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Topological Radon transforms with modified kernels on Grassmann manifolds

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Abstract: In this note, we survey our results on inversion formulas for topological Radon transforms on Grassmann manifolds [6]. We generalized Schapira's results in [10]. Moreover, we also introduce several topological Radon transforms whose incidence relations are different from the above ones and give their inversion formulas.

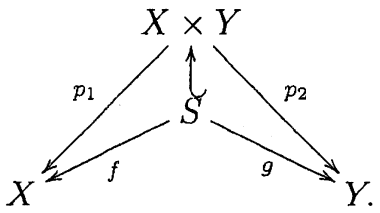
1 Introduction

Let X be a real analytic manifold. We say that a $\mathbb{Z}$-valued function $\varphi\colon X \longrightarrow \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}_{i\in I}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_{i\in I} c_i \mathbf{1}_{X_i} \quad (c_i \in \mathbb{Z}).$$

Here $\mathbf{1}_{X_i}$ denotes the characteristic function of X_i.

Let us consider the diagram:



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Here X and Y are real analytic manifolds, S is a subanalytic subset of $X \times Y$ and f and g are restrictions of natural projections p_1 and p_2 to S respectively.

In [5, 11, 12], several operations, such as direct and inverse images etc., on constructible functions were introduced. See Section 2.1 for the precise definitions. Therefore, for a constructible function φ on X we can define the topological Radon transform $\mathcal{R}_S(\varphi)$ of φ by

$$\mathcal{R}_S(\varphi) = \int_g f^*\varphi,$$

where f^* denotes the inverse image by f and $\int_g$ denotes the direct image by g. Now let X be a projective space $\mathbb{P}_N$, Y its dual space $\mathbb{P}_N^*$, and S the incidence submanifold of $X \times Y$. Note that $Y = \mathbb{P}_N^*$ is naturally identified with the set of hyperplanes in $X = \mathbb{P}_N$. In this situation, for a subanalytic subset K of X, the topological Radon transform $\mathcal{R}_S(\mathbf{1}_K)$ of $\mathbf{1}_K$ satisfies

$$\mathcal{R}_S(\mathbf{1}_K)(H) = \chi(K \cap H)$$

for any hyperplane $H \in Y = \mathbb{P}_N^*$. Namely, the values of our topological Radon transform $\mathcal{R}_S(\mathbf{1}_K)$ are the topological Euler characteristics of hyperplane sections of K.

In this note, we survey our results on inversion formulas for topological Radon transforms on Grassmann manifolds [6]. Our situations are more complicated than those in [10]. An intuitive meaning of one of our results is as follows. For an integer $0 \leq q \leq N-1$ and a subanalytic subset K of $X = \mathbb{P}_N$, we can reconstruct K from the Euler characteristics of the sections of K by q-dimensional linear subspaces $L \simeq \mathbb{P}_q$ in $X = \mathbb{P}_N$ under appropriate conditions. Moreover, we also give small generalizations of them. Namely, we introduce several new topological Radon transforms whose incidence relations are different from the above ones and give their inversion formulas.

In [7, 8], we studied the image of (standard) topological Radon transforms and their applications to projective duality. We hope that topological Radon transforms with modified kernels also might have good applications.

2 Preliminaries

2.1 Constructible functions

Definition 2.1. Let X be a real analytic manifold. We say that a function $\varphi\colon X \longrightarrow \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}_{i\in I}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_i c_i \mathbf{1}_{X_i} \qquad (c_i \in \mathbb{Z}).$$

Here $\mathbf{1}_{X_i}$ denotes the characteristic function of X_i. We denote the abelian group of constructible functions on X by $CF(X)$.

We define several operations on constructible functions in the following way.

Definition 2.2 ([5, 12]). Let X and Y be real analytic manifolds and $f\colon Y \longrightarrow X$ a real analytic map from Y to X.

(i) (The inverse image) For $\varphi \in CF(X)$, we define an inverse image $f^*\varphi \in CF(Y)$ of φ by f by

$$f^*\varphi(y) := \varphi(f(y)).$$

(ii) (The integral) Let $\varphi = \sum_i c_i \mathbf{1}_{X_i} \in CF(X)$ be a constructible function on X and assume that its support $\mathrm{supp}(\varphi)$ is compact. Then we define a topological (Euler) integral $\int_X \varphi \in \mathbb{Z}$ of φ by

$$\int_X \varphi := \sum_i c_i \cdot \chi(X_i),$$

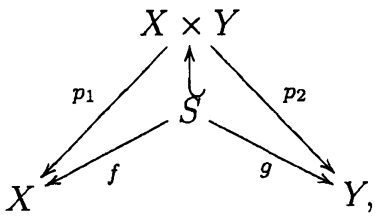
where $\chi(X_i)$ is the topological Euler characteristic of X_i.

(iii) (The direct image) Let $\psi \in CF(Y)$ such that $f|_{\mathrm{supp}(\psi)}\colon \mathrm{supp}(\psi) \longrightarrow X$ is proper. Then we define a direct image $\int_f \psi \in CF(X)$ of ψ by f by

$$\left(\int_f \psi\right)(x) := \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

2.2 Topological Radon transforms

Let X and Y be real analytic manifolds and S a real analytic submanifold of $X \times Y$. Consider the diagram:



(2.1)

where p_1 and p_2 are natural projections and f and g are restrictions of p_1 and p_2 to S respectively.

Definition 2.3. Let $\varphi \in CF(X)$. We define the topological Radon transform $\mathcal{R}_S(\varphi) \in CF(Y)$ of φ by

$$\mathcal{R}_S(\varphi) := \int_g f^*\varphi = \int_{p_2} \mathbf{1}_S \cdot p_1^* f.$$

We consider $\mathbf{1}_S$ as the kernel function of the topological Radon transform $\mathcal{R}_S$.

We denote the projective space of dimension N over a field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) by $\mathbb{P}_N$ and its dual space by $\mathbb{P}_N^*$. Then we have the following identifications.

$$\mathbb{P}_N = \{l \mid l \text{ is a line in } \mathbb{K}^{N+1} \text{ through the origin}\},$$
$$\mathbb{P}_N^* = \{H' \mid H' \text{ is a hyperplane in } \mathbb{K}^{N+1} \text{ through the origin}\}.$$

Note that if we projectivize a hyperplane H' in $\mathbb{K}^{N+1}$ we obtain a hyperplane $H \simeq \mathbb{P}_{N-1}$ in $\mathbb{P}_N$. Therefore we identify the dual projective space $\mathbb{P}_N^*$ with the set

$$\{H \mid H \text{ is a hyperplane in } \mathbb{P}_N\}.$$

Example 2.4. Let $X = \mathbb{P}_N$, $Y = \mathbb{P}_N^*$, $S = \{(l, H) \in X \times Y \mid l \subset H\}$ and K a subanalytic subset of $X = \mathbb{P}_N$. Then for any hyperplane $H \in Y$ we have

$$\mathcal{R}_S(\mathbf{1}_K)(H) = \chi(K \cap H).$$

Namely, the values of our topological Radon transform $\mathcal{R}_S(\mathbf{1}_K)$ are the topological Euler characteristics of hyperplane sections of K.

3 Inversion formulas for topological Radon transforms

In this section, we introduce our results in [6].

For $0 \leq k \leq N-1$, we denote by $\mathbb{G}_{N,k}$ the Grassmann manifold consisting of k-dimensional linear subspaces $L \simeq \mathbb{P}_k$ in $\mathbb{P}_N$. Namely we set

$$\mathbb{G}_{N,k} = \left\{ L' \;\middle|\; \begin{array}{l} L' \text{ is a } (k+1)\text{-dimensional linear subspace in } \mathbb{K}^{N+1} \\ \text{through the origin} \end{array} \right\}$$
$$= \{L \mid L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{P}_N\}.$$

Let $0 \leq p < q \leq N-1$ and let us consider the diagram (2.1) for $X = \mathbb{G}_{N,p}$, $Y = \mathbb{G}_{N,q}$ and $S = \{(L_p, L_q) \in \mathbb{G}_{N,p} \times \mathbb{G}_{N,q} \mid L_p \subset L_q\}$.

In this case, unfortunately the formal dual ${}^t\mathcal{R}_S = \int_f g^*$ of $\mathcal{R}_S$ is not a left inverse of our topological Radon transform $\mathcal{R}_S = \int_g f^*$ in general. By modifying the kernel function of the formal dual ${}^t\mathcal{R}_S$ by the Schubert calculus on Grassmann manifolds, we could construct a left inverse of $\mathcal{R}_S$ as follows.

Theorem 3.1 ([6]). *Assume that one of the following conditions are satisfied.*

(i) $\mathbb{K} = \mathbb{C}$ *and* $p + q \leq N-1$,

(ii) $\mathbb{K} = \mathbb{R}$, $p + q \leq N-1$ *and* $q - p$ *is even.*

Then there exist a group homomorphism $\widehat{\mathcal{R}}\colon CF(Y) \longrightarrow CF(X)$ *and a constant* $C_{p,q} \neq 0$ *which depends only on* p *and* q *such that*

$$\widehat{\mathcal{R}} \circ \mathcal{R}_S(\varphi) = C_{p,q} \cdot \varphi \quad \text{for any} \quad \varphi \in CF(X).$$

Note that our construction of the left inverse $C_{p,q}^{-1} \cdot \widehat{\mathcal{R}}$ of $\mathcal{R}_S$ in [6] is quite explicit. Namely, we construct the left inverse $C_{p,q}^{-1} \cdot \widehat{\mathcal{R}}$ of $\mathcal{R}_S$ by combining several topological Radon transforms with modified kernels (see Section 4). By our theorem, we can completely reconstruct the original function $\varphi \in CF(X)$ from its topological Radon transform $\mathcal{R}_S(\varphi)$. In particular,

when $\mathbb{K} = \mathbb{R}$, $p = 0$ and q is even (i.e. when $X = \mathbb{P}_N$, $Y = \mathbb{G}_{N,q}$), Theorem 3.1 implies that for any subanalytic set K of $X = \mathbb{P}_N$ we can reconstruct K from the topological Euler characteristics of its sections by q-dimensional linear subspaces $L_q \simeq \mathbb{P}_q$ in $X = \mathbb{P}_N$.

Remark 3.2. The meaning of our integrations is not the (usual) analytic one but the topological one based on Euler characteristics. Nevertheless, our results above are very similar to the ones obtained in the case of analytic Radon transforms. For example, by using invariant differential operators, Kakehi [4] obtained an inversion formula for analytic Radon transforms of C^∞-functions on $\mathbb{G}_{N,p}$ under the same condition that $\mathbb{K} = \mathbb{R}$ and $q - p$ is even. Namely, in spite of the difference of the definitions of integrations, the sufficient conditions under which we obtain an inversion formula coincide with each other. It would be an interesting problem to investigate the reason why we need the same condition. Note that in [3] Grinberg and Rubin constructed an inversion formula for analytic Radon transforms of C^∞-functions on $\mathbb{G}_{N,p}$ for $\mathbb{K} = \mathbb{R}$ and any p, q by using the Gårding-Gindikin fractional integrals.

In some special cases, we can prove also that the left inverse $C_{p,q}^{-1} \cdot \widehat{\mathcal{R}}$ in Theorem 3.1 is actually the inverse of $\mathcal{R}_S$ as follows.

Theorem 3.3 ([6]). *Assume that one of the following conditions are satisfied.*

(i) $\mathbb{K} = \mathbb{C}$ *and* $p + q = N - 1$,

(ii) $\mathbb{K} = \mathbb{R}$, $p + q = N - 1$ *and* $q - p$ *is even.*

Then the topological Radon transform $\mathcal{R}_S$ *induces a non-trivial group isomorphism between* $CF(\mathbb{G}_{N,p})$ *and* $CF(\mathbb{G}_{N,q})$.

In the special case where $\mathbb{K} = \mathbb{R}$, $p = 0$, $q = N - 1$ (i.e. when $X = \mathbb{P}_N$, $Y = \mathbb{P}_N^*$) and N is odd, Schapira [10] already proved that

$$ {}^t\mathcal{R}_S \circ \mathcal{R}_S(\varphi) = \varphi \quad \text{for any} \quad \varphi \in CF(X). $$

Hence our result is a generalization of this result to Grassmann cases.

4 Inversion formulas for topological Radon transforms with modified kernels

In this section, we introduce a small generalization of Theorem 3.1. Let $0 \leq p \leq q \leq N-1$ and set $X = \mathbb{G}_{N,p}$, $Y = \mathbb{G}_{N,q}$. In this section, we consider new incidence varieties. For $r = -1, 0, \ldots, p$, we set

$$S_r := \begin{cases} \{(L_p, L_q) \in \mathbb{G}_{N,p} \times \mathbb{G}_{N,q} \mid \dim(L_p \cap L_q) = r\} & (r = 0, 1, \ldots, p), \\ \{(L_p, L_q) \in \mathbb{G}_{N,p} \times \mathbb{G}_{N,q} \mid L_p \cap L_q = \emptyset \text{ in } \mathbb{P}_N\} & (r = -1). \end{cases}$$

Note that S_p is the incidence manifold considered in Section 3. Let us consider the diagram (2.1) for $X = \mathbb{G}_{N,p}$, $Y = \mathbb{G}_{N,q}$ and $S = S_r$. We denote the restrictions of p_1 (resp. p_2) to S_r by f_r (resp. g_r) and set

$$\mathcal{R}_{S_r} = \int_{g_r} {f_r}^* = \int_{p_2} \mathbf{1}_{S_r} \cdot p_1^* \colon CF(X) \longrightarrow CF(Y).$$

Note that $\mathcal{R}_{S_p}$ is nothing but the (standard) topological Radon transform in Section 3 and the others are new ones. We call $\mathcal{R}_{S_{-1}}, \ldots, \mathcal{R}_{S_{p-1}}$ the topological Radon transforms with modified kernels. We also define the formal dual of $\mathcal{R}_{S_r}$ by

$${}^t\mathcal{R}_{S_r} = \int_{f_r} {g_r}^* = \int_{p_1} \mathbf{1}_{S_r} \cdot p_2^* \colon CF(Y) \longrightarrow CF(X).$$

In [6], we construct a left inverse transform of $\mathcal{R}_{S_p}$ by using not only its formal dual ${}^t\mathcal{R}_{S_p}$ but also ${}^t\mathcal{R}_{S_{-1}}, \ldots, {}^t\mathcal{R}_{S_{p-1}}$. In this section, we discuss about an inversion formulas for each $\mathcal{R}_{S_r}$ $(r = -1, 0, 1, \ldots, p)$. In order to state our theorem, let us define a number $C_{p,q,r}$ by the following way. We use the generalized binomial coefficient defined by

$$\binom{u}{v} := \begin{cases} \binom{u}{v} & (0 \leq v \leq u), \\ 0 & (\text{otherwise}). \end{cases}$$

(i) In the case $\mathbb{K} = \mathbb{C}$, we set

$$c_{i,j} := \sum_{l=-1}^{r} \binom{j+1}{l+1} \binom{p-j}{r-l} \binom{p-j}{i-l} \binom{N-2p+j}{q-r-i+l}, \tag{4.1}$$

$$C_{p,q,r} := \det(c_{i,j})_{-1 \leq i,j \leq p}. \tag{4.2}$$

(ii) In the case $\mathbb{K} = \mathbb{R}$, we set

$$c_{i,j} = \sum_{l=-1}^{r} a^{p+1,r+1}_{j+1,l+1}\, b^{N-r,q-r,p-r}_{p-l,i-l,j-l}, \tag{4.3}$$

$$C_{p,q,r} := \det(c_{i,j})_{-1\le i,j\le p}, \tag{4.4}$$

where the sequence $\{a^{x,y}_{u,v}\}$ is defined by

$$a^{x,y}_{u,v} := \begin{cases} 0 & \begin{pmatrix}(y-v)(x-u-y+v)\\ \text{or } v(u-v) \text{ is odd}\end{pmatrix}, \\ (-1)^{(x-1)yuv}\binom{[\frac{u}{2}]}{[\frac{v}{2}]}\binom{[\frac{x-u}{2}]}{[\frac{y-v}{2}]} & \text{(otherwise)} \end{cases}$$

and the sequence $\{b^{x,y,z}_{u,v,w}\}$ is determined by the following recursive formula:

$$b^{x,y,z}_{u,v,w} = \begin{cases} 0 & (z,w<0 \text{ or } z<w), \\ a^{x,y}_{u,v} & (z=w=0), \\ a^{x,y}_{u,v} - \displaystyle\sum_{m=1}^{z}\sum_{n=0}^{w} a^{z,m}_{w,n} b^{x-m,y-m,z-m}_{u-n,v-n,w-n} & (z\ge 1, z\ge w\ge 0). \end{cases}$$

The complexity of the definitions (in particular in the case $\mathbb{K} = \mathbb{R}$) comes from that of the topological Euler characteristics of Grassmann manifolds $\mathbb{G}_{N,q}$. Note that $C_{p,q,r}$ depends only on p, q and r. Then we have the following result.

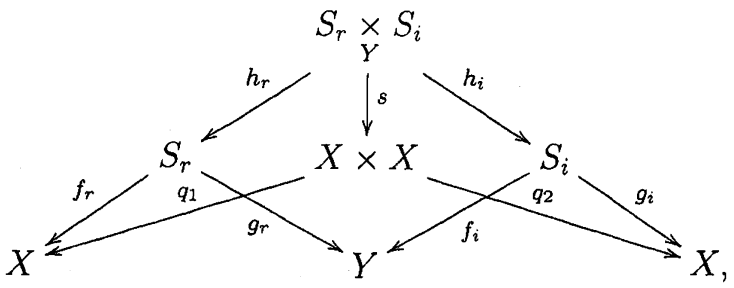
Theorem 4.1. *Assume that $C_{p,q,r}$ does not vanish. Then there exists a group homomorphism $\widehat{\mathcal{R}_r}\colon CF(Y) \longrightarrow CF(X)$ such that*

$$\widehat{\mathcal{R}_r} \circ \mathcal{R}_{S_r}(\varphi) = C_{p,q,r} \cdot \varphi \qquad \textit{for } \varphi \in CF(X).$$

In the case $r = p$, Theorem 4.1 is nothing but Theorem 3.1. Although the condition of Theorem 4.1 is complicated, in some cases such as Theorem 3.1 we can obtain good one.

Let us explain briefly our proof of Theorem 4.1. The outline of proof is same as that of Theorem 3.1 (in [6]) although we need much more complicated calculations. For $i = -1, 0, \ldots, p$, let us consider the following

commutative diagram:



where q_1, q_2 (resp. h_r, h_i) are the natural projections from $X \times X$ to each X (resp. from $S_r \underset{Y}{\times} S_i$ to S_r and S_i respectively) and $s\colon S_r \underset{Y}{\times} S_i \longrightarrow X \times X$ is the natural embedding. Then we have

$$ {}^t\mathcal{R}_{S_i} \circ \mathcal{R}_{S_r}(\varphi) = \int_{q_2} \left(\int_s \mathbf{1}_{S_r \underset{Y}{\times} S_i} \right) q_1^* \varphi. $$

We calculate $\displaystyle\int_s \mathbf{1}_{S_r \underset{Y}{\times} S_i}$ as follows. For $j = -1, 0, \ldots, p$, we set

$$ Z_j := \begin{cases} \{(x_1, x_2) \in X \times X \mid \dim(x_1 \cap x_2) = j\} & (j = 0, 1, \ldots, p), \\ \{(x_1, x_2) \in X \times X \mid x_1 \cap x_2 = \emptyset\} & (j = -1). \end{cases} $$

Since a function $\chi(s^{-1}(x_1, x_2))$ is constant on Z_j for each j, we set this number $c_{i,j}$. This is calculated by (4.1) or (4.3). Then we have

$$ \begin{aligned} \left(\int_s \mathbf{1}_{S_r \underset{Y}{\times} S_i} \right)(x_1, x_2) &= \sum_{j=-1}^{p} \left(\int_{S_r \underset{Y}{\times} S_i} \mathbf{1}_{s^{-1}(x_1, x_2) \cap s^{-1}(Z_j)} \right) \mathbf{1}_{Z_j}(x_1, x_2) \\ &= \sum_{j=-1}^{p} c_{i,j} \mathbf{1}_{Z_j}(x_1, x_2). \end{aligned} $$

Thus we have

$$ {}^t\mathcal{R}_{S_i} \circ \mathcal{R}_{S_r}(\varphi) = \sum_{j=-1}^{p} c_{i,j} \left(\int_{q_2} \mathbf{1}_{Z_j} \cdot q_1^* \varphi \right) \qquad (i = -1, 0, \ldots, p). \quad (4.5) $$

By the Cramer's formula, if $C_{p,q,r}$ does not vanish, then (4.5) can be solved with respect to $\displaystyle\int_{q_2} \mathbf{1}_{Z_p} \cdot q_1^* \varphi$. Since we have $\displaystyle\int_{q_2} \mathbf{1}_{Z_p} \cdot q_1^* \varphi = \varphi$, we obtain an inversion formula for $\mathcal{R}_{S_r}$. Note that our left inverse transform $\widehat{\mathcal{R}_r}$ of $\mathcal{R}_{S_r}$ is constructed by a linear combination of ${}^t\mathcal{R}_{S_{-1}}, \ldots, {}^t\mathcal{R}_{S_p}$.

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