数理解析研究所講究録
第 1764 巻 2011 年 149-164

Basic Topics on Tropical Geometry and Singularities

(トロピカル幾何と特異性に関する基本的トピックス)

Goo Ishikawa
Department of Mathematics, Hokkaido University, Japan
(石川剛郎, 北海道大学数学教室, 日本)
e-mail: ishikawa@math.sci.hokudai.ac.jp

Motivated from real algebraic geometry, Viro, Mikhalkin, Shustin, Itenberg, and other mathematicians have developed the "tropical (algebraic) geometry" [8]. Algebraic curves are tropicalized to piecewise-linear curves. The method was used to construct topological types of real algebraic curves in Hilbert's 16th problem [24].

In this rough sketch, we present several basic topics of tropical geometry, in particular, the notion of hyperfields introduced by Viro recently [25].

【 Tropical Limits of Operations 】

Let $\mathbf{R}_+$ denote the set of non-negative real numbers.

We fix $h > 0$ and consider the bijection

$$h \log : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$$

defined by

$$u \mapsto h \log u, \quad e^{\frac{x}{h}} \leftarrow x.$$

On $\mathbf{R} \cup \{-\infty\}$, two operations

$$\begin{cases} x +_h y & := & h \log\left(e^{\frac{x}{h}} + e^{\frac{y}{h}}\right) \\ x \times_h y & := & h \log\left(e^{\frac{x}{h}} \cdot e^{\frac{y}{h}}\right) = x + y \end{cases}$$

are induced from the summension and the multiplication on $\mathbf{R}_+$.

Set $m = \max\{x,y\}$. Then we have

$$h \log\left(e^{\frac{m}{h}}\right) \leq x +_h y \leq h \log\left(e^{\frac{m}{h}} + e^{\frac{m}{h}}\right),$$

namely,

$$m \leq x +_h y \leq m + h \log 2.$$

Therefore we have that

$$\lim_{h \downarrow 0} (x +_h y) = \max\{x,y\}.$$

【 Tropical Semi-Ring 】

$\mathbf{R}_{\text{trop}} = \mathbf{R} \cup \{-\infty\}$ with the two operations

$$\text{“}x + y\text{”} := \max\{x,y\}, \quad \text{“}x \cdot y\text{”} := x + y,$$

is called the *tropical semi-ring* (or the *max-plus algebra*).

Moreover we set “x/y” $= x - y$ if $y \neq -\infty$. Note that there is no tropical subtraction. The tropical sum is idempotent :

$$\text{“}x + x\text{”} = x,$$

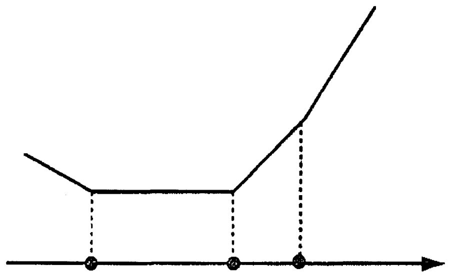
$-\infty$ being the tropical zero.

【 Tropical Polynomials 】

For a finite subset $A \subset \mathbf{Z}^n$, consider a “tropical” (Laurent) polynomial

$$F(x) = \text{“} \sum_{j \in A} c_j x^j \text{”} = \max\{c_j + j \cdot x \mid j \in A\},$$

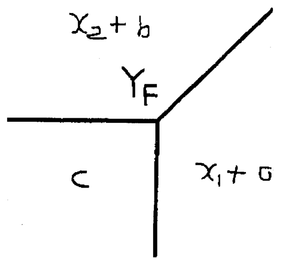
($c_j \in \mathbf{R}$), which is a PL-function on $\mathbf{R}^n$. Then the tropical hypersurface $Y_F \subset \mathbf{R}^n$ is defined by F as the *corner locus* of F.

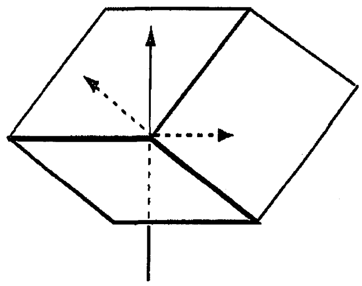


Example 1. (tropical line). We consider

$$F(x_1,x_2) = \text{“}ax_1 + bx_2 + c\text{”} = \max\{x_1 + a,\ x_2 + b,\ c\}.$$

Then Y_F consists of three half-lines meeting at one point.





【 Tropical Hyperfields 】

We define a *multi-valued* addition $\curlyvee$ on $\mathbf{R} \cup \{-\infty\}$: For $a,b \in \mathbf{R} \cup \{-\infty\}$, we set

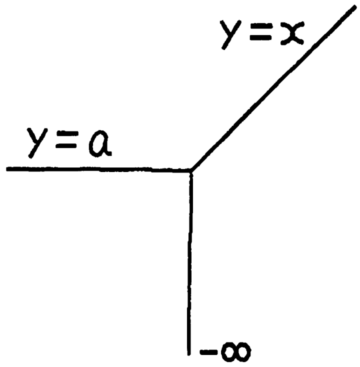
$$a \curlyvee b := \begin{cases} \max\{a,b\}, & (a \neq b) \\ \{y \in \mathbf{R} \cup \{-\infty\} \mid y \leq a\}, & (a = b) \end{cases}$$

The multiplication is defined by the ordinary addition.
We set $\mathbb{Y} = (\mathbf{R} \cup \{-\infty\}, \curlyvee, +)$ and we call it the *tropical hyperfield.*

This implies the natural definition of “tropical zero”.

Example 2. For $a \in \mathbf{R}$, we define the function $x \curlyvee a : \mathbb{Y} \to \mathbb{Y}$. Then we have

$$\text{graph}(x \curlyvee a) = \{y = a, x < a\} \cup \{y = x, a < x\} \cup \{y \leq a, x = a\}.$$



【 Definition of Hyperfields 】

Suppose there are given, on a set X, a multi-valued binary operation $\top$ and a single-valued binary operation $\cdot$.

Then $(X, \top, \cdot)$ is called a *hyperfield* if

- $a \top b = b \top a, \quad a \top (b \top c) = (a \top b) \top c$
- $\exists 0 \in X,\ 0 \top a = a$, for any $a \in X$,
- $\forall a \in X, \exists_1\ -a \in X$ (minus a) such that $0 \in a \top (-a)$.
- $c \in a \top b \iff (-c) \in (-a) \top (-b)$
- The operation $\cdot$ is commutative, associative and
  $0 \cdot a = 0$ holds for any $a \in X$,
- $(X \setminus \{0\}, \cdot)$ is a commutative group, which will be denoted by $X^{\times}$,
- the "distributive law" holds:
  $a \cdot (b \top c) = (a \cdot b) \top (a \cdot c), \quad (b \top c) \cdot a = (b \cdot a) \top (c \cdot a).$

Lemma 3. *The tropical hyperfield* $\mathbb{Y} = (\mathbf{R} \cup \{-\infty\}, \curlyvee, +)$ *is a hyperfield.*

In fact we have

- The zero-element is $-\infty$.
- For $a \in \mathbb{Y}$, $-a$ equals a, since $-\infty \in a \curlyvee b \Leftrightarrow b = a$.
- The commutative group $\mathbb{Y}^{\times} = (\mathbf{R}, +)$, the unit being $0 \in \mathbf{R}$.

[Tropical Hypersurfaces and Newton Polyhedra]

For a tropical Laurent polynomial $F(x) = `` \sum_{j \in A} c_j x^j ''$, we define

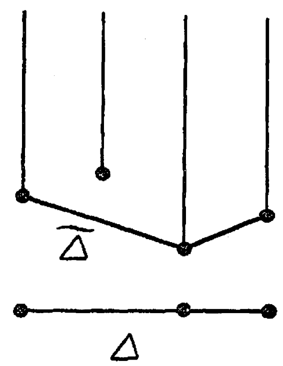
$$v = -c : A \to \mathbf{R}$$

by $v(j) := -c_j, (j \in A)$. Then we set

$$\sqcup(v) := \text{convex hull } \{(j,y) \in \mathbf{R}^n \times \mathbf{R} \mid j \in A, y \geqq v(j)\} \subset \mathbf{R}^{n+1}$$

We set $\Delta = \Delta_F = \text{convex hull } (A) \subset \mathbf{R}^n$, and $\widetilde{\Delta}$ the union of compact faces of $\sqcup(v)$. We call $\Delta = \Delta_F$ the Newton polyhedra of F.

Then $\widetilde{\Delta}$ projects to Δ in bijection by $\pi : \widetilde{\Delta} \to \mathbf{R}^n$, $\pi(j,y) := j$. An integral subdivision of Δ is induced from $\widetilde{\Delta}$. We obtain the convex function $\overline{v} : \Delta \to \mathbf{R}$ having $\widetilde{\Delta}$ as its graph.



The tropical hypersurface Y_F is an $(n-1)$-dimensional regular polyhedral complex. (Regularity condition: the boundary of each i-cell is a union of $(i-1)$-cells.)

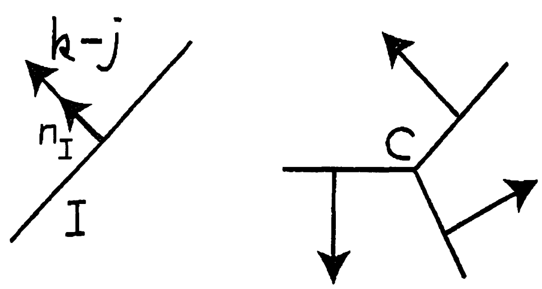
Along each $(n-1)$-cell I, two functions $c_j + j \cdot x, c_k + k \cdot x$ have the same value. From $c_j + j \cdot x = c_k + k \cdot x$, we have the equation

$$(k-j)x + (c_k - c_j) = 0$$

of the hyperplane containing I. Then the integer vector $k - j$ is orthogonal to I. Then there exist the unique positive integer w_I and the primitive integer vector n_I such that $k - j = w_I n_I$.

For each $(n-2)$-cell C, and $(n-1)$-cells $I_1, I_2, \ldots, I_m$ adjacent to C, if we fix a co-orientation of C and take primitive orthogonal vectors n_{I_j}, then we have the *balanced condition*

$$w_{I_1} n_{I_1} + w_{I_2} n_{I_2} + \cdots + w_{I_m} n_{I_m} = 0.$$



Thus the tropical hypersurface Y is an $(n-1)$-dimensional weighted rational polyhedral complex satisfying the regularity condition and the balanced condition.

Tropical hypersurface Y_F is invariant under the deformations, called the *fundamental deformations*, of the tropical Laurent polynomial F.

(1) Replace c by $c' : A \to \mathbf{R}, c'(j) = c(j) + \text{const.}$.

(2) Replace A by $A' = A + j_0, j_0 \in \mathbf{Z}^n$ and c by $c' : A' \to \mathbf{R}$, $c'(j + j_0) = c(j)$.

(3) Replace $c : A \to \mathbf{R}$ by $c' : A' \to \mathbf{R}$ such that convex hull $A' = \Delta$ and the convex function $\overline{-c'} = \overline{-c}$.

【 Legendre Transformations 】

Consider the contact manifold $M = \mathbf{R}^{2n+1}$ with coordinates

$$(x, y, p) = (x_1, \ldots, x_n, y, p_1, \ldots, p_n)$$

and with the contact form $\theta = dy - \sum_{i=1}^{n} p_i dx_i$.

Note that $-\theta = d(\sum_{i=1}^{n} p_i x_i - y) - \sum_{i=1}^{n} x_i dp_i$. Then we have the *double Legendrian fibration*:

$$\mathbf{R}^{n+1} \xleftarrow{\pi_1} \mathbf{R}^{2n+1} \xrightarrow{\pi_2} \mathbf{R}^{n+1},$$

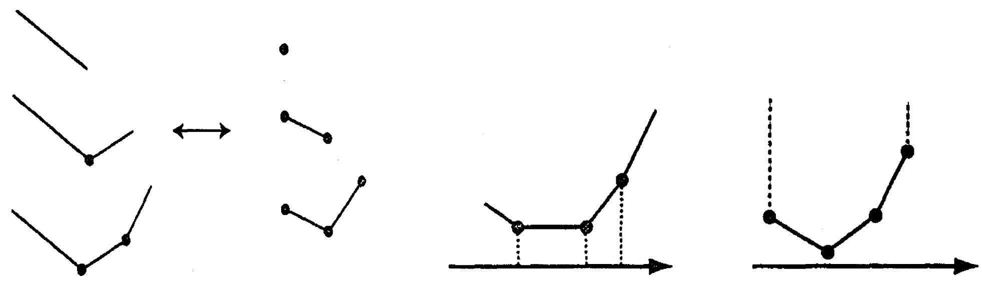
$\pi_1(x, y, p) = (x, y), \ \ \pi_2(x, y, p) = (\tilde{y}, p), \ \ \tilde{y} = \sum_{i=1}^{n} p_i x_i - y.$

For a function $h : \Delta \to \mathbf{R}$ on a convex set $\Delta \subset \mathbf{R}^n$, the *Legendre transformation* of h is defined as the set of *supporting hyperplanes* of the *epi-graph* of h.

Lemma 4. *The graph of tropical polynomial function*

$$F(x) = \text{“}\sum_{j\in A} c_j x^j\text{”}$$

and the graph of the convex function $\overline{-c} : \mathbf{R}^n \to \mathbf{R}$ *are the Legendre transformations to each other.*



We consider the topological classification problem of tropical polynomial functions preserving corner loci.

Definition 5. Two tropical polynomials $F(x)$ and $G(x)$ are called *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ and $\Psi : \mathbf{R} \to \mathbf{R}$ such that

$$\Psi(F(x)) = G(\Phi(x)),\ \Phi(Y_F) = Y_G.$$

Proposition 6. *There exists a semialgebraic set* $\Sigma \subset \mathbf{R}^A$ *of codim* > 0 *such that, for any* $c \in \mathbf{R}^A \setminus \Sigma$, *the decomposition of* Δ *is simplicial.*

For each connected component U *of* $\mathbf{R}^A \setminus \Sigma$, *the family* $F_c(x), c \in U$ *of tropical polynomial functions is topologically trivial.*

【 Topological Bifurcations of Singularities 】

The topology of a tropical polynomial with a non-simplicial decomposition bifurcates into a generic tropical polynomial.

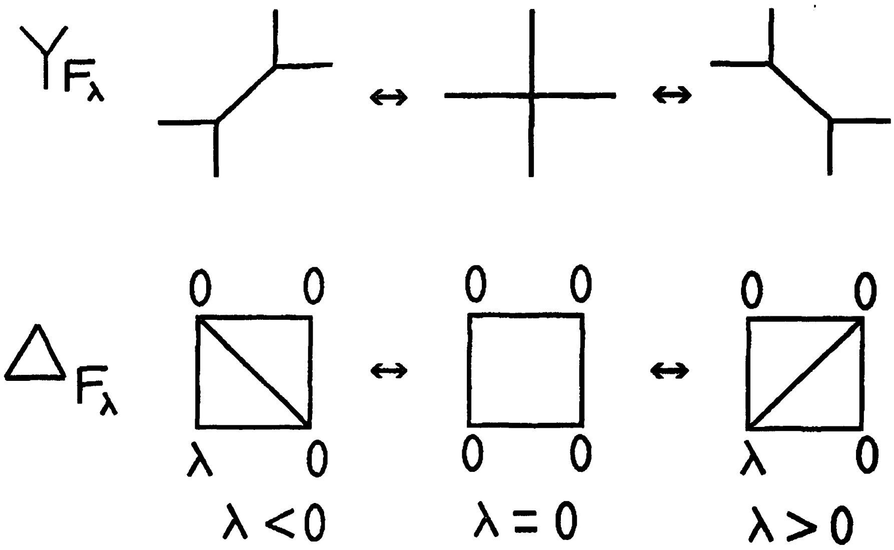
Example 7. Let us consider the tropical polynomial

$$F = \text{“ } 0 + 0x_1 + 0x_2 + 0x_1x_2 \text{ ”} = \max\{0, x_1, x_2, x_1 + x_2\}.$$

Then F has the deformation:

$$F_\lambda = \text{“}\lambda + 0x_1 + 0x_2 + 0x_1x_2\text{''} = \max\{\lambda, x_1, x_2, x_1 + x_2\}, (\lambda \in \mathbf{R}, \lambda \neq 0).$$

The tropical curve Y_F bifurcates into $Y_{F_\lambda}(\lambda > 0, \lambda < 0)$. The decomposition of Newton polyhedron Δ_F bifurcates into $\Delta_{F_\lambda}(\lambda > 0, \lambda < 0)$.



【 Amoeba and Pachworking 】

For a complex Laurent polynomial

$$f(z) = \sum_{j \in A} b_j z^j \in \mathbf{C}[z_1^{\pm}, \ldots, z_n^{\pm}], \quad b_j \in \mathbf{C}^{\times},$$

we have a hypersurface

$$Z_f = \{z \in (\mathbf{C}^{\times})^n \mid f(z) = 0\} \subset (\mathbf{C}^{\times})^n$$

in the complex torus $(\mathbf{C}^{\times})^n$.

For a given function $v : A \to \mathbf{R}$, consider the family of polynomials,

$$f_t = f_t^v(z) := \sum_{j \in A} b_j t^{-v(j)} z^j, \quad (t > 0).$$

We call it the *patchworking polynomial* induced by f and v. Note that $f_1 = f$.

Let us define $\mathrm{Log}_t : \mathbf{C}^n \to (\mathbf{R} \cup \{-\infty\})^n$ by

$$\mathrm{Log}_t(z_1, \ldots, z_n) = (\log_t |z_1|, \ldots, \log_t |z_n|).$$

We set $\mathcal{A}_f = \mathrm{Log}(Z_f) \subset \mathbf{R}^n$ and we call it the *amoeba* of Z_f.

Proposition 8. (Viro, Kapranov)

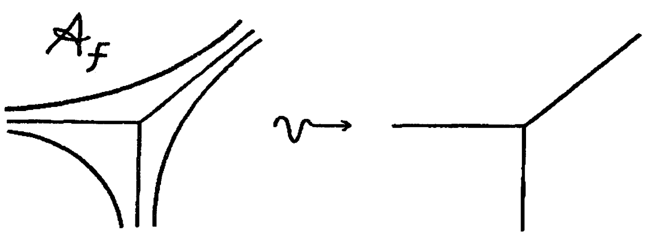
$$\lim_{t\to\infty} \text{Hausdorff-dist}(\mathrm{Log}_t(Z_{f_t}), Y_{f^v_{\mathrm{trop}}}) = 0$$

where

$$f^v_{\mathrm{trop}}(x) := \text{“} \sum_{j\in A} (-v(j)) x^j \text{”} = \max_{j\in A}(j \cdot x - v(j))$$

(Legendre transformation of v).

Example 9. *Amoeba of $f(z_1, z_2) = z_1 + z_2 + 1$.*



【Puiseux Series and Non-Archimedean Amoeba】

Let us denote by $\mathbf{C}[\mathbf{R}]$ the group algebra of the additive group $\mathbf{R}$ over $\mathbf{C}$. We consider its formal version:

A *Puiseux-Laurent series* of real power (Hahn series[4]) is given by

$$a = a(s) = \sum_{p\in I} \alpha_p s^p$$

where $\alpha_p \in \mathbf{C}^\times$ and the support $I = I_a \subset \mathbf{R}$ of a is a well-ordered subset.

We set

$$\mathbf{C}((\mathbf{R})) := \{a(s) \mid a(s) : \text{Puiseux-Laurent series of real power}\} \cup \{0\}.$$

Lemma 10. $\mathbf{K} = \mathbf{C}((\mathbf{R}))$ *is an algebraically closed field.*

Define the valuation $\mathrm{val} : \mathbf{C}((\mathbf{R})) \to \mathbf{R} \cup \{\infty\}$ on $\mathbf{C}((\mathbf{R}))$ by

$$\mathrm{val}(a) := \min I_a \in \mathbf{R}, \ (a \in \mathbf{C}((\mathbf{R})) \setminus \{0\}), \quad \mathrm{val}(0) = \infty,$$

Then we have that $\mathrm{val}(a) = \infty$ if and only if $a = 0$, and that

$$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b), \quad \mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}.$$

We define the *non-Archimedes norm* on $\mathbf{C}((\mathbf{R}))$ by

$$\|a\| := e^{-\mathrm{val}(a)} \quad (a \in \mathbf{C}((\mathbf{R}))^\times), \qquad \|0\| = 0.$$

Then we have the *tropical triangular inequality*

$$\|a+b\| \ \leq \ \max\{\|a\|, \|b\|\} \ = \ \text{“ } \|a\| + \|b\| \text{ ”}$$

Define $\mathrm{Log} : \mathbf{C}((\mathbf{R}))^n \to (\mathbf{R} \cup \{-\infty\})^n$ by

$$\begin{aligned} \mathrm{Log}(a_1, \ldots, a_n) &:= (\log\|a_1\|, \ldots, \log\|a_n\|) \\ &= (-\mathrm{val}(a_1), \ldots, -\mathrm{val}(a_n)). \end{aligned}$$

Given a Laurent polynomial $f(z) = \sum_j a_j z^j \in \mathbf{K}[z, z^{-1}]$, we define

$$Z_f := \{z \in (\mathbf{K}^\times)^n \mid f(z) = 0\} \subset (\mathbf{K}^\times)^n.$$

Its Log-image $\mathbf{A}_f := \mathrm{Log}(Z_f) \subset (\mathbf{R} \cup \{-\infty\})^n$ is called the *non-Archimedean amoeba* of Z_f.

Define a tropical Laurent polynomial

$$\begin{aligned} f_{\mathrm{trop}}(x) &:= \text{“ } \textstyle\sum_{j \in A} \log\|a_j\| x^j \text{ ”} = \text{“ } \textstyle\sum_{j \in A}(-\mathrm{val}(a_j)) x^j \text{ ”} \\ &= \max_{j \in A}(j \cdot x - \mathrm{val}(a_j)). \end{aligned}$$

We call $f_{\mathrm{trop}}(x)$ the *tropicalization* of $f(z)$.

Proposition 11. (Kapranov) *Non-Archimedean amoeba is a tropical hypersurface: We have* $\mathbf{A}_f = Y_{f_{\mathrm{trop}}}$.

【 Triangle hyperfield 】

On $\mathbf{R}_+$, define the multi-valued addition

$$\begin{aligned} a\nabla b &:= \{c \in \mathbf{R}_+ \mid |a-b| \le c \le a+b\} \\ &= \{|z+w| \mid |z|=a, |w|=b\}. \end{aligned}$$

This reminds us the superposition of waves.

Then $\mathbf{R}_+^{\text{tri}} = (\mathbf{R}_+, \nabla, \cdot)$ is a hyperfield.

【 Amoeba hyperfield 】

By the bijection $\log : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$, we have the hyperfield

$$\log(\mathbf{R}_+^{\text{tri}}) := (\mathbf{R} \cup \{-\infty\}, \ \curlyvee \ , +),$$

which is called the *amoeba hyperfield*:

$$a \curlyvee b := \{c \in \mathbf{R} \cup \{-\infty\} \mid \log(|e^a - e^b|) \le c \le \log(e^a + e^b)\}.$$

【 Tropical Limits of Amoeba Hyperfield 】

Define, on $\mathbf{R} \cup \{-\infty\}$,

$$\begin{aligned} a \curlyvee_h b &:= h(\tfrac{a}{h} \curlyvee \tfrac{b}{h}) \\ &= \{c \in \mathbf{R} \cup \{-\infty\} \mid \\ &\qquad h\log(|e^{\frac{a}{h}} - e^{\frac{b}{h}}|) \le c \le h\log(e^{\frac{a}{h}} + e^{\frac{b}{h}}),\} \end{aligned}$$

$$\begin{aligned} a \curlyvee_h b &= \{c \in \mathbf{R} \cup \{-\infty\} \mid -\infty \le c \le a + h\log 2\} \\ &= [-\infty, a] =: a \curlyvee a. \end{aligned}$$

If $a \neq b$, then $a \curlyvee_h b \to \{ \max\{a,b\} \}$.

$$\lim_{h\to 0} a \curlyvee_h b = a \curlyvee b,$$

$$\lim_{h\to 0} \log(\mathbf{R}_+^{\text{tri}})_h \to \mathbb{Y} \text{ (: tropical hyperfield)}.$$

【 Complex Tropical hyperfield 】

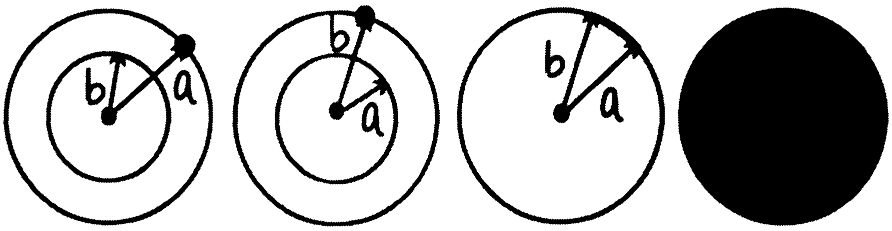
We define a multi-valued addition $\smile$ on $\mathbf{C}$: Let $a,b \in \mathbf{C}$. If $|a| \neq |b|$, then we set $a \smile b := a$ if $|a| > |b|$, and $a \smile b := b$ if $|a| < |b|$.

Suppose $|a| = |b|$. If $b \neq -a$, then

$$a \smile b \quad := \quad [\text{ the shortest arc connecting a and b on the circle } \{z \in \mathbf{C} \mid |z| = |a|\}\ ].$$

If $b = -a$, then set

$$a \smile b \quad := \quad \{z \in \mathbf{C} \mid |z| \leq |a|\}.$$



We define the *complex tropical hyperfield* by

$$\mathcal{T}\mathbf{C} := (\mathbf{C}, \smile, \text{the usual multiplication}).$$

On $\mathbf{C}$, we consider the bijection $S_h : \mathbf{C} \to \mathbf{C}$ defined by

$$S_h(z) := \begin{cases} |z|^{\frac{1}{h}} \frac{z}{|z|} & (z \neq 0), \\ 0 & (z = 0). \end{cases}$$

and we define

$$z +_h w := S_h^{-1}(S_h(z) + S_h(w)).$$

Then we have a family of fields $(\mathbf{C}, +_h, \times)$, $h > 0$.

Theorem 12. (Viro [25]) *Let*

$$\Gamma = \{(z, w, z +_h w, h) \in \mathbf{C}^3 \times \mathbf{R}_+ \mid (z, w, h) \in \mathbf{C}^2 \times \mathbf{R}_+\}.$$

Then

$$\overline{\Gamma} \cap (\mathbf{C}^3 \times \{0\}) = \{(a,\ b,\ a \smile b,\ 0) \mid (a,b) \in \mathbf{C}^2\}.$$

【 Viro's Diagram 2010】

Thus we have the diagram:

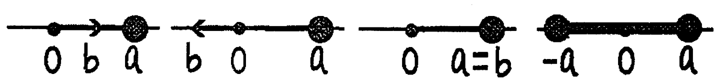
Complex Algebraic Geometry		Complex Tropical Geometry
$\mathbf{C} \cong \mathbf{C}_h$	$\xrightarrow{h \to 0}$	$\mathbf{C}_0 = \mathcal{T}\mathbf{C}$
$z \mapsto \lvert z \rvert \downarrow$		$\downarrow z \mapsto \lvert z \rvert$
$\mathbf{R}_+^{\mathrm{tri}} \cong \mathbf{R}_{+h}^{\mathrm{tri}}$	$\xrightarrow{h \to 0}$	$\mathbf{R}_{+0}^{\mathrm{tri}} = \mathbb{Y}_\times$
$x \mapsto \log x \downarrow$		$\downarrow x \mapsto \log x$
$\log(\mathbf{R}_+^{\mathrm{tri}}) \cong \log(\mathbf{R}_+^{\mathrm{tri}})_h$	$\xrightarrow{h \to 0}$	$\mathbb{Y}$
Amoeba Geometry		Tropical Geometry

【 Real Tropical Hyperfield 】

Question: What is the real counterpart of the complex tropical hyperfield ?

We are naturally led to define the multi-valued addition $\smile_{\mathbf{R}}$ on $\mathbf{R}$ induced from $\smile$ on $\mathbf{C}$: For $a, b \in \mathbf{R}$, we set

$$\begin{cases} a \smile_{\mathbf{R}} b & := & a & \text{if } |a| > |b|, \\ a \smile_{\mathbf{R}} b & := & b & \text{if } |a| < |b|, \\ a \smile_{\mathbf{R}} a & := & a, & \\ a \smile_{\mathbf{R}} (-a) & := & [-a, a]. & \end{cases}$$



Theorem 13. $(\mathbf{R}, \smile_{\mathbf{R}}, \times)$ *is a hyperfield. Moreover let*

$$\Gamma_{\mathbf{R}} = \{(a, b, a +_h b, h) \in \mathbf{R}^3 \times \mathbf{R}_+ \mid (a, b, h) \in \mathbf{R}^2 \times \mathbf{R}_+\}.$$

Then we have

$$\overline{\Gamma_{\mathbf{R}}} \cap (\mathbf{R}^3 \times \{0\}) = \{(a,\ b,\ a \smile_{\mathbf{R}} b,\ 0) \mid (a, b) \in \mathbf{R}^2\}.$$

The real tropical hyperfield is, in some sense, a "double covering" of the tropical hyperfield via $x \mapsto \log|x|$. Therefore, "real tropical geometry" can be constructed as a "double covering" of tropical geometry.

【 Several Questions 】

Question: Is the complex tropical hyperfield $\mathcal{T}\mathbf{C}$ is algebraically closed, in an appropriate sense ?

Question: Are the real tropical hyperfield and the tropical hyperfield $\mathbb{Y}$ real closed ?

Question: What is the real tropical algebraic geometry ?

In Amoeba geometry, it is known the *Ronkin function*

$$N_f(x) := \frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathrm{Log}^{-1}(x)} \log|f(z)|\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

is linear on each connected component of $\mathbf{R}^n \setminus \mathcal{A}_f$. We have $\mathrm{grad}N_f : \mathbf{R}^n \setminus \mathcal{A}_f \to \Delta \cap \mathbf{Z}^n$ and $\mathrm{grad}N_f$ separates every connected components of $\mathbf{R}^n \setminus \mathcal{A}_f$.

Question: Can the Ronkin function be described in terms of the amoeba hyperfield ?

References

[1] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean Analysis,* Grund. math. Wiss., **261**, Springer-Verlag Berlin, (1984).

[2] A.D. Bruno, *Power Geometry in Algebraic and Differential Equations,* North-Holland Math. Library 57, Elsevier, (2000).

[3] J. Draisma, *A tropical approach to secant dimensions,* arXiv:math.AG/0605345, (2006).

[4] P. Ehrlich, *Hahn's Über die nichtarchimedischen Grössensysteme and the developement of the modern theory of magnitudes and numbers to measure them,* in Essays

on the Development of the Foundations of Mathematics, ed. by Jaakko Hintikka, Kluwer Academic Publishers, (1995), pp. 165–213.

[5] M. Einsiedler, M. Kapranov, D. Lind, *Non-Archimedean amoebas and tropical varieties,* J. reine angew. Math., **601** (2006), 139–157.

[6] M. Forsberg, M. Passare, A. Tsikh, *Laurent determinants and arrangements of hyperplane amoebas,* Advances in Math., 151 (2000), 45–70.

[7] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants,* (Mathematics : Theory and Applications). Springer Verlag ; ISBN: 0817636609 ; (1996/07/10).

[8] I. Itenberg, G. Mikhalkin, E. Shustin, *Tropical Algebraic Geometry,* Oberbolfach Seminars **35**, Birkhäuser (2007).

[9] V.N. Kolokoltsov, *Idempotency structures in optimization,* Journal Math. Sci., **104–1** (2001), 847–880.

[10] M. Kontsevich, Y. Soibelman, *Affine structure and non-archmedean analytic spaces,* arXiv: math.AG/0406564 v1 2004/6/28.

[11] G.L. Litvinov, V.P. Maslov (ed.), *Idempotent Mathematics and Mathematical Physics,* Contemporary Mathematics, **377** (2005), ISBN 0-8218-3538-6.

[12] G. Mikhalkin, *Real algebraic curves, moment map and amoebas,* Ann. of Math., 151 (2000), 309–326.

[13] G. Mikhalkin, *Counting curves via lattice paths in polygons,* C. R. Math. Acad. Sci. Paris **336–8** (2003), 629–634.

[14] G. Mikhalkin, *Amoebas of algebraic varieties and tropical geometry,* arXiv: math.AG/0403015. Different Faces of Geometry, International Mathematical Series,Vol.3(Donaldson, Simon; Eliashberg,Yakov; Gromov, Mikhael(Eds.)) (2004), 257–300.

[15] G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces,* Topology **43–5** (2004), 1035–1065.

[16] G. Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbf{R}^2$, arXiv: math.AG/0312530. Journal of Amer. Math. Soc., **18** (2005), 313–377.

[17] G. Mikhalkin, *Tropical geometry and its application,* arXiv:math.AG/0601041, (2006). Proceedings of the ICM 2006, Madrid.

[18] M. Passare, H. Rullgard, *Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope,* Duke Math. J. **121 − 3** (2004), 481–507.

[19] J.-E. Pin, *Tropical semirings, Idempotency,* Publ. Newton Inst., **11**, Cambridge Univ. Press, Cambridge, (1998), 50–69.

[20] P. Ribenboim, *Fields: Algebraically closed and others,* manuscripts math., **75** (1992), 115–150.

[21] J. Richter-Geber, B. Sturmfels, T. Theobald, *First steps in tropical geometry,* arXiv: math.AG/0306366. in [11], 289–317.

[22] F.J. Rayner, *Algebraically closed fields analogous to fields of Puiseux series,* J. London Math. Soc., **8** (1974), 504–506.

[23] E. Shustin, *Patchworking singular algebraic curves, non-Archimedean amoebas and enumerative geometry,* arXiv math.AG/0211278.

[24] O. Viro, *Dequantization of real algebraic geometry on logarithmic paper,* European Congress Math., **1** Progress in Math., **201**, Birkhäuser, Basel, (2001), pp. 135–146.

[25] O. Viro, *Hyperfields for tropical geometry I. Hyperfields and dequantization,* arXiv: math.AG 1006.3034.