(V') \arrow[d, "h'"] \\ \mathfrak{T}(X) \arrow[r, "\mathfrak{T}(g_t)"'] & \mathfrak{T}(X') \end{tikzcd}$$

Definition 3.6. Let $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ be a cospan and let $(\mathfrak{S} \downarrow \mathfrak{T})$ be the above comma category associated to the cospan. We define the canonical *projection functors* as follows:

(1) $\pi_t : (\mathfrak{S} \downarrow \mathfrak{T}) \to \mathcal{C}_t$ is defined by
   - for an object (V, X, h), $\pi_t((V, X, h)) := X$,
   - for a morphism $(g_s, g_t) : (V, X, h) \to (V', X', h')$, $\pi_t((g_s, g_t)) := g_t$.

(2) $\pi_s : (\mathfrak{S} \downarrow \mathfrak{T}) \to \mathcal{C}_s$ is defined by
   - for an object (V, X, h), $\pi_s((V, X, h)) := V$,
   - for a morphism $(g_s, g_t) : (V, X, h) \to (V', X', h')$, $\pi_s((g_s, g_t)) := g_s$.

Namely *a cospan of categories* $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ induces *a span of categories*

$$\mathcal{C}_s \xleftarrow{\pi_s} (\mathfrak{S} \downarrow \mathfrak{T}) \xrightarrow{\pi_t} \mathcal{C}_t.$$

Definition 3.7. (e.g. see [22]) Let $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ be a functor of two categories. Then, for an object $B \in Obj(\mathcal{D})$, *the fiber category* of $\mathfrak{F}$ over B, denoted by $\mathfrak{F}^{-1}(B)$, is defined to be the category consisting of

- $Obj(\mathfrak{F}^{-1}(B)) = \{X \in Obj(\mathcal{C}) \mid \mathfrak{F}(X) = B\}$,
- $Hom_{\mathfrak{F}^{-1}(B)}(X, X') = \{f \in Hom_{\mathcal{C}}(X, X') \mid \mathfrak{F}(f) = id_B\}$.

(In this sense it would be better to denote the fiber category by $\mathfrak{F}^{-1}(B, id_B)$ instead of $\mathfrak{F}^{-1}(B)$.)

Example 3.8. As above, let us consider a cospan of categories and its associated span of categories:

$$\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t, \qquad \mathcal{C}_s \xleftarrow{\pi_s} (\mathfrak{S} \downarrow \mathfrak{T}) \xrightarrow{\pi_t} \mathcal{C}_t.$$

(1) For an object $X \in \mathcal{C}_t$, the fiber category $\pi_t^{-1}(X)$ is nothing but the $\mathfrak{S}$*-over category* $(\mathfrak{S} \downarrow \mathfrak{T}(X))$, whose objects are *objects* $\mathfrak{S}$*-over* $\mathfrak{T}(X)$, i.e., the triple (V, X, h), and for two triples (V, X, h) and (V', X, h') a morphism from (V, X, h) to (V', X, h') is $g_s \in Hom_{\mathcal{C}_s}(V, V')$ such that the following triangle commutes:

$$\begin{tikzcd} \mathfrak{S}(V) \arrow[rr, "\mathfrak{S}(g_s)"] \arrow[dr, "h"'] & & \mathfrak{S}(V') \arrow[dl, "h'"] \\ & \mathfrak{T}(X). & \end{tikzcd}$$

(2) Furthermore, if $\mathcal{C}_s = \mathcal{B}$ and $=id_{\mathcal{B}}$ is the identity functor, then the above *S-over category* $(\mathfrak{S} \downarrow X)$ is the standard *over category* $(\mathcal{B} \downarrow X)$, whose objects are objects over X, i.e., morphisms $h : V \to X$, and for two tmorphisms $h : V \to X$ and $h : V' \to X$ a morphism from $h : V \to X$ to $h : V' \to X$ is $g \in Hom_{\mathcal{B}}(V, V')$ such that the following triangle commutes:

$$\begin{tikzcd} V \arrow[rr, "g"] \arrow[dr, "h"'] & & V' \arrow[dl, "h'"] \\ & X. & \end{tikzcd}$$

(3) For an object $V \in \mathcal{C}_s$, the fiber category $\pi_s^{-1}(V)$ is nothing but the $\mathfrak{T}$*-under category* $(\mathfrak{S}(V) \downarrow \mathfrak{T})$, whose objects are *objects* $\mathfrak{T}$*-under* $\mathfrak{S}(V)$, i.e., the triple (V, X, h), and for two triples (V, X, h) and (V, X', h') a morphism from (V, X, h) to (V, X', h') is $g_t \in Hom_{\mathcal{C}_t}(X, X')$ such that the following triangle commutes:

$$\begin{tikzcd} & \mathfrak{S}(V) \arrow[dl, "h"'] \arrow[dr, "h'"] & \\ \mathfrak{T}(X) \arrow[rr, "\mathfrak{T}(g_t)"'] & & \mathfrak{T}(X'). \end{tikzcd}$$

Similarly, we can think of the $\mathfrak{T}$*-under category* $(V \downarrow \mathfrak{T})$ and the *under category* $(V \downarrow \mathcal{B})$.

Proposition 3.9. *Let $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ be a cospan of categories. Then a morphism $f \in Hom_{\mathcal{C}_t}(X_1, X_2)$ gives rise to the functor between the corresponding fiber categories:*

$$\mathfrak{T}(f)_* : \pi_t^{-1}(X_1) \to \pi_t^{-1}(X_2),$$

which is defined by

(1) *For an object* (V, X_1, h), $\mathfrak{T}(f)_*((V, X_1, h)) := (V, X_2, \mathfrak{T}(f) \circ h)$.
(2) *For a morphism* $(g_s, id_{X_1}) : (V, X_1, h) \to (V', X_1, h')$ *with* $g_s \in Hom_{\mathcal{C}_s}(V, V')$,

$$\mathfrak{T}(f)_*((g_s, id_{X_1})) := (g_s, id_{X_2}) : (V, X_2, \mathfrak{T}(f) \circ h) \to (V', X_2, \mathfrak{T}(f) \circ h').$$

$$\begin{tikzcd}
\mathfrak{S}(V) \arrow[rr, "g"] \arrow[dr, "h"] \arrow[ddr, "\mathfrak{T}(f)\circ h"'] & & \mathfrak{S}(V') \arrow[dl, "h'"'] \arrow[ddl, "\mathfrak{T}(f)\circ h'"] \\
& \mathfrak{T}(X_1) \arrow[d, "\mathfrak{T}(f)"] & \\
& T(X_2) &
\end{tikzcd}$$

Lemma 3.10. *Let* $\mathcal{C}_s, \mathcal{C}_t, \mathcal{B}$ *be two categories equipped with coproduct structures and let* $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ *be a cospan of categories. Assume that both functors* $\mathfrak{S}$ *and* $\mathfrak{T}$ *preserve the coproduct structures strongly, i.e.,* $\mathfrak{S}(V \sqcup V') = \mathfrak{S}(V) \sqcup \mathfrak{S}(V')$ *and* $\mathfrak{T}(X \sqcup X') = \mathfrak{T}(X) \sqcup \mathfrak{T}(X')$. *Then for each object* $X \in Obj(\mathcal{C}_t)$ *the fiber category* $\pi_t^{-1}(X)$, *i.e., the* $\mathfrak{S}$*-over category* $(\mathfrak{S} \downarrow \mathfrak{T}(X))$ *is a category equipped with the coproduct structure*

$$(V, X, h) \sqcup (V', X, h') := (V \sqcup V', X, h + h').$$

Corollary 3.11. *Let the situation be as above. A morphism* $f \in Hom_{\mathcal{C}_t}(X_1, X_2)$ *gives rise to the canonical group homomorphism*

$$\mathfrak{T}(f)_* =: K(\pi_t^{-1}(X_1)) \to K(\pi_t^{-1}(X_2)),$$

and

$$K(\pi_t^{-1}(-)) : \mathcal{C}_t \to \mathcal{AB}$$

is a covariant functor from the category $\mathcal{C}_t$ *to the category of abelian groups.*

Definition 3.12 (*Generalized relative Grothendieck groups with respect to a cospan of categories*).

(1) Let $\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$ be functors of categories equipped with coproduct structures and for an object $X \in \mathcal{C}_t$, the Grothendieck group of the fiber category of the projection functor $\pi_t : (\mathfrak{S} \downarrow \mathfrak{T}) \to \mathcal{C}_t$ is denoted by

$$K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X)) := K(\pi_t^{-1}(X)))$$

and called *the generalized* $(\mathfrak{S}, \mathfrak{T})$*-relative Grothendieck group of* X. This is a covariant functor from $\mathcal{C}_t$ to $\mathcal{AB}$.

(2) If $\mathcal{C}_t = \mathcal{B}$ and $T = id_{\mathcal{B}}$, then $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X))$ is simply denoted by $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/X)$.

(3) If $\mathfrak{S} = \mathfrak{T} = id_{\mathcal{C}_s} : \mathcal{C}_s \to \mathcal{C}_s$ is the identity functor, then the above $id_{\mathcal{C}}$-relative Grothendieck group $K(\mathcal{C} \xrightarrow{id_{\mathcal{C}_s}} \mathcal{C}_s/X)$ is simply denoted by

$$K(\mathcal{C}_s/X)$$

and called *the relative Grothendieck group of* X.

Remark 3.13. If X is the terminal object pt in the category of $\mathcal{C}_t$, then all the above relative Grothendieck groups is isomophic to the Grothendieck group $K(\mathcal{C}_s)$ of the category $\mathcal{C}_s$:

$$K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(pt)) \cong K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/pt) \cong K(\mathcal{C}_s/pt) \cong K(\mathcal{C}_s).$$

Proposition 3.14. *Let $\mathcal{C}_s, \mathcal{C}'_s, \mathcal{C}_t, \mathcal{C}'_t, \mathcal{B}, \mathcal{B}'$ be categories equipped with coproduct structures and suppose we have the following commutative diagrams of functors among them:*

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r, "\mathfrak{S}"] \arrow[d, "\Phi_s"] & \mathcal{B} \arrow[d, "\Phi_b"] & \mathcal{C}_t \arrow[l, "\mathfrak{T}"'] \arrow[d, "\Phi_t"] \\
\mathcal{C}'_s \arrow[r, "\mathfrak{S}'"'] & \mathcal{B}' & \mathcal{C}'_t, \arrow[l, "\mathfrak{T}'"]
\end{tikzcd}$$

(1) *We have the canonical functor of two comma categories $(\mathfrak{S} \downarrow \mathfrak{T})$ and $(\mathfrak{S}' \downarrow \mathfrak{T}')$:*

$$\Phi : (\mathfrak{S} \downarrow \mathfrak{T}) \to (\mathfrak{S}' \downarrow \mathfrak{T}'),$$

which is defined naturally as follows:

(a) *for an object $(V, X, h) \in Obj((\mathfrak{S} \downarrow \mathfrak{T}))$,*

$$\Phi((V, X, h)) := (\Phi_s(V), \Phi_t(X), \Phi_b(h)),$$

(b) *for a morphism $g; : (V, X, h) \to (V', X', h')$ with $g \in Hom_{\mathcal{C}_s}(V, V')$*

$$\Phi(g) := \Phi_s(g).$$

(2) *In the following special case*

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r, "\mathfrak{S}"] \arrow[d, "id_{\mathcal{C}_s}"] & \mathcal{B} \arrow[d, "\Phi_b"] & \mathcal{C}_t \arrow[l, "\mathfrak{T}"'] \arrow[d, "id_{\mathcal{C}_t}"] \\
\mathcal{C}_s \arrow[r, "\mathfrak{S}'"'] & \mathcal{B}' & \mathcal{C}_t, \arrow[l, "\mathfrak{T}'"]
\end{tikzcd}$$

the covariant functor $\Phi : (\mathfrak{S} \downarrow \mathfrak{T}) \to (\mathfrak{S}' \downarrow \mathfrak{T}')$ gives rise to the canonical natural transformation from the functor $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-)) : \mathcal{C}_t \to \mathcal{AB}$ to the functor $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}'} \mathcal{B}'/\mathfrak{T}'(-)) : \mathcal{C}_t \to \mathcal{AB}$:

$$\Phi_* : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-)) \to K(\mathcal{C}_s \xrightarrow{\mathfrak{S}'} \mathcal{B}'/\mathfrak{T}'(-))$$

i.e., for a morphism $f \in Hom_{\mathcal{C}_t}(X_1, X_2)$ the following diagram commutes in the category $\mathcal{AB}$:

$$\begin{tikzcd}
K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X_1)) \arrow[r, "\Phi_*"] \arrow[d, "\mathfrak{T}(f)_*"'] & K(\mathcal{C}_s \xrightarrow{\mathfrak{S}'} \mathcal{B}'/\mathfrak{T}'(X_1)) \arrow[d, "\mathfrak{T}'(f)_*"] \\
K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X_2)) \arrow[r, "\Phi_*"'] & K(\mathcal{C}_s \xrightarrow{\mathfrak{S}'} \mathcal{B}'/\mathfrak{T}'(X_2)),
\end{tikzcd}$$

Here $\Phi_* : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X)) \to K(\mathcal{C}_s \xrightarrow{\mathfrak{S}'} \mathcal{B}'/\mathfrak{T}'(X))$ is defined by

$$\Phi_*([(V, X, h)] := [(V, X, \Phi_b(h))].$$

Theorem 3.15 (A "categorification" of an additive function on the objects). *Let the situation be as in Proposition 3.14 and suppose that $\mathcal{B}'$ is the category $\mathcal{AB}$ of abelian groups. Furthermore suppose that there is a function α on $Obj(\mathcal{C}_s)$ such that*

- $\alpha(V) \in \mathfrak{S}'(V)$

- α *is* additive, *i.e..,* $\alpha(V \sqcup V') = \alpha(V) + \alpha(V')$, *more precisely,*
$$\alpha(V \sqcup V') = \mathfrak{S}'(\iota_V)(\alpha(V)) + \mathfrak{S}'(\iota_{V'})(\alpha(V')),$$
where $\iota_V : V \to V \sqcup V'$ *and* $\iota_{V'} : V' \to V \sqcup V'$ *are the inclusions.*

Then the function α *can be turned into the following two natural transformations:*

(1) $\tau_\alpha : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-)) \to \mathfrak{T}'(-)$ *on the category* $\mathcal{C}_t$,
$$\tau_\alpha([V, X, h]) := \Phi_b(h)(\alpha(V)) \in \mathfrak{T}'(X).$$

(2) $\tau_\alpha : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{S}(-)) \to \mathfrak{S}'(-)$ *on the category* $\mathcal{C}_s$,
$$\tau_\alpha([V, X, h]) := \Phi_b(h)(\alpha(V)) \in \mathfrak{S}'(X).$$

Here we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}_s & \xrightarrow{\mathfrak{S}} & \mathcal{B} & \xleftarrow{\mathfrak{S}} & \mathcal{C}_s \\ \downarrow{\scriptstyle id_{\mathcal{C}_s}} & & \downarrow{\scriptstyle \Phi_b} & & \downarrow{\scriptstyle id_{\mathcal{C}_s}} \\ \mathcal{C}_s & \xrightarrow[\mathfrak{S}']{} & \mathcal{B}' & \xleftarrow[\mathfrak{S}']{} & \mathcal{C}_s, \end{array}$$

And if there is a natural transformation $\tau'_\alpha : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{S}(-)) \to \mathfrak{S}'(-)$ *satisfying the condition that*
$$\tau_\alpha([V, V, id_V]) = \alpha(V) \in \mathfrak{S}'(X),$$
then $\tau'_\alpha([V, X, S(h)]) = \tau_\alpha([V, X, \mathfrak{S}(h)])$ *for any morphism* $h \in Hom_{\mathcal{C}_t}(V, X)$.

(3) *If* $\mathfrak{S} : \mathcal{C}_s \to \mathcal{B}$ *is a full functor, then a natural transformation* $\tau_\alpha : K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{S}(-)) \to \mathfrak{S}'(-)$ *on the category* $\mathcal{C}_s$ *satisfying the condition that*
$$\tau_\alpha([V, V, id_V]) = \alpha(V) \in \mathfrak{S}'(X)$$
is unique.

4. A CATEGORIFICATION OF AN ADDITIVE HOMOLOGY CLASS

From now we will treat categories of topological spaces with some extra structures, such as the category of closed oriented smooth manifolds, the category of complex algebraic varieties, the category of finite CW-complexes, etc. The category $\mathcal{B}'$ is the category $\mathcal{AB}$ of abelian groups and the functor $\Phi_s : \mathcal{C}_s \to \mathcal{AB}$, etc, is the homology functor.

Since we use the homological pushforward $f_* : H_*(X) \to H_*(Y)$ for a continuous map $f : X \to Y$, we require the properness of f. So, we modify the previous generalized relative Grothendieck group with respect to a cospan of categories slightly.

Definition 4.1. (*Generalized "proper" relative Grothendieck groups*) Let $\mathcal{C}_s, \mathcal{C}_t$ and $\mathcal{B}$ be some categories of topological spaces with extra structures which are possibly different respectively, and let
$$\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B} \xleftarrow{\mathfrak{T}} \mathcal{C}_t$$
be a cospan of functors, which are, for example, forgetful functors or inclusion functors, etc. For a space $X \in Obj(\mathcal{C}_t)$
$$K^{prop}(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X))$$
is defined to be the subgroup of the the generalized realtive Grothendieck group $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(X))$ generated by
$$[(V, X, h)]$$

with $h : \mathfrak{S}(V) \to \mathfrak{T}(X)$ being *a proper map*. Similarly we have the "proper" versions:

$$K^{prop}(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/X) \quad \text{and} \quad K^{prop}(\mathcal{C}_s/X).$$

Proposition 4.2. (1) *Let $\mathcal{C}^\infty$ be the category of smooth manifolds and let $\mathfrak{f} : \mathcal{C}^\infty \to \mathcal{TOP}$ be the forgetful functor. Then $K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X)$ (also $K(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X)$) has a cross product structure on the category $\mathcal{TOP}$:*

$$K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X) \otimes K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X) \xrightarrow{\times} K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X \times Y);$$

$$[(V, X, h)] \times [(W, Y, k)] = [V \times W, X \times Y, h \times k)].$$

(2) *Let α be an additive R-homology-class-valued function (simply called* an additive homology class*) on $Obj(\mathcal{C}^\infty)$ with R being a commutative ring, i.e., it satisfies that*

- $\alpha(V) \in H_*(V; R)$ *and*
- $\alpha(V \sqcup V') = (\iota_V)_*(\alpha(V)) + (\iota_{V'})_*(\alpha(V'))$
  where $\iota_V : V \to V \sqcup V'$ and $\iota_{V'} : V \to V \sqcup V'$ are the inclusions.

Then there exists a unique natural transformation

$$\tau_\alpha : K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-; R).$$

satisfying the condition that for a differentiable manifold $V \in Obj(\mathcal{C}^\infty)$

$$\tau_\alpha([(V, \mathfrak{f}(V), id_{\mathfrak{f}(V)})]) = \alpha(V).$$

(3) *If furthermore the additive homology class α is multiplicative, i.e.,*

$$\alpha(V \times V') = \alpha(V) \times \alpha(V'),$$

then $\tau_\alpha : K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-; R)$ commutes with the cross product, i.e., the following diagram commutes:

$$\begin{array}{ccc}
K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X) \otimes K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/Y) & \xrightarrow{\tau_\alpha \times \tau_\alpha} & H_*(X; R) \otimes H_*(Y; R) \\
{\scriptstyle\times}\downarrow & & \downarrow{\scriptstyle\times} \\
K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X \times Y) & \xrightarrow[\tau_\alpha]{} & H_*(X \times Y; R),
\end{array}$$

Corollary 4.3. *Let $\mathcal{C}^\infty_{\mathbb{C}}$ be the category of complex smooth manifolds and let $c\ell(E) \in H^*(X; R)$ be any multplicative characteristic class of complex vector bundles, i.e., $c\ell(E \oplus F) = c\ell(E) \cup c\ell(F)$ for complex vector bundles E, F over the same space. Then there exists a unique natural transformation*

$$\tau_{c\ell} : K^{prop}(\mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-; R)$$

such that for a smooth complex manifold V

$$\tau_{c\ell}([(V, \mathfrak{f}(V), id_{\mathfrak{f}(V)})]) = c\ell(TV) \cap [V].$$

And $\tau_{c\ell}$ is also multiplicative, i.e., for any $[(V, X, h)]$, and $[(W, Y, k)]$ we have

$$\tau_{c\ell}([(V, X, h)] \times [(W, Y, k)]) = \tau_{c\ell}([(V, X, h)]) \times \tau_{c\ell}([(W, Y, k)]).$$

Corollary 4.4. *Let $\mathcal{V}_{\mathbb{C}}$ be the category of complex algebraic varieties and let $\mathcal{SV}_{\mathbb{C}}$ be the category of smooth varieties, which is a full subcategory of $\mathcal{V}_{\mathbb{C}}$ and let $\iota : \mathcal{SV}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}}$ be the inclusion functor and consider the cospan*

$$\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}} \xleftarrow{id_{\mathcal{V}_{\mathbb{C}}}} \mathcal{V}_{\mathbb{C}}.$$

Let $c\ell(E) \in H^*(X;R)$ be any multplicative characteristic class of complex vector bundles. Then there exists a unique natural transformation

$$\tau_{c\ell} : K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/-) \to H_*(-;R)$$

such that for a smooth variety V

$$\tau_{c\ell}([(V,V,id_V)]) = c\ell(TV) \cap [V].$$

And $\tau_{c\ell}$ is also multiplicative, i.e., for any $[(V,X,h)]$, and $[(W,Y,k)]$ we have

$$\tau_{c\ell}([(V,X,h)] \times [(W,Y,k)]) = \tau_{c\ell}([(V,X,h)]) \times \tau_{c\ell}([(W,Y,k)]).$$

Definition 4.5. As above, let $c\ell$ be any multiplicative characteristic class of complex vector bundles. For a complex algebraic variety X *the $c\ell$-Mather homology class $c\ell_*^{Ma}(X)$* is defined to be

$$c\ell_*^{Ma}(X) := \nu_*(c\ell(\widehat{TX}) \cap [\widehat{X}]).$$

Here $\nu : \widehat{X} \to X$ is the Nash blow-up and $\widehat{TX}$ is he tautological Nash tangent bundle over $\widehat{X}$.

Corollary 4.6. *Let the situation be as above.*

(1) *There exists a unique natural transformation*

$$\tau_{c\ell_*^{Ma}} : K^{prop}(\mathcal{V}_{\mathbb{C}}/-) \to H_*(-;R)$$

such that for any variety X we have $\tau_{c\ell_*^{Ma}}([X \xrightarrow{id_X} X]) = c\ell_*^{Ma}(X)$.

(2) *When $c\ell = c$ the Chern class, then the following diagram commutes:*

$$\begin{tikzcd}
K^{prop}(\mathcal{V}_{\mathbb{C}}/X) \arrow[rr, "\mathcal{E}u"] \arrow[dr, "\tau_{c_*^{Ma}}"'] & & F(X) \arrow[dl, "c_*^{Mac}"] \\
& H_*(X,\mathbb{Z}) . &
\end{tikzcd}$$

Here the natural transformation $\mathcal{E}u : K^{prop}(\mathcal{V}_{\mathbb{C}}/X) \to F(X)$ is defined by

$$\mathcal{E}u([V \xrightarrow{h} X]) := h_* Eu_V$$

where Eu_V is the local Euler obstruction of V.

Remark 4.7. (1) Using resolution of singularities one can show that there are finitely many subvarieties V's and integers a_V's such that $1\!\!1_X = \sum_{V \subset X} a_V Eu_V$, thus $c_*^{Mac}(1\!\!1_X) = \sum_{V \subset X} a_V c_*^{Ma}(V)$. Whether X is singular or not, $c_*^{Mac}(1\!\!1_X)$ is called MacPherson's Chern class or Chern–Schwarzt–MacPherson class of X (see [13, 30, 31]), denoted by $c_*^{Mac}(X)$. It follows from the naturality of the transformation that the degree of the 0-dimensional component of $c_0^{Mac}(X)$ is equal to the Euler–Poincaré characteristic:

$$\int_X c_0^{Mac}(X) = \chi(X).$$

(2) On the other hand, the degree of the 0-dimensional component of the Chern–Mather class $c_*^{Ma}(X)$ is the *global Euler obstruction $Eu(X)$*, which was introduced and studied in [32]:

$$\int_X c_0^{Ma}(X) = Eu(X).$$

(3) The above "motivic $c\ell$-Mather class" transformation $\tau_{c\ell_*^{Ma}} : K^{prop}(\mathcal{V}_{\mathbb{C}}/-) \to H_*(-; R)$ could be considered as a very naïve theory of characteristic classes of possibly singular complex algebraic varieties.

So far we dealt with the covariance of the functor $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-))$. Here we discuss the contravariance. In the above general set-up, it seems that $K(\mathcal{C}_s \xrightarrow{\mathfrak{S}} \mathcal{B}/\mathfrak{T}(-))$ cannot become a contravariant functor with a reasonable pullback. So we consider some special cases.

Lemma 4.8. *The functor $K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/-)$ becomes a contravariant functor for smooth morphisms on the category $\mathcal{V}_{\mathbb{C}}$, where for a smooth morphism $f : X \to Y$ the pullback homomorphism*

$$f^* : K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/Y) \to K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/X)$$

is defined by

$$f^*([(V, Y, h)]) := [V', X, h'],$$

where we use the following fiber square

$$\begin{array}{ccc} V' & \xrightarrow{f'} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & V. \end{array}$$

Theorem 4.9 (Verdier-type Riemann–Roch Theorem). *Let the situation be as in Lemma 4.8. Let $c\ell$ be any multiplicative characetristic R-cohomology class of complex vector bundles. Then the natural transformation $\tau_{c\ell} : K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/-) \to H_*(- : R)$ on the category $\mathcal{V}_{\mathbb{C}}$ satisfies the following Verdier-type Riemann–Roch formula: For a smooth morphism $f : X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc} K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/Y) & \xrightarrow{\tau_{c\ell}} & H_*(Y; R) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c\ell(T_f)\cap f^*} \\ K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/X) & \xrightarrow[\tau_{c\ell}]{} & H_*(X; R). \end{array}$$

Now we consider a smaller group

$$K^{prop.sm}(\mathcal{SV}_{\mathbb{C}}/X)$$

which is a subgroup of $K^{prop}(\mathcal{SV}_{\mathbb{C}}/X)$, generated by $[(V, X, h)]$ with $h : V \to X$ being *a proper and smooth map.*

Theorem 4.10 (SGA 6 -type Riemann–Roch Theorem). *On the category $\mathcal{SV}_{\mathbb{C}}$ let us define*

$$T_{c\ell} : K^{prop.sm}(\mathcal{SV}_{\mathbb{C}}/X) \to H^*(X)$$

by

$$T_{c\ell}([V \xrightarrow{h} X]) = \mathcal{PD}_X{}^{-1}(h_*(c\ell(T_h) \cap [V])).$$

Here $\mathcal{PD}_X : H^*(X) \to H_*(X)$ is the Poincaré duality isomorphism given by taking the cup product with the fundamental class. Then the following diagram is commutative for a *smooth morphism* $f : X \to Y$*:*

$$\begin{array}{ccc} K^{prop.sm}(\mathcal{SV}_{\mathbb{C}}/X) & \xrightarrow{T_{c\ell}} & H^*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_!(c\ell(T_f)\cup\ \ )} \\ K^{prop.sm}(\mathcal{SV}_{\mathbb{C}}/Y) & \xrightarrow[T_{c\ell}]{} & H^*(Y). \end{array}$$

Here the Gysin homomorphism $f_! : H^*(X) \to H^*(Y)$ *is defined by*

$$f_! = \mathcal{PD}_Y{}^{-1} \circ f_* \circ \mathcal{PD}_X.$$

5. EXAMPLES

5.1. **The case of fundamental class.** The fundamental class $[-]$ is certainly an additive (and multiplicative) homology class and we have the unique natural transformation on the category $\mathcal{TOP}$ of topological spaces:

$$\tau_{[\ \ ]} : K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-).$$

The classical *Steenrod's realization problem* is asking if the homomorphism $\tau_{[\ \ ]}$ is surjective or not.

The following results are known (see [27]):

- ([35] and [26, Chapter IV, Theorem 7.37])
$$\tau_{[\ \ ]} : K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X) \to \bigoplus_{0\le i\le 6} H_i(X)$$
is surjective.
- ([21]) Let $\mathcal{C}^{\text{Poincaré}}$ be the category of Poincaré complexes, i.e., topological spaces which satisfies the Poincaré duality. Then the following is surjective:
$$\tau_{[\ \ ]} : K^{prop}(\mathcal{C}^{\text{Poincaré}} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X) \to \bigoplus_{i\neq 3} H_i(X).$$
- ([33] and [26, Chapter VIII, Example 1.25(a)]) Let $\mathcal{C}^{\text{pseudo}}$ be the category of pseudo-manifolds. Then the following is surjective:
$$\tau_{[\ \ ]} : K^{prop}(\mathcal{C}^{\text{pseudo}} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/X) \to H_*(X).$$

5.2. **The case of Stiefel–Whitney class.** Let V be a differentiable manifold. For a polynomial $P(w) = P(w_1, w_2, \cdots)$ of Stiefel–Whitney classes $w^*(TV) \in H^*(V,\mathbb{Z}_2)$, we let $P(w)_*(V) \in H_*(V,\mathbb{Z}_2)$ be the Poincaré dual $P(w) \cap [V]$ of $P(w)$. $P(w)_*(V)$ is an additive homology class and we have a unique natural transformation on the category $\mathcal{TOP}$ of topological spaces

$$P(w)_* : K^{prop}(\mathcal{C}^\infty \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-,\mathbb{Z}_2)$$

such that for a differentiable manifold X we have

$$P(w)_*([(X, \mathfrak{f}(X), id_{\mathfrak{f}(X)})]) = P(w)_*(X).$$

In particular the Stiefel–Whitney class w_* is a typical one.

If we restrict ourselves to the category $\mathcal{V}_{\mathbb{R}}$ of real algebraic varieties and we let $\mathcal{SV}_{\mathbb{R}}$ be its full subcategory of smooth real algebraic varieties, then we have a finer natural transformation on the category $\mathcal{V}_{\mathbb{R}}$

$$P(w)_* : K^{prop}(\mathcal{SV}_{\mathbb{R}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{R}}/-) \to H_*(-,\mathbb{Z}_2).$$

In the case when $P(w) = w$, we have the following more geometric "realization" on the category $\mathcal{V}_{\mathbb{R}}$ through *constructible functions*:

$$\begin{tikzcd}
K^{prop}(\mathcal{SV}_{\mathbb{R}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{R}}/X) \arrow[rr, "const"] \arrow[dr, "w_*"'] & & F(X) \arrow[dl, "w_*"] \\
& H_*(X,\mathbb{Z}_2) . &
\end{tikzcd}$$

Remark 5.1. For a Poincaré space Thom constructed a Whitney class using a relation with Steenrod squares [36] (see [24]). Let us call this class Thom-Whitney class, denoted by $w_*^{Th}(X) \in H_*(X;\mathbb{Z}_2)$. Then we have the natural transformation

$$\tau_{w_*}^{Th} : K^{prop}(\mathcal{C}^{\text{Poincaré}} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-;\mathbb{Z}_2)$$

defined by

$$\tau_{w_*}^{Th}([(V,X,h)]) = h_* w_*^{Th}(V).$$

If we consider the above Whitney class natural transformation

$$w_* : K^{prop}(\mathcal{C}^{\infty} \xrightarrow{\mathfrak{f}} \mathcal{C}^{\text{Poincaré}}/-) \to H_*(-;\mathbb{Z})$$

on the category $\mathcal{C}^{\text{Poincaré}}$ of Poincaré spaces, then for a given Poincaré space X it is a natural problem to find a class $\alpha \in K^{prop}(\mathcal{C}^{\infty} \xrightarrow{\mathfrak{f}} \mathcal{C}^{\text{Poincaré}}/X)$ such that

$$w_*(\alpha) = w_*^{Th}(X).$$

5.3. **The case of Pontryagin class.** Let V be a differentiable manifold and let $P(p)_*(V) \in H_*(V,\mathbb{Z})$ be the Poincaré dual of a $\mathbb{Q}$-coefficient polynomial $P(p) = P(p_1, p_2, \cdots)$ of Pontryagin classes $p^*(TV) \in H^*(V,\mathbb{Q})$. $P(p)_*(V)$ is an additive homology class with $\mathbb{Q}$-coefficients: $H_*(-,\mathbb{Q})$ and we have a unique natural transformation on the category $\mathcal{TOP}$

$$P(p)_* : K^{prop}(\mathcal{C}^{\infty} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-,\mathbb{Q})$$

such that for a differentiable manifold V we have

$$P(p)_*([(V,\mathfrak{f}(V), id_{\mathfrak{f}(V)})]) = P(p)_*(V).$$

Here of course we can consider a $\mathbb{Z}$-coefficient polynomial.

Furthermore we have a finer natural transformation on the category $\mathcal{V}_{\mathbb{R}}$

$$P(p)_* : K^{prop}(\mathcal{SV}_{\mathbb{R}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{R}}/-) \to H_*(-,\mathbb{Q}).$$

If we further restrict ourselves to the categories $\mathcal{V}_{\mathbb{C}}$ and $\mathcal{SV}_{\mathbb{C}}$, then we have another finer natural transformation on the category $\mathcal{V}_{\mathbb{C}}$

$$P(p)_* : K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/-) \to H_*(-,\mathbb{Q}).$$

In the case when $P(p) = L$ is Hirzebruch's L-class, we have the following more geometric "realization" on the category $\mathcal{V}_{\mathbb{C}}$ through *Cappell–Shaneson–Youssin–Balmer's* *cobordism groups* $\Omega_*(X)$(see [15], [3], [48]):

$$\begin{tikzcd} K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/X) \arrow[rr, "sd"] \arrow[dr, "L_*"'] & & \Omega(X) \arrow[dl, "L_*"] \\ & H_*(X,\mathbb{Q}) . & \end{tikzcd}$$

Remark 5.2. As in Remark 5.1, for a Poincaré space Thom constructed a Pontryagin class using a relation with the signature (see [24]). Let us call this class Thom–Pontryaging class, denoted by $p_*^{Th}(X) \in H_*(X)$. Then we have the natural transformation

$$\tau_{p_*}^{Th} : K^{prop}(\mathcal{C}^{\text{Poincaré}} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-;\mathbb{Z})$$

defined by

$$\tau_{p_*}^{Th}([(V, X, h)]) = h_* p_*^{Th}(V).$$

If we consider the above Pontryagin class natural transformation

$$p_* : K^{prop}(\mathcal{C}^{\infty} \xrightarrow{\mathfrak{f}} \mathcal{C}^{\text{Poincaré}}/-) \to H_*(-)$$

on the category $\mathcal{C}^{\text{Poincaré}}$ of Poincaré spaces, then for a given Poincaré space X it is a natural problem to find a class $\alpha \in K^{prop}(\mathcal{C}^{\infty} \xrightarrow{\mathfrak{f}} \mathcal{C}^{\text{Poincaré}}/X)$ such that

$$p_*(\alpha) = p_*^{Th}(X).$$

5.4. **The case of Chern class.** Let V be a complex smooth manifold and let $P(c)_*(V) \in H_*(V, \mathbb{Z})$ be the Poincaré dual of a $\mathbb{Z}$-coefficient polynomial $P(c) = P(c_1, c_2, \cdots)$ of Chern classes $c^*(TV) \in H^*(V, \mathbb{Z})$. $P(c)_*(V)$ is an additive $\mathcal{H}$-class with $\mathcal{H} = H_*(-, \mathbb{Z})$ and we have a unique natural transformation on the category $\mathcal{TOP}$

$$P(c)_* : K^{prop}(\mathcal{C}^{\infty}_{\mathbb{C}} \xrightarrow{\mathfrak{f}} \mathcal{TOP}/-) \to H_*(-, \mathbb{Z})$$

such that for a smmoth complex manifold X we have

$$P(c)_*([X \xrightarrow{\mathrm{id}_X} X]) = P(c)_*(X).$$

Similarly we get

$$P(c)_* : K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/-) \to H_*(-, \mathbb{Z}).$$

In the case when $P(c) = c$ is the Chern class, then we have the following more geometric "realization" on the category $\mathcal{V}_{\mathbb{C}}$ through *constructible functions* via MacPherson's theorem:

$$\begin{tikzcd} K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/X) \arrow[rr, "const"] \arrow[dr, "c_*"'] & & F(X) \arrow[dl, "c_*^{Mac}"] \\ & H_*(X,\mathbb{Z}) . & \end{tikzcd}$$

Here $const : K^{prop}(\mathcal{SV}_{\mathbb{C}} \xrightarrow{\iota} \mathcal{V}_{\mathbb{C}}/X) \to F(X)$ is defined by

$$const([V \xrightarrow{h} X]) := h_* 1\!\!1_V.$$

5.5. **Banagl's theory of Intersection Spaces.** Before finishing the paper we want to mention a possible application of the recent theory of *Intersection Spaces*, which has been introduced by Markus Banagl [5] (also see [4, 6] and [7, 8]). Given a psuedomanifold X he modifies the space along the singular locus of X without doing anything off the singular locus of X, which is a kind of "modification" of singularities, depending on the perversity $\overline{p}$. The resulting space is called *the intersection space associated to the perversity* $\overline{p}$ and denoted by $I^{\overline{p}}X$. The reduced ordinary homology $H_*(I^{\overline{p}}X)$ of the intersection space $I^{\overline{p}}X$, which is denoted by $HI_*^{\overline{p}}X$, turns out not to be isomorphic to the intersection homology $IH_*^{\overline{p}}(X)$, but a striking thing about $HI_*^{\overline{p}}X$ is that $(HI_*(X), IH_*(X))$ forms *a mirror pair* in the sense of mirror symmetry in algebraic geometry.

For certain pseudomanfiolds (not in a full generality), such as complex projective algebraic varieties, the set $\{I^{\overline{p}}X\}$ of the intersection spaces of X associated to the perversities $\overline{p}$'s satisfy the generalized Poincaré duality, i.e., for the complementary perversities $\overline{p}$ and $\overline{q}$ (which means that $\overline{p}+\overline{q}=\overline{t}$) there exists a non-degenerate intersection pairing

$$H_i(I^{\overline{p}}X;\mathbb{Q})\otimes H_{n-i}(I^{\overline{p}}X;\mathbb{Q})\to\mathbb{Q},$$

where $n=\dim X$. In particular, for the middle perversity $\overline{m}$, the intersection space $I^{\overline{m}}X$ becomes a (rational) Poincaré space, since $\overline{m}$ is self-complementary, i.e., $\overline{m}+\overline{m}=\overline{t}$.

Since there is a canonical map $q: I^{\overline{m}}X\to X$, one could consider some distinguished homology class $\gamma^{\overline{m}}(X)\in HI_*^{\overline{m}}(X)$ (which is supposed to be a reasonable and interesting invariant in the mirror symmetry) and pushforward it to the original space X:

$$q_*(\gamma^{\overline{m}}(X))\in H_*(X).$$

We hope or speculate that one could do similar procedures as above and could get a certain natural transformation of some reasonable classes related to the intersection spaces. Note that no theory of characteristic classes with values in intersection-homology groups is available yet.

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