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# NOTES ON DING-IOHARA ALGEBRA AND AGT CONJECTURE

H. AWATA, B. FEIGIN, A. HOSHINO, M. KANAI, J. SHIRAISHI AND S. YANAGIDA

ABSTRACT. We study the representation theory of the Ding-Iohara algebra $\mathcal{U}$ to find $q$-analogues of the Alday-Gaiotto-Tachikawa (AGT) relations. We introduce the endomorphism $T(u,v)$ of the Ding-Iohara algebra, having two parameters $u$ and $v$. We define the vertex operator $\Phi(w)$ by specifying the permutation relations with the Ding-Iohara generators $x^{\pm}(z)$ and $\psi^{\pm}(z)$ in terms of $T(u,v)$. For the level one representation, all the matrix elements of the vertex operators with respect to the Macdonald polynomials are factorized and written in terms of the Nekrasov factors for the $K$-theoretic partition functions as in the AGT relations. For higher levels $m=2,3,\ldots$, we present some conjectures, which imply the existence of the $q$-analogues of the AGT relations.

## 1. INTRODUCTION

The aim of this note is to continue our study on the representation theory of the Ding-Iohara algebra $\mathcal{U}$ [DI] on positive integer levels, and to search a connection with the findings of Alday, Gaiotto and Tachikawa (AGT) [AGT]. Authors' previous discussions on $\mathcal{U}$ are found in [FHHSY] and [FHSSY]. As for the related works, see [FT], [SV], [FFJMM1], [FFJMM2] and [Sc].

In [FHHSY], we studied the level one action of the Ding-Iohara algebra $\mathcal{U}$ on the space of Macdonald symmetric functions $P_\lambda(x;q,t)$, namely on the Fock space $\mathcal{F}_u$ (see §2.1, §2.2 and §2.3). In [FHSSY], we showed that for positive integer levels $m=2,3,\ldots$, the Ding-Iohara algebra is realized on the $m$-fold tensor space $\mathcal{F}_{u_1}\otimes\mathcal{F}_{u_2}\otimes\cdots\otimes\mathcal{F}_{u_m}$ by the deformed $\mathcal{W}_m$ algebra together with an extra Heisenberg algebra. In this note, we introduce several bases on the $m$-fold tensor representation space. The first is the Macdonald-type basis $(|P_{\boldsymbol\lambda}\rangle)$ (see §3.3). Here $\boldsymbol\lambda=(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(m)})$, and each component $\lambda^{(i)}$ is a partition. Next we introduce the 'Poincaré-Birkoff-Witt-type basis' $(|X_{\boldsymbol\lambda}\rangle)$, and the 'integral basis' $(|K_{\boldsymbol\lambda}\rangle)$ (see §2.4 and §3.4). In the level one case, we can show that $(|K_\lambda\rangle)$ essentially gives the integral form $J_\lambda(x;q,t)$ (see Proposition 2.11). Unfortunately, at this moment, we do not have proofs that $(|X_{\boldsymbol\lambda}\rangle)$ and $(|K_\lambda\rangle)$ are bases for higher level cases $m=2,3,\ldots$.

We introduce an endomorphism $T(u,v)$ acting on the Ding-Iohara algebra having two parameters $u$ and $v$ (see Definition 2.3). In the level one case, we define the vertex operator $\Phi(w):\mathcal{F}_u\to\mathcal{F}_v$ by the normalization $\Phi(w)\,|0\rangle=|0\rangle+\cdots$, and the permutation relations $T(vw,q^{-1}tuw)(a)\Phi(w)=\Phi(w)T(q^{-1}tvw,uw)(a)$ for all $a\in\mathcal{U}$ (see Definition 2.12). Then we claim that

(1) (Proposition 2.13) the $\Phi(w)$ exists uniquely as

$$\Phi(w)=\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{v^n-(t/q)^nu^n}{1-q^n}a_{-n}w^n\Big)\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\frac{v^{-n}-u^{-n}}{1-q^{-n}}a_nw^{-n}\Big),$$

(2) (Proposition 2.14) all the matrix elements $\langle K_\lambda|\,\Phi(w)\,|K_\mu\rangle$ are factorized as

$$\langle K_\lambda|\,\Phi(w)\,|K_\mu\rangle=N_{\lambda,\mu}(qv/tu)(-tuvw/q)^{|\lambda|}(tvw/q)^{-|\mu|}u^{|\mu|}t^{-n(\mu)}q^{n(\mu')}.$$

Here $a_n$'s denote the Heisenberg generators satisfying $[a_m, a_n] = \delta_{m+n,0} m(1-q^{|m|})/(1-t^{|m|})$, and we have used the notation for the '$K$-theoretic Nekrasov factor' (see Definition 2.10)

$$\begin{aligned} N_{\lambda,\mu}(u) &:= \prod_{(i,j)\in\lambda} (1 - uq^{-\mu_i+j-1}t^{-\lambda'_j+i}) \cdot \prod_{(k,l)\in\mu} (1 - uq^{\lambda_k-l}t^{\mu'_l-k+1}) \\ &= \prod_{\square\in\lambda} (1 - uq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)}) \cdot \prod_{\blacksquare\in\mu} (1 - uq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1}). \end{aligned}$$

See §2.2 for the combinatorial symbols used here. Hence we found a $q$-analogue of the AGT relation [AGT] for the case the gauge group is U(1).

For higher level cases, we define the vertex operator $\Phi(w)$ in a similar manner (see Definition 3.12). Then we present our main conjecture about the properties of $\Phi(w)$ (see Conjecture 3.13). Our conjecture implies that we have $q$-deformed AGT relation for the case the gauge group is $\mathrm{U}(m)$.

This note is organised as follows. In Section 2, we recall the definition of the Ding-Iohara algebra $\mathcal{U}$, the Macdonald polynomials, and the level one representation of $\mathcal{U}$ on the Fock space $\mathcal{F}_u$. We give the definitions of the integral basis $|K_\lambda\rangle$ and the vertex operator $\Phi(w)$. Then we state the properties of $\Phi(w)$ in Proposition 2.14. In Section 3, we study the level $m$ representation given on the $m$-fold tensor space $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2} \otimes \cdots \otimes \mathcal{F}_{u_m}$. In Conjecture 3.13, we summarize our observation about the vertex operator $\Phi(w)$. Section 4 is devoted to a brief review of the AGT conjecture, Whittaker or Gaiotto state, and their five dimensional version. In Section 5, we study the Whittaker vectors for the Ding-Iohara algebra. In Section 6, we give some examples of calculating the matrix elements of $\Phi(w)$ for the level one case.

## 2. Level One Representation

**2.1. Ding-Iohara algebra.** Recall the Ding-Iohara algebra [DI]. Let $q, t$ be independent indeterminates and $\mathbb{F} := \mathbb{Q}(q,t)$. Let $g(z)$ be the formal series

$$g(z) := \frac{G^+(z)}{G^-(z)} \in \mathbb{F}[[z]], \qquad G^\pm(z) := (1-q^{\pm1}z)(1-t^{\mp1}z)(1-q^{\mp1}t^{\pm1}z).$$

We have $g(z) = g(z^{-1})^{-1}$ as is required.

**Definition 2.1.** Let $\mathcal{U}$ be the unital associative algebra over $\mathbb{F}$ generated by the Drinfeld currents $x^\pm(z) = \sum_{n\in\mathbb{Z}} x_n^\pm z^{-n}$, $\psi^\pm(z) = \sum_{\pm n\in\mathbb{Z}_{\geq 0}} \psi_n^\pm z^{-n}$ and the central element $\gamma^{\pm1/2}$, satisfying the defining relations

$$\begin{aligned} &\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z), && \psi^+(z)\psi^-(w) = \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^-(w)\psi^+(z), \\ &\psi^+(z)x^\pm(w) = g(\gamma^{\mp1/2}w/z)^{\mp1}x^\pm(w)\psi^+(z), && \psi^-(z)x^\pm(w) = g(\gamma^{\mp1/2}z/w)^{\pm1}x^\pm(w)\psi^-(z), \end{aligned}$$

$$[x^+(z), x^-(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\left(\delta(\gamma^{-1}z/w)\psi^+(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^-(\gamma^{-1/2}w)\right),$$

$$G^\mp(z/w)x^\pm(z)x^\pm(w) = G^\pm(z/w)x^\pm(w)x^\pm(z).$$

**Fact 2.2.** The algebra $\mathcal{U}$ has a formal Hopf algebra structure. The formulas for the coproduct read $\Delta(\gamma^{\pm1/2}) = \gamma^{\pm1/2} \otimes \gamma^{\pm1/2}$ and

$$\begin{aligned} \Delta(\psi^\pm(z)) &= \psi^\pm(\gamma_{(2)}^{\pm1/2}z) \otimes \psi^\pm(\gamma_{(1)}^{\mp1/2}z), \\ \Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(\gamma_{(1)}^{1/2}z) \otimes x^+(\gamma_{(1)}z), \\ \Delta(x^-(z)) &= x^-(\gamma_{(2)}z) \otimes \psi^+(\gamma_{(2)}^{1/2}z) + 1 \otimes x^-(z), \end{aligned}$$

where $\gamma_{(1)}^{\pm 1/2} := \gamma^{\pm 1/2} \otimes 1$ and $\gamma_{(2)}^{\pm 1/2} := 1 \otimes \gamma^{\pm 1/2}$. Since we do not use the antipode $a$ and the counit $\varepsilon$ in this paper, we omit them.

When the central element takes the value $\gamma^{\pm 1/2} = (t/q)^{\pm m/4}$ on a representation space with some $m \in \mathbb{Q}$, we call it of *level m*.

Now we introduce our main tool in the present paper.

**Definition 2.3.** For generic parameters $u$ and $v$, define the endomorphism $T(u,v)$ of $\mathcal{U}$ by

$$\begin{aligned} T(u,v)(x^+(z)) &= (1-u/z)x^+(z), \\ T(u,v)(x^-(z)) &= (1-\gamma v/z)x^-(z), \\ T(u,v)(\psi^\pm(z)) &= (1-\gamma^{\mp 1/2}u/z)(1-\gamma^{1\pm 1/2}v/z)\psi^\pm(z), \end{aligned}$$

where $\gamma$ is the central element. In Fourier modes, we have

$$\begin{aligned} T(u,v)(x_n^+) &= x_n^+ - ux_{n-1}^+, \\ T(u,v)(x_n^-) &= x_n^+ - \gamma v x_{n-1}^-, \\ T(u,v)(\psi_n^\pm) &= \psi_n^\pm - (\gamma^{\mp 1/2}u + \gamma^{1\pm 1/2}v)\psi_{n-1}^\pm + \gamma uv\psi_{n-2}^\pm. \end{aligned}$$

The endomorphism $T(u,v)$ will be used for giving the defining relations for our vertex operator $\Phi(w)$. See Definition 2.12 and Definition 3.12 below.

**Remark 2.4.** The image $T(u,v)(\mathcal{U})$ is strictly smaller than $\mathcal{U}$. Formally we can write $T(u,v)^{-1}(x_n^+) = x_n^+ + ux_{n-1}^+ + u^2x_{n-2}^+ + \cdots$ but this does not belong to $\mathcal{U}$ because of the infinite sum. It might be an interesting problem to find some meaning to the formal inverse, however, we will not consider it in this paper.

2.2. **Macdonald polynomials.** We basically follow [M] for the notations. A partition $\lambda$ is a series of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ with finitely many nonzero entries. We use the following symbols: $|\lambda| := \sum_{i\geq 1}\lambda_i$, $n(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i$. If $\lambda_l > 0$ and $\lambda_{l+1} = 0$, we write $\ell(\lambda) := l$ and call it the length of $\lambda$. The conjugate partition of $\lambda$ is denoted by $\lambda'$ which corresponds to the transpose of the diagram $\lambda$. The empty sequence is denoted by $\emptyset$. The dominance ordering is defined by $\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\sum_{k=1}^i \lambda_k \geq \sum_{k=1}^i \mu_k$ for all $i = 1, 2, \dots$.

We also follow [M] for the convention of the Young diagram. Namely, the first coordinate $i$ (the row index) increases as one goes downwards, and the second coordinate $j$ (the column index) increases as one goes rightwards. We denote by $\square = (i,j)$ the box located at the coordinate $(i,j)$. For a box $\square = (i,j)$ and a partition $\lambda$, we use the following notations:

$$i(\square) := i, \quad j(\square) := j, \quad a_\lambda(\square) := \lambda_i - j, \quad \ell_\lambda(\square) := \lambda'_j - i.$$

Let $\Lambda$ be the ring of symmetric functions in $x = (x_1, x_2, \dots)$ over $\mathbb{Z}$, and let $\Lambda_{\mathbb{F}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$. Let $m_\lambda$ be the monomial symmetric functions. Denote the power sum function by $p_n = \sum_{i\geq 1} x_i^n$. For a partition $\lambda$, we write $p_\lambda = \prod_i p_{\lambda_i}$. Macdonald's scalar product on $\Lambda_{\mathbb{F}}$ is

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \qquad z_\lambda = \prod_{i\geq 1} i^{m_i} \cdot m_i!, \tag{2.1}$$

Here we denote by $m_i$ the number of entries in $\lambda$ equal to $i$.

**Fact 2.5.** The Macdonald symmetric function $P_\lambda(x;q,t)$ is uniquely characterized by the conditions [M, Chap. VI, (4.7)].

$$\begin{aligned} &P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu \qquad (u_{\lambda\mu} \in \mathbb{F}), \\ &\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \qquad (\lambda \neq \mu). \end{aligned}$$

Denote $Q_\lambda := P_\lambda/\langle P_\lambda, P_\lambda\rangle_{q,t}$. Then $(Q_\lambda)$ and $(P_\lambda)$ are dual bases of $\Lambda_{\mathbb{F}}$.
The integral form $J_\lambda$ is defined by [M, Chap. VI, (8.1),(8.1′),(8.3)].

$$J_\lambda := c_\lambda P_\lambda = c'_\lambda Q_\lambda,$$
$$c_\lambda := \prod_{\square\in\lambda}(1 - q^{a_\lambda(\square)}t^{\ell_\lambda(\square)+1}), \quad c'_\lambda := \prod_{\square\in\lambda}(1 - q^{a_\lambda(\square)+1}t^{\ell_\lambda(\square)}). \tag{2.2}$$

As for the norms of $P_\lambda$ and $J_\lambda$, we have [M, Chap. VI, (6.19)]

$$\langle P_\lambda, P_\lambda\rangle_{q,t} = c'_\lambda/c_\lambda, \quad \langle J_\lambda, J_\lambda\rangle_{q,t} = c'_\lambda c_\lambda. \tag{2.3}$$

2.3. **Level one representation of $\mathcal{U}$.** Recall the level one representation constructed over the space of Macdonald polynomials [FHHSY]. Set $\widetilde{\mathbb{F}} := \mathbb{Q}(q^{1/4}, t^{1/4})$.

Let $\mathcal{H}$ be the Heisenberg algebra over $\widetilde{\mathbb{F}}$ with generators $\{a_n \mid n \in \mathbb{Z}\}$ satisfying

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}\, a_0.$$

Let $|0\rangle$ be the vacuum state satisfying the annihilation conditions for the positive Fourier modes $a_n|0\rangle = 0$ $(n \in \mathbb{Z}_{>0})$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we denote $|a_\lambda\rangle = a_{-\lambda_1}a_{-\lambda_2}\cdots|0\rangle$ for short. Denote by $\mathcal{F}$ the Fock space having the basis $(|a_\lambda\rangle)$.

As graded vector spaces, the space of the symmetric functions $\Lambda_{\widetilde{\mathbb{F}}}$ and the Fock space $\mathcal{F}$ are isomorphic. We denote the isomorphism by $\iota$. It is defined by

$$\iota : \mathcal{F} \xrightarrow{\sim} \Lambda_{\widetilde{\mathbb{F}}}, \quad |a_\lambda\rangle \mapsto p_\lambda. \tag{2.4}$$

We give an $\mathcal{H}$-module structure on $\Lambda_{\widetilde{\mathbb{F}}}$ by setting $a_0 v = v$ and

$$a_{-n}v = p_n v, \quad a_n v = n\frac{1-q^n}{1-t^n}\frac{\partial v}{\partial p_n}, \qquad (n > 0, v \in \Lambda_{\widetilde{\mathbb{F}}}).$$

In what follows we identify $\mathcal{F}$ and $\Lambda_{\widetilde{\mathbb{F}}}$ as $\mathcal{H}$ module via $\iota$.

**Fact 2.6** ([FHHSY, Prop. A.6]). Set

$$\eta(z) := \exp\Big(\sum_{n=1}^\infty \frac{1-t^{-n}}{n}a_{-n}z^n\Big)\exp\Big(-\sum_{n=1}^\infty \frac{1-t^n}{n}a_n z^{-n}\Big),$$
$$\xi(z) := \exp\Big(-\sum_{n=1}^\infty \frac{1-t^{-n}}{n}(t/q)^{n/2}a_{-n}z^n\Big)\exp\Big(\sum_{n=1}^\infty \frac{1-t^n}{n}(t/q)^{n/2}a_n z^{-n}\Big),$$
$$\varphi^+(z) := \exp\Big(-\sum_{n=1}^\infty \frac{1-t^n}{n}(1-t^nq^{-n})(t/q)^{-n/4}a_n z^{-n}\Big),$$
$$\varphi^-(z) := \exp\Big(\sum_{n=1}^\infty \frac{1-t^{-n}}{n}(1-t^nq^{-n})(t/q)^{-n/4}a_{-n} z^{n}\Big).$$

Let $u \in \widetilde{\mathbb{F}}^*$. We have a level one representation $\rho_u(\cdot)$ of $\mathcal{U}$ on $\mathcal{F}$ by setting

$$\rho_u(\gamma^{\pm 1/2}) = (t/q)^{\pm 1/4}, \quad \rho_u(\psi^\pm(z)) = \varphi^\pm(z), \quad \rho_u(x^+(z)) = u\,\eta(z), \quad \rho_u(x^-(z)) = u^{-1}\xi(z).$$

We denote this left $\mathcal{U}$-module by $\mathcal{F}_u$.

**Fact 2.7** ([AMOS][Sh]). The $x_0^+$ is identified with the first-order Macdonald difference operator (under the isomorphism $\iota : \mathcal{F}_u \xrightarrow{\sim} \Lambda_{\widetilde{\mathbb{F}}} : |P_\lambda\rangle \mapsto P_\lambda$, see (2.4))

$$x_0^+|P_\lambda\rangle = u\varepsilon_\lambda|P_\lambda\rangle, \quad \varepsilon_\lambda := 1 + (t-1)\sum_{i=1}^{\ell(\lambda)}(q^{\lambda_i}-1)t^{-i}. \tag{2.5}$$

The dual Fock space $\mathcal{F}^*$ is defined in a similar manner. Let $\langle 0|$ be the dual vacuum state satisfying the annihilation conditions for the negative Fourier modes $\langle 0|a_n = 0$ $(n \in \mathbb{Z}_{<0})$. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, write $\langle a_\lambda| = \langle 0| \cdots a_{\lambda_2} a_{\lambda_1}$ for short. The $(\langle a_\lambda|)$ is a basis of $\mathcal{F}^*$. By the homomorphism $\rho_u$, $\mathcal{F}^*$ becomes a right $\mathcal{U}$-module.

We have the compatibility between the Macdonald scalar product and the Fock pairing:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \langle a_\lambda | a_\mu \rangle.$$

2.4. **Integral basis $|K_\lambda\rangle$ for the level one case.** One of our motivations of this paper is to study the integral form $J_\lambda = c_\lambda P_\lambda$ of the Macdonald symmetric function, and its higher level analogues, from the point of view of the Ding-Iohara algebra $\mathcal{U}$. A point is how one can understand the mysterious normalization of $J_\lambda$.

The standard normalization of the Macdonald symmetric function is based on the lower triangular expansion $P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu$ with respect to the dominance ordering. Set the integral form by $J_\lambda = c_\lambda P_\lambda$, then the scalar product $\langle J_\lambda, J_\lambda \rangle_{q,t} = c'_\lambda c_\lambda$ is a polynomial in $q$ and $t$. As we will observe shortly, we have a similar polynomiality in all the matrix elements of our vertex operator with respect to the integral forms. At first glance, it seems that we need to face the problem of understanding the $c_\lambda$ from the algebra $\mathcal{U}$. We, however, bypass it by introducing a Poincaré-Birkhoff-Witt-type basis for $\mathcal{F}_u$.

For simplicity of display, we treat separately the level one case here. We omit writing the symbol $\rho_u$ from our formulas. For any partition $\lambda$, set $|X_\lambda\rangle$ by

$$|X_\lambda\rangle = x^+_{-\lambda_1} x^+_{-\lambda_2} \cdots x^+_{-\lambda_{\ell(\lambda)}} |0\rangle.$$

For the dual space, we set

$$\langle X_\lambda| = \langle 0| x^+_{\lambda_{\ell(\lambda)}} \cdots x^+_{\lambda_2} x^+_{\lambda_1}.$$

**Proposition 2.8.** The $(|X_\lambda\rangle)$ (resp. $(\langle X_\lambda|)$) is a basis of $\mathcal{F}$ (resp. $\mathcal{F}^*$).

On $\mathcal{F}_u$, we can expand the eigenfunctions of the operator $x^+_0$, namely the $|P_\lambda\rangle$'s, with respect to the basis $(|X_\lambda\rangle)$. Set

$$|K_\lambda\rangle = \left|X_{(1^{|\lambda|})}\right\rangle + \sum_{\mu > (1^{|\lambda|})} c_{\lambda\mu}(u) \left|X_\mu\right\rangle, \qquad x^+_0 |K_\lambda\rangle = u\varepsilon_\lambda |K_\lambda\rangle,$$

were $c_{\lambda\mu}(u) \in \mathbb{F}[u]$. Namely, we normalize the eigenfunctions $|K_\lambda\rangle$ in such a way that the coefficient of $\left|X_{(1^{|\lambda|})}\right\rangle$ is one.

Similarly on the dual space $\mathcal{F}^*_u$, set

$$\langle K_\lambda| = \left\langle X_{(1^{|\lambda|})}\right| + \sum_{\mu > (1^{|\lambda|})} c_{\lambda\mu}(u) \left\langle X_\mu\right|, \qquad \langle K_\lambda| x^+_0 = u\varepsilon_\lambda \langle K_\lambda|.$$

**Example 2.9.** For $|\lambda| \le 2$, we have

$$\begin{aligned}
\left|K_{(1)}\right\rangle &= \left|X_{(1)}\right\rangle = -t^{-1}u \left|J_{(1)}\right\rangle, \\
\left|K_{(2)}\right\rangle &= \left|X_{(1^2)}\right\rangle + \frac{(q-1)u}{t} \left|X_{(2)}\right\rangle = t^{-2}u^2 \left|J_{(2)}\right\rangle, \\
\left|K_{(1^2)}\right\rangle &= \left|X_{(1^2)}\right\rangle + \frac{q(t-1)u}{t} \left|X_{(2)}\right\rangle = t^{-3}u^2 \left|J_{(1^2)}\right\rangle.
\end{aligned}$$

In this paper we use the following notation for the so-called 'Nekrasov factor.'

**Definition 2.10.** For a pair of partitions $(\lambda, \mu)$ and an indeterminate $u$, set

$$N_{\lambda,\mu}(u) := \prod_{(i,j)\in\lambda} \left(1 - uq^{-\mu_i+j-1} t^{-\lambda'_j+i}\right) \cdot \prod_{(k,l)\in\mu} \left(1 - uq^{\lambda_k-l} t^{\mu'_l-k+1}\right)$$

$$= \prod_{\square\in\lambda}(1 - uq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)}) \cdot \prod_{\blacksquare\in\mu}(1 - uq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1}).$$

**Proposition 2.11.** We have

$$|K_\lambda\rangle = (-u/t)^{|\lambda|}t^{-n(\lambda)}\,|J_\lambda\rangle, \quad \langle K_\lambda| = (-u)^{|\lambda|}t^{-n(\lambda)}\,\langle J_\lambda|,$$
$$\langle K_\lambda|K_\lambda\rangle = (-u^2)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}N_{\lambda,\lambda}(q/t).$$

The proof is due to the specialization technique of [M, Chap. VI, (6.17)]. The detail will appear elsewhere.

2.5. **Vertex operator for the level one case.** We state our definition of the level one vertex operator $\Phi_u^v(w)$ in terms of the endomorphism $T(u,v)$.

**Definition 2.12.** Define the vertex operator $\Phi(w)$ by the conditions

$$\Phi(w) = \Phi_u^v(w) : \mathcal{F}_u \longrightarrow \mathcal{F}_v,$$
$$\Phi(w)\,|0\rangle = |0\rangle + O(w),$$
$$T(vw, q^{-1}tuw)(a)\Phi(w) = \Phi(w)T(q^{-1}tvw, uw)(a) \quad (\forall\, a \in \mathcal{U}).$$

In terms of $\eta(z), \xi(z), \varphi^\pm(z)$, the permutation relations are explicitly written as

$$(1 - vw/z)v\eta(z)\Phi(w) = (1 - q^{-1}tvw/z)\Phi(w)u\eta(z),$$
$$(1 - (t/q)^{3/2}uw/z)v^{-1}\xi(z)\Phi(w) = (1 - (t/q)^{1/2}uw/z)\Phi(w)u^{-1}\xi(z),$$
$$(1 - (t/q)^{-1/4}vw/z)(1 - (t/q)^{7/4}uw/z)\varphi^+(z)\Phi(w)$$
$$= (1 - (t/q)^{3/4}vw/z)(1 - (t/q)^{3/4}uw/z)\Phi(w)\varphi^+(z),$$
$$(1 - (t/q)^{1/4}vw/z)(1 - (t/q)^{5/4}uw/z)\varphi^-(z)\Phi(w)$$
$$= (1 - (t/q)^{5/4}vw/z)(1 - (t/q)^{1/4}uw/z)\Phi(w)\varphi^-(z).$$

From these, one immediately finds that the $\Phi(w)$ can be uniquely expressed in terms of a normal ordered exponent of the Heisenberg generators.

**Proposition 2.13.** We have

$$\Phi(w) = \exp\Big(-\sum_{n=1}^\infty \frac{1}{n}\frac{v^n - (t/q)^n u^n}{1-q^n}a_{-n}w^n\Big)\exp\Big(\sum_{n=1}^\infty \frac{1}{n}\frac{v^{-n} - u^{-n}}{1-q^{-n}}a_n w^{-n}\Big). \quad (2.6)$$

Now we are ready to state our main result.

**Proposition 2.14.** Let $J_\lambda$ be the integral form of the Macdonald polynomial. Then we have

$$\langle J_\lambda|\,\Phi(w)\,|J_\mu\rangle = N_{\lambda,\mu}(qv/tu)w^{|\lambda|-|\mu|}(tu/q)^{|\lambda|}(-v/q)^{-|\mu|}t^{n(\lambda)}q^{n(\mu')}.$$

This Proposition and Proposition 2.11 give us

$$\langle K_\lambda|\,\Phi(w)\,|K_\mu\rangle = N_{\lambda,\mu}(qv/tu)(-tuvw/q)^{|\lambda|}(tvw/q)^{-|\mu|}u^{|\mu|}t^{-n(\mu)}q^{n(\mu')}.$$

**Remark 2.15.** Proposition 2.14 is nothing but the $K$-theoretic analogue of [CO]. In fact, one can prove this based on their argument and the geometric realization of Ding-Iohara algebra on $\oplus_n K^{\mathbb{T}}(\mathrm{Hilb}_n(\mathbb{C}^2))$. The $\Phi(w)$ is essentially the same with the operator constructed from certain virtual bundle in [SV]. The proof will appear elsewhere.

Consider the composition of the vertex operators

$$\Phi_v^w(z_1)\Phi_u^v(z_2) : \mathcal{F}_u \longrightarrow \mathcal{F}_v \longrightarrow \mathcal{F}_w.$$

We have from Proposition 2.14

$$\langle 0| \Phi_v^w(z_1)\Phi_u^v(z_2) |0\rangle = \sum_\lambda \frac{N_{\emptyset,\lambda}(qw/tv)N_{\lambda,\emptyset}(qv/tu)}{N_{\lambda,\lambda}(q/t)}(uz_2/wz_1)^{|\lambda|}. \tag{2.7}$$

The right hand side of (2.7) coincides with the instanton part of the 5D U(1) Nekrasov partition function with $N_f = 2$ fundamental matters (see [AY2, §5]). See Remark 3.14 below as for the higher level case.

2.6. **Examples of the calculation of the matrix elements of $\Phi(w)$.** We show some examples of calculating the matrix elements of $\Phi(w)$.

On $\mathcal{F}_u$, we have

$$\sum_{l\geq 0} f_l x_{m-l}^+ x_{n+l}^+ = \sum_{l\geq 0} f_l x_{n-l}^+ x_{m+l}^+, \tag{2.8}$$

$$f_0 = 1, \qquad f_l = \frac{(1-q)(1-t^{-1})(1-q^l t^{-l})}{1-qt^{-1}} \quad \text{for } l = 1,2,3,\ldots.$$

The permutation rule for $x_n^+$ and $\Phi(w) = \Phi_u^v(w)$ reads

$$(x_n^+ - vwx_{n-1}^+)\Phi(w) = \Phi(w)(x_n^+ - q^{-1}tvwx_{n-1}^+). \tag{2.9}$$

We have $\langle 0| \Phi(w) |0\rangle = 1$, $x_0^+ |0\rangle = u |0\rangle$, $x_n^+ |0\rangle = 0$ $(n = 1,2,\ldots)$, and $\langle 0| x_0^+ = v \langle 0|$, $\langle 0| x_{-n}^+ = 0$ $(n = 1,2,\ldots)$.

From (2.9) written for $n = 1$, we have $\langle 0| (x_1^+ - vwx_0^+)\Phi(w) |0\rangle = \langle 0| \Phi(w)(x_1^+ - q^{-1}tvwx_0^+) |0\rangle$. Hence we have

$$\langle X_{(1)}| \Phi(w) |X_\emptyset\rangle = \langle 0| x_1^+\Phi(w) |0\rangle = vw(v - q^{-1}tu).$$

From (2.8) written for $m = 1, n = 0$ and $m = 1, n = -1$, we have $\langle 0| x_1^+ x_0^+ = v(1 - f_1) \langle 0| x_1^+$ and $\langle 0| x_1^+ x_{-1}^+ = -v^2 f_1 \langle 0|$. Then, from (2.9) written for $n = 0$, we have $\langle 0| x_1^+(x_0^+ - vwx_{-1}^+)\Phi(w) |0\rangle = \langle 0| x_1^+\Phi(w)(x_0^+ - q^{-1}tvwx_{-1}^+) |0\rangle$. Hence we have

$$\begin{aligned} &v(1-f_1) \langle 0| x_1^+\Phi(w) |0\rangle + v^3 w f_1 \langle 0| \Phi(w) |0\rangle \\ &= u \langle 0| x_1^+\Phi(w) |0\rangle - q^{-1}tvw \langle 0| x_1^+\Phi(w)x_{-1}^+ |0\rangle, \end{aligned}$$

namely

$$\langle X_{(1)}| \Phi(w) |X_{(1)}\rangle = \langle 0| x_1^+\Phi(w)x_{-1}^+ |0\rangle = -u^2(1-qv/u)(1-v/tu).$$

## 3. Level $m$ representation

One can easily guess what should be the higher level counterparts of the intertwining properties in Definition 2.12, and the integral basis $|K_\lambda\rangle$. By some brute force computations, we observed that the AGT phenomena may exist also for the higher level cases, namely, all the matrix elements of the vertex operator with respect to $|K_\lambda\rangle$ are factorized and written in terms of the function $N_{\lambda,\mu}(u)$.

3.1. **'PBW-type basis' for the level $m$ case.** Let $m$ be a positive integer and $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ be an $m$-tuple of parameters. Consider the $m$-fold tensor representation $\rho_{u_1} \otimes \rho_{u_2} \otimes \cdots \otimes \rho_{u_m}$ on $\mathcal{F}^{\otimes m}$. Define $\Delta^{(m)}$ inductively by $\Delta^{(1)} :=$ id, $\Delta^{(2)} := \Delta$ and $\Delta^{(m)} := (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \circ \Delta^{(m-1)}$.

**Definition 3.1.** Define the morhpism $\rho_{\mathbf{u}}^{(m)}$ by

$$\rho_{\mathbf{u}}^{(m)} := (\rho_{u_1} \otimes \rho_{u_2} \otimes \cdots \otimes \rho_{u_m}) \circ \Delta^{(m)}.$$

We denote by $\mathcal{F}_{\mathbf{u}}$ (resp. $\mathcal{F}_{\mathbf{u}}^*$) the left (resp. right) $\mathcal{U}$-module on $\mathcal{F}^{\otimes m}$ (resp. $\mathcal{F}^{*\otimes m}$) given by $\rho_{\mathbf{u}}^{(m)}$. These representations are of level $m$, and we call them *the* level $m$ representations.

Set

$$X^{(1)}(z) := \rho_{\mathbf{u}}^{(m)}(x^+(z)) = (\rho_{u_1} \otimes \rho_{u_2} \otimes \cdots \otimes \rho_{u_m}) \circ \Delta^{(m)}(x^+(z)).$$

Then we have

$$X^{(1)}(z) = \sum_{i=1}^{m} u_i \widetilde{\Lambda}_i(z), \tag{3.1}$$

where

$$\widetilde{\Lambda}_i(z) := \varphi^-(p^{-1/4}z) \otimes \varphi^-(p^{-3/4}z) \otimes \cdots \otimes \varphi^-(p^{-(2i-3)/4}z) \otimes \eta(p^{-(i-1)/2}z) \otimes 1 \otimes \cdots \otimes 1. \tag{3.2}$$

Here $p := q/t$ and $\eta(p^{-(i-1)/2}z)$ sits in the $i$-th tensor component. (See [FHSSY, Lemma 2.6].) For $k = 2, 3, \ldots$, set further

$$X^{(k)}(z) := X^{(1)}(p^{k-1}z) \cdots X^{(1)}(pz) X^{(1)}(z).$$

Then for $k = 1, 2, \ldots, m$ we have

$$X^{(k)}(z) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} u_{i_1} u_{i_2} \cdots u_{i_k} : \widetilde{\Lambda}_{i_1}(z) \widetilde{\Lambda}_{i_2}(pz) \cdots \widetilde{\Lambda}_{i_k}(p^{k-1}z) :,$$

and $0 = X^{(m+1)}(z) = X^{(m+2)}(z) = \cdots$. Here $: * :$ denotes the usual normal ordering in the Heisenberg algebra $\mathcal{H}$. Define the Fourier components $X_i^{(k)}$ of $X^{(k)}(z)$ by

$$X^{(k)}(z) = \sum_{i \in \mathbb{Z}} X_i^{(k)} z^{-i}.$$

**Remark 3.2.** As for the connection between the $X^{(i)}(z)$'s and the deformed $\mathcal{W}_m$ generators, see [FHSSY].

**Definition 3.3.** Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ be an $m$-tuple of partitions with $\lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots)$. We set

$$|X_{\boldsymbol{\lambda}}\rangle := X^{(1)}_{-\lambda_1^{(1)}} X^{(1)}_{-\lambda_2^{(1)}} \cdots X^{(2)}_{-\lambda_1^{(2)}} X^{(2)}_{-\lambda_2^{(2)}} \cdots X^{(m)}_{-\lambda_1^{(m)}} X^{(m)}_{-\lambda_2^{(m)}} \cdots |\mathbf{0}\rangle,$$
$$\langle X_{\boldsymbol{\lambda}}| := (q/t)^{\sum_{k=1}^{m}(k-1)|\lambda^{(k)}|} \langle \mathbf{0}| \cdots X^{(m)}_{\lambda_2^{(m)}} X^{(m)}_{\lambda_1^{(m)}} \cdots X^{(2)}_{\lambda_2^{(2)}} X^{(2)}_{\lambda_1^{(2)}} \cdots X^{(1)}_{\lambda_2^{(1)}} X^{(1)}_{\lambda_1^{(1)}},$$

where $|\mathbf{0}\rangle := |0\rangle^{\otimes m}$ and $\langle \mathbf{0}| := \langle 0|^{\otimes m}$.

**Conjecture 3.4.** The $(|X_{\boldsymbol{\lambda}}\rangle)$ (resp. $(\langle X_{\boldsymbol{\lambda}}|)$) is a basis of $\mathcal{F}_{\mathbf{u}}$ (resp. $\mathcal{F}_{\mathbf{u}}^*$).

3.2. **Partial orderings.** As in the case of level one representation, we study the eigenfunctions of the operator $X_0^{(1)} = \rho_{\mathbf{u}}^{(m)}(x_0^+)$ on the spaces $\mathcal{F}_{\mathbf{u}}$ and $\mathcal{F}_{\mathbf{u}}^*$ . A remark is in order. We can not regard the $X_0^{(1)}$ as a self adjoint operator, because of the structure of the coproduct. Hence we need to consider the left eigenfunctions in $\mathcal{F}_{\mathbf{u}}$ and the right eigenfunctions in $\mathcal{F}_{\mathbf{u}}^*$ separately.

For $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ with $\lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots)$, we denote the total number of boxes by $|\boldsymbol{\lambda}| := \sum_{k=1}^{m} |\lambda^{(k)}|$.

**Definition 3.5.** Introduce two partial orderings $\geq^R$ and $\geq^L$ on the $m$-tuples of partitions by

$$\begin{aligned}
&\boldsymbol{\lambda} \geq^R \boldsymbol{\mu} \overset{\text{def}}{\Longleftrightarrow} |\boldsymbol{\lambda}| = |\boldsymbol{\mu}| \text{ and} \\
&\qquad |\lambda^{(1)}| + \cdots + |\lambda^{(j-1)}| + \sum_{k=1}^{i} \lambda_k^{(j)} \geq |\mu^{(1)}| + \cdots + |\mu^{(j-1)}| + \sum_{k=1}^{i} \mu_k^{(j)} \\
&\qquad \text{for all } i \geq 1, 1 \leq j \leq m, \\
&\boldsymbol{\lambda} \geq^L \boldsymbol{\mu} \overset{\text{def}}{\Longleftrightarrow} |\boldsymbol{\lambda}| = |\boldsymbol{\mu}| \text{ and} \\
&\qquad |\lambda^{(m)}| + \cdots + |\lambda^{(j+1)}| + \sum_{k=1}^{i} \lambda_k^{(j)} \geq |\mu^{(m)}| + \cdots + |\mu^{(j+1)}| + \sum_{k=1}^{i} \mu_k^{(j)} \\
&\qquad \text{for all } i \geq 1, 1 \leq j \leq m.
\end{aligned} \tag{3.3}$$

**Example 3.6.** We consider the case $>^L$ and denote it by $>$ for short. In the case $m = 2$ and $|\boldsymbol{\lambda}| \leq 3$, we have

$$(\emptyset, (1)) > ((1), \emptyset),$$

$$(\emptyset, (2)) > (\emptyset, (1^2)) > ((1), (1)) > ((2), \emptyset) > ((1^2), \emptyset),$$

$$(\emptyset, (3)) > (\emptyset, (21)) \begin{matrix} > ((1), (2)) > \\ > (\emptyset, (1^3)) > \end{matrix} ((1), (1^2)) > ((2), (1)) \begin{matrix} > ((3), (\emptyset)) > \\ > ((1^2), (1)) > \end{matrix} ((21), \emptyset) > ((1^3), \emptyset).$$

3.3. **Eigenfunctions.** For an $m$-tuple of partitions $\boldsymbol{\lambda}$, set

$$m_{\boldsymbol{\lambda}} = m_{\lambda^{(1)}} \otimes m_{\lambda^{(2)}} \otimes \cdots \otimes m_{\lambda^{(m)}} \in \Lambda^{\otimes m},$$

where $m_{\lambda^{(i)}}$'s are the monomial symmetric functions. Via the isomorphism $\iota^{\otimes m}$ (see (2.4)), we identify $m_{\boldsymbol{\lambda}} \in \Lambda^{\otimes m}$ with the corresponding vector $|m_{\boldsymbol{\lambda}}\rangle \in \mathcal{F}_{\mathbf{u}}$ or $\langle m_{\boldsymbol{\lambda}}| \in \mathcal{F}_{\mathbf{u}}^*$.

**Proposition 3.7.** We have

$$X_0^{(1)} |m_{\boldsymbol{\lambda}}\rangle = \sum_{\boldsymbol{\mu} \leq^L \boldsymbol{\lambda}} \alpha_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) |m_{\boldsymbol{\mu}}\rangle,$$

$$\langle m_{\boldsymbol{\lambda}}| X_0^{(1)} = \sum_{\boldsymbol{\mu} \leq^R \boldsymbol{\lambda}} \beta_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) \langle m_{\boldsymbol{\mu}}|,$$

for some $\alpha_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}), \beta_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) \in \widetilde{\mathbb{F}}[u_1, u_2, \ldots, u_m]$.

**Proposition 3.8.** (1) For any $m$-tuples of partitions $\boldsymbol{\lambda}$, a vector $|P_{\boldsymbol{\lambda}}\rangle \in \mathcal{F}_{\mathbf{u}}$ is uniquely characterized by

$$|P_{\boldsymbol{\lambda}}\rangle = |m_{\boldsymbol{\lambda}}\rangle + \sum_{\boldsymbol{\mu} <^L \boldsymbol{\lambda}} a_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) |m_{\boldsymbol{\mu}}\rangle, \qquad (a_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) \in \widetilde{\mathbb{F}}(u_1, u_2, \ldots, u_m)),$$

$$X_0^{(1)} |P_{\boldsymbol{\lambda}}\rangle = \varepsilon_{\boldsymbol{\lambda},\mathbf{u}} |P_{\boldsymbol{\lambda}}\rangle, \qquad \varepsilon_{\boldsymbol{\lambda},\mathbf{u}} := \sum_{k=1}^{m} u_k \varepsilon_{\lambda^{(k)}}.$$

(2) For any $m$-tuples of partitions $\boldsymbol{\lambda}$, a vector $\langle P_{\boldsymbol{\lambda}}| \in \mathcal{F}_{\mathbf{u}}^*$ is uniquely characterized by

$$\langle P_{\boldsymbol{\lambda}}| = \langle m_{\boldsymbol{\lambda}}| + \sum_{\boldsymbol{\mu} <^R \boldsymbol{\lambda}} b_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) \langle m_{\boldsymbol{\mu}}|, \qquad (b_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\mathbf{u}) \in \widetilde{\mathbb{F}}(u_1, u_2, \ldots, u_m)),$$

$$\langle P_{\boldsymbol{\lambda}}| X_0^{(1)} = \varepsilon_{\boldsymbol{\lambda},\mathbf{u}} \langle P_{\boldsymbol{\lambda}}|.$$

(3) We have

$$\langle P_{\boldsymbol{\lambda}} | P_{\boldsymbol{\mu}} \rangle = \prod_{k=1}^{m} \frac{c'_{\lambda^{(k)}}}{c_{\lambda^{(k)}}} \cdot \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}.$$

For (1) and (2), it is enough to prove Proposition 3.7. The detail will appear elsewhere. Once we have (1) and (2), for the proof of (3) we only need to recall the norm of $P_\lambda$ (2.3).

The vector $|P_{\boldsymbol{\lambda}}\rangle$ can be considered as a higher level analogue of the Macdonald symmetric function $P_\lambda$.

**Example 3.9.** Consider the case $m=2$. We denote by $|P_{\lambda_1}\otimes P_{\lambda_2}\rangle$ the image of $P_{\lambda_1}\otimes P_{\lambda_2}$ in $\mathcal{F}_{u_1}\otimes\mathcal{F}_{u_2}$ under the isomorphism $\iota^{\otimes 2}$. Below we give some examples of the vectors $|P_{(\lambda_1,\lambda_2)}\rangle$ expanded in terms of $(|P_{\lambda_1}\otimes P_{\lambda_2}\rangle)$.

First we trivially have $|P_{(\emptyset,\emptyset)}\rangle=|1\otimes 1\rangle$. For $|\boldsymbol{\lambda}|=1$, we have

$$
\begin{aligned}
|P_{((1),\emptyset)}\rangle &= |P_{(1)}\otimes 1\rangle,\\
|P_{(\emptyset,(1))}\rangle &= |1\otimes P_{(1)}\rangle + (q/t)^{1/2}\frac{(t-q)u_2}{q(u_1-u_2)}\,|P_{(1)}\otimes 1\rangle.
\end{aligned}
$$

For $|\boldsymbol{\lambda}|=2$, we have

$$
\begin{aligned}
|P_{((1^2),\emptyset)}\rangle &= |P_{(1^2)}\otimes 1\rangle,\\
|P_{((2),\emptyset)}\rangle &= |P_{(2)}\otimes 1\rangle,\\
|P_{((1),(1))}\rangle &= |P_{(1)}\otimes P_{(1)}\rangle + (q/t)^{1/2}\frac{(1-q)(t+1)t(t-q)u_2}{q(1-qt)(u_1-tu_2)}\,|P_{(1^2)}\otimes 1\rangle\\
&\quad + (q/t)^{1/2}\frac{(t-q)u_2}{q(qu_1-u_2)}\,|P_{(2)}\otimes 1\rangle,\\
|P_{(\emptyset,(1^2))}\rangle &= |1\otimes P_{(1^2)}\rangle + (q/t)^{1/2}\frac{(t-q)u_2}{q(tu_1-u_2)}\,|P_{(1)}\otimes P_{(1)}\rangle\\
&\quad + \frac{(t-q)((q^2t-qt-q+t^2)u_2-qt(t^2-1)u_1)u_2}{qt(u_1-u_2)(1-qt)(tu_1-u_2)}\,|P_{(1^2)}\otimes 1\rangle\\
&\quad - \frac{(t-q)u_2}{q(tu_1-u_2)}\,|P_{(2)}\otimes 1\rangle,\\
|P_{(\emptyset,(2))}\rangle &= |1\otimes P_{(2)}\rangle - (q/t)^{1/2}\frac{(t-1)(1+q)(t-q)u_2}{(1-qt)(u_1-qu_2)}\,|P_{(1)}\otimes P_{(1)}\rangle\\
&\quad + \frac{(q-t)(q(q^2t-q+qt-t^2)u_2+(1-q^2)tu_1)u_2}{qt(1-qt)(u_1-u_2)(qu_2-u_1)}\,|P_{(2)}\otimes 1\rangle\\
&\quad - \frac{(t^2-1)(1-q^2)(t-q)u_2}{(qu_2-u_1)(1-qt)^2}\,|P_{(1^2)}\otimes 1\rangle.
\end{aligned}
$$

For the case $|\boldsymbol{\lambda}|=3$, the partial ordering $>^L$ is not a total ordering. Here we give five examples for the sake of demonstration:

$$
\begin{aligned}
|P_{((1^3),\emptyset)}\rangle &= |P_{(1^3)}\otimes 1\rangle,\\
|P_{((2,1),\emptyset)}\rangle &= |P_{(2,1)}\otimes 1\rangle,\\
|P_{((3),\emptyset)}\rangle &= |P_{(3)}\otimes 1\rangle,\\
|P_{((1^2),(1))}\rangle &= |P_{(1^2)}\otimes P_{(1)}\rangle + (q/t)^{1/2}\frac{(t-q)u_2}{q(qu_1-u_2)}\,|P_{(2,1)}\otimes 1\rangle\\
&\quad + (q/t)^{1/2}\frac{(1-q)(t-q)(1-t^3)t^2u_2}{q(1-qt^2)(1-t)(u_1-t^2u_2)}\,|P_{(1^3)}\otimes 1\rangle,\\
|P_{((2),(1))}\rangle &= |P_{(2)}\otimes P_{(1)}\rangle + (q/t)^{1/2}\frac{(t-q)u_2}{q(q^2u_1-u_2)}\,|P_{(3)}\otimes 1\rangle
\end{aligned}
$$

$$+ (q/t)^{1/2}\frac{(1-q^2)(t-q)(1-qt^2)tu_2}{q(1-qt)(1-q^2t)(u_1-tu_2)}\left|P_{(2,1)}\otimes 1\right\rangle.$$

As for the dual eigenvectors, we have $\left\langle P_{(\emptyset,\emptyset)}\right| = \langle 1\otimes 1|$, and

$$\left\langle P_{((1),\emptyset)}\right| = \left\langle P_{(1)}\otimes 1\right| - (q/t)^{1/2}\frac{(t-q)u_2}{q(u_1-u_2)}\left\langle 1\otimes P_{(1)}\right|,$$
$$\left\langle P_{(\emptyset,(1))}\right| = \left\langle 1\otimes P_{(1)}\right|$$

for $|\boldsymbol{\lambda}|=1$.

3.4. **'Integral basis' $|K_{\boldsymbol{\lambda}}\rangle$ for the level $m$ case.** As in the level one case, we introduce the following normalization of the eigenvectors.

**Definition 3.10.** Define the integral form $|K_{\boldsymbol{\lambda}}\rangle \in \mathcal{F}_{\mathbf{u}}$ by

$$X_0^{(1)}\left|K_{\boldsymbol{\lambda}}\right\rangle = \varepsilon_{\boldsymbol{\lambda},\mathbf{u}}\left|K_{\boldsymbol{\lambda}}\right\rangle, \qquad \left|K_{\boldsymbol{\lambda}}\right\rangle = \left((X_{-1}^{(1)})^{|\boldsymbol{\lambda}|}+\cdots\right)|\mathbf{0}\rangle.$$

Similarly we define $\langle K_{\boldsymbol{\lambda}}| \in \mathcal{F}_{\mathbf{u}}^*$ by

$$\left\langle K_{\boldsymbol{\lambda}}\right| X_0^{(1)} = \varepsilon_{\boldsymbol{\lambda},\mathbf{u}}\left\langle K_{\boldsymbol{\lambda}}\right|, \qquad \left\langle K_{\boldsymbol{\lambda}}\right| = \langle\mathbf{0}|\left((X_1^{(1)})^{|\boldsymbol{\lambda}|}+\cdots\right).$$

**Conjecture 3.11.** We have

$$\begin{aligned}\langle K_{\boldsymbol{\lambda}}|K_{\boldsymbol{\lambda}}\rangle &\overset{?}{=}\left((-1)^m(t/q)^{m-1}e_m(\mathbf{u})\right)^{|\boldsymbol{\lambda}|}\\ &\times\prod_{k=1}^m u_k^{-(m-2)|\lambda^{(k)}|}q^{-(m-2)n(\lambda^{(k)\prime})}t^{(m-2)n(\lambda^{(k)})}\times\prod_{i,j=1}^m N_{\lambda^{(i)},\lambda^{(j)}}(qu_i/tu_j).\end{aligned} \tag{3.4}$$

with $e_m(\mathbf{u}) := u_1u_2\cdots u_m$.

3.5. **Vertex operator for the level $m$ case.** We extend the construction of the vertex operator $\Phi(w)$ for higher level cases.

**Definition 3.12.** Let $\mathbf{u} = (u_1,u_2,\cdots,u_m)$ and $\mathbf{v} = (v_1,v_2,\cdots,v_m)$. Define the vertex operator $\Phi(w) = \Phi_{\mathbf{u}}^{\mathbf{v}}(w)$ by

$$\begin{aligned}&\Phi(w):\mathcal{F}_{\mathbf{u}} = \mathcal{F}_{u_1}\otimes\mathcal{F}_{u_2}\otimes\cdots\otimes\mathcal{F}_{u_m}\longrightarrow\mathcal{F}_{\mathbf{v}} = \mathcal{F}_{v_1}\otimes\mathcal{F}_{u_2}\otimes\cdots\otimes\mathcal{F}_{v_m},\\ &\Phi(w)|\mathbf{0}\rangle = |\mathbf{0}\rangle + O(w),\\ &T(e_m(\mathbf{v})w, q^{-1}te_m(\mathbf{u})w)(a)\Phi(w) = \Phi(w)T(q^{-1}te_m(\mathbf{v})w, e_m(\mathbf{u})w)(a)\quad(\forall\, a\in\mathcal{U}).\end{aligned}$$

Here we used the symbols $e_m(\mathbf{v}) := v_1v_2\cdots v_m$ and $e_m(\mathbf{u}) := u_1u_2\cdots u_m$.

Now we state our main conjecture.

**Conjecture 3.13.** (1) The $\Phi(w)$ exists uniquely.

(2) We have the factorized matrix elements with respect to the integral forms as

$$\begin{aligned}\langle K_{\boldsymbol{\lambda}}|\,\Phi(w)\,|K_{\boldsymbol{\mu}}\rangle &\overset{?}{=}\left((-1)^m(t/q)^m e_m(\mathbf{u})e_m(\mathbf{v})w\right)^{|\boldsymbol{\lambda}|}\left((t/q)e_m(\mathbf{v})w\right)^{-|\boldsymbol{\mu}|}\\ &\times\prod_{k=1}^m v_k^{-(m-1)|\lambda^{(k)}|}u_k^{|\mu^{(k)}|}q^{-(m-1)n(\lambda^{(k)\prime})+n(\mu^{(k)\prime})}t^{(m-1)n(\lambda^{(k)})-n(\mu^{(k)})}\\ &\times\prod_{i,j=1}^m N_{\lambda^{(i)},\mu^{(j)}}(qv_i/tu_j).\end{aligned} \tag{3.5}$$

Conjectures (3.5) and (3.4) imply

$$\langle\mathbf{0}|\,\Phi_{\mathbf{v}}^{\mathbf{w}}(z_2)\Phi_{\mathbf{u}}^{\mathbf{v}}(z_1)\,|\mathbf{0}\rangle = \sum_{\boldsymbol{\lambda}}\frac{\langle K_{\emptyset}|\,\Phi_{\mathbf{v}}^{\mathbf{w}}(z_2)\,|K_{\boldsymbol{\lambda}}\rangle\langle K_{\boldsymbol{\lambda}}|\,\Phi_{\mathbf{u}}^{\mathbf{v}}(z_1)\,|K_{\emptyset}\rangle}{\langle K_{\boldsymbol{\lambda}}|K_{\boldsymbol{\lambda}}\rangle} \tag{3.6}$$

$$\stackrel{?}{=}\sum_{\lambda}\left(\frac{e_m(\mathbf{u})z_1}{e_m(\mathbf{w})z_2}\right)^{|\lambda|}\prod_{i,j=1}^{m}\frac{N_{\emptyset,\lambda^{(j)}}(qw_i/tv_j)N_{\lambda^{(i)},\emptyset}(qv_i/tu_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(qv_i/tv_j)}$$

$$=\sum_{\lambda}\left(\frac{t^m}{q^m}\frac{e_m(\mathbf{u})z_1}{e_m(\mathbf{w})z_2}\right)^{|\lambda|}\prod_{i,j=1}^{m}\frac{N_{\emptyset,\lambda^{(j)}}(qw_i/tv_j)N_{\lambda^{(i)},\emptyset}(qv_i/tu_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(v_i/v_j)}.$$

**Remark 3.14.** The left hand side of (3.6) can be understood as a $q$-analogue of the four point correlation function of CFT. The right hand side coincides with the instanton part of the 5D $U(m)$ Nekrasov partition function with $N_f = 2m$ fundamental matters (see [AY2, §5]). Our main conjecture 3.13 implies that we have a description of the five dimensional analogue of the AGT conjecture in terms of the level $m$ representation of the Ding-Iohara algebra.

**Remark 3.15.** In [AFLT], a good understanding is found about the primary fields of the conformal field theory, the integrable structure, and the AGT conjecture. Their ideas and the main points are summarized as:

- to consider the extended algebra $\mathcal{A} :=$ (Virasoro algebra) $\otimes$ (Heisenberg algebra),
- to study the integrable structure in $\mathcal{A}$ and the complete eigenfunctions,
- matrix elements of the primary field with respect to the eigen-basis,
- factorization of the matrix elements in temrs of the Nekrasov function.

Note that on level two ($m = 2$), $\mathcal{U}$ is regarded as (deformed Virasoro) $\otimes$ (Heisenberg algebra). Hence, it is expected that our level two case ($m = 2$) can be regarded as a $q$-deformation of [AFLT].

## 4. Interlude: AGT conjecture

4.1. **Four dimensional version.** In [AGT] a remarkable proposal, now called the AGT conjecture/relation, was given on the equivalence between the conformal block of the Liouville theory and the Nekrasov partition function. Among the related investigations, Gaiotto proposed several degenerated versions in [G]. Its simplest case claims that the inner product $\langle G|G\rangle$ of a certain element $|G\rangle$ in the Verma module of Virasoro algebra coincides with the instanton part of the Nekrasov partition function $Z^{\rm inst}_{\text{pure SU}(2)}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)$ for the four dimensional $\mathcal{N}=2$ super-symmetric pure SU(2) gauge theory [N].

4.1.1. *Whittaker vector for Virasoro algebra.* Recall the notion of the Whittaker vector for a finite dimensional Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be a maximal nilpotent Lie subalgebra of $\mathfrak{g}$ and $\chi : \mathfrak{n} \to \mathbb{C}$ be a character. Let $V$ be any $U(\mathfrak{g})$-module. Then a vector $w \in V$ is called a Whittaker vector with respect to $\chi$ if $xw = \chi(x)w$ for all $x \in \mathfrak{n}$.

In [G], analogue of Whittaker vectors was considered for the Verma module of the Virasoro algebra. Let $\mathrm{Vir} := \mathbb{C}C \oplus \bigoplus_{n\in\mathbb{Z}}\mathbb{C}L_n$ be the Virasoro algebra with the relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m+n,0}\,C, \quad [C, \mathrm{Vir}] = 0.$$

We have a triangular decomposition $\mathrm{Vir} = \mathrm{Vir}_{>0}\oplus\mathrm{Vir}_0\oplus\mathrm{Vir}_{<0}$ with $\mathrm{Vir}_{>0} := \bigoplus_{n\in\mathbb{Z}_{>0}}\mathbb{C}L_n$, $\mathrm{Vir}_0 := \mathbb{C}C\oplus\mathbb{C}L_0$ and $\mathrm{Vir}_{<0} := \bigoplus_{n\in\mathbb{Z}_{<0}}\mathbb{C}L_n$. The Verma module $M_{c,h}$ is a representation of Vir induced from $\mathbb{C}_{c,h} = \mathbb{C}\,|c,h\rangle$, the one dimensional representation of $\mathrm{Vir}_{>0}\oplus\mathrm{Vir}_0$ where $\mathrm{Vir}_{>0}$ acts trivially, $L_0$ acts by multiplication of $h$ and $C$ acts by multiplication of $c$.

Note that the elements $L_1, L_2 \in \mathrm{Vir}_{>0}$ generate $\mathrm{Vir}_{>0}$. Thus if we set $\mathfrak{n} := \mathrm{Vir}_{>0}$ in the above definition of the Whittaker vector, then the homomorphism $\chi : \mathrm{Vir}_{>0} \to \mathbb{C}$ is determined by $\chi_1 := \chi(L_1)$ and $\chi_2 := \chi(L_2)$. Then the Whittaker vector $v$ is an element of the completed Verma module $\widehat{M_{c,h}}$ satisfying

$$L_1 v = \chi_1 v, \quad L_2 v = \chi_2 v.$$

The simplest case Gaiotto considered is the choice $\chi_2 = 0$, and we denote the corresponding Whittaker vector by $|G\rangle$. Imposing a normalization condition and changing parameter $\chi_1$, we have

$$L_1 |G\rangle = \Lambda^2 |G\rangle, \quad |G\rangle = |c, h\rangle + \cdots. \tag{4.1}$$

In fact such $|G\rangle$ is uniquely determined.

4.1.2. *Four dimensional Nekrasov partition function.* Recall [N] that Nekrasov's partition function $Z^{\text{inst}}_{\text{pure}\,SU(N)}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ for four dimensional pure $SU(N)$ gauge theory is defined to be the generating function of equivariant integrals over the $SU(N)$ instanton moduli spaces $M_{N,n}$, where $n$ is the instanton number:

$$Z^{\text{inst}}_{\text{pure}\,SU(N)}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) := \sum_{n=0}^{\infty} \Lambda^{2nN} \text{“} \int_{M_{N,n}} 1 \text{”}.$$

Here the quoted integral is justified as follows (see also [NY1]). Let $M_0(N,n)$ be the Uhlenbeck partial compactification of the framed instanton moduli space of rank $N$ and instanton number $n$. It is a (singular) affine variety and its complex dimension is $2nN$. It has an action of $\mathbb{T} := (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{N-1}$, where $(\mathbb{C}^*)^2$ acts on $\mathbb{CP}^2$ and $(\mathbb{C}^*)^{N-1}$ acts on the framing. The fixed point set $M_0(N,n)^{\mathbb{T}}$ consists of one point. We denote by $\iota_{0*} : M_0(N,n)^{\mathbb{T}} \hookrightarrow M_0(N,n)$ the inclusion map. Now consider the fundamental class $[M_0(N,n)] \in H^{\mathbb{T}}_{4Nn}(M_0(N,n), \mathbb{C})$ of the moduli in the (Borel-Moore) equivariant homology group. The inclusion map induces the pushforward $\iota_{0*} : H^{\mathbb{T}}_*(M_0(N,n)^{\mathbb{T}}) \to H^{\mathbb{T}}_*(M_0(N,n))$. By the localization theorem it is an isomorphism after tensoring the quotient field of $H^*_{\mathbb{T}}(\text{pt})$. Now we define

$$Z^{\text{inst}}_{\text{pure}\,SU(N)}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) := \sum_{n=0}^{\infty} \Lambda^{2nN} \iota_{0*}^{-1}[M_0(N,n)].$$

The obtained function can be considered as an element of the quotient field $H^*_{\mathbb{T}}(\text{pt})$. We will write $H^*_{\mathbb{T}}(\text{pt}) = \mathbb{C}[\varepsilon_1, \varepsilon_2, \vec{a}]$, where $(\varepsilon_1, \varepsilon_2)$ corresponds to $(\mathbb{C}^*)^2$ acting on $\mathbb{CP}^2$, and $\vec{a}$ corresponds to $(\mathbb{C}^*)^{N-1}$ acting on the framing. Thus $Z^{\text{inst}}_{\text{pure}\,SU(N)}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ is an element of $\mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{a})[[\Lambda^{2N}]]$.

4.1.3. *AGT relation and its generalization.* The simplest AGT relation proposed in [G] is

$$\langle G|G\rangle = Z^{\text{inst}}_{\text{pure}\,SU(2)}(\varepsilon_1, \varepsilon_2, a; \Lambda), \tag{4.2}$$

where $|G\rangle$ is determined by (4.1), and the parameters correspond as $c = 13 + 6(\varepsilon_1/\varepsilon_2 + \varepsilon_2/\varepsilon_1)$ and $h = \big((\varepsilon_1 + \varepsilon_2)^2 - a^2\big)/4\varepsilon_1\varepsilon_2$. Several AGT relations including the above (4.2) were proved by [FL] and [HJS]. A special case of the original AGT conjecture for the conformal block was proved in [MMS].

The AGT conjecture implies an action of Virasoro algebra on the equivariant cohomology of the rank two instanton moduli. The word ‘AGT conjecture/relation’ means a conjectural existence of $\mathcal{W}(p, \widehat{\mathfrak{g}})$-algebra on the equivariant (intersection) cohomology on the moduli of parabolic ${}^L G$-sheaves. Roughly speaking, this conjecture suggests a realization of ‘$\mathcal{W}$-algebra’ as the hidden symmetry of the ‘instanton moduli space’. See, for example, [BFRF].

4.2. **Five dimensional version.** Let us mention another generalization of the AGT conjecture: $K$-theoretic analogue. The paper [AY1] proposed a conjecture which relates the instanton part of Nekrasov's five dimensional (or $K$-theoretic) pure $SU(2)$ partition function $Z^{\text{inst}K}_{\text{pure}\,SU(2)}(Q, q, t; \Lambda)$ (see §4.2.2 below) to the deformed Virasoro algebra [SKAO]. The conjecture claims that $Z^{\text{inst}K}_{\text{pure}\,SU(2)}(Q, q, t; \Lambda)$ coincides with the inner product $\langle G; q, t|G; q, t\rangle$ of certain Whittaker vector $|G; q, t\rangle$ in the Verma module of the deformed Virasoro algebra. This state is a natural $q$-deformed analogue of (4.1).

4.2.1. *Recollection of the deformed Virasoro algebra.* First we introduce the deformed Virasoro algebra $\mathrm{Vir}_{q,t}$, its Verma module $M_h$ and the Whittaker vector.

Let $q,t$ be two generic complex parameters. Set $p := q/t$ for simplicity. The deformed Virasoro algebra $\mathrm{Vir}_{q,t}$ [SKAO] is defined to be the associative $\mathbb{C}$-algebra generated by $\{T_n \mid n \in \mathbb{Z}\}$ and 1 with relations

$$[T_n, T_m] = -\sum_{l=1}^{\infty} f_l (T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p}(p^n - p^{-n})\delta_{m+n,0},$$

where the coefficients $f_l$'s are determined by the following generating function:

$$\sum_{l=0}^{\infty} f_l x^l = \exp\Big(\sum_{n=1}^{\infty} \frac{(1-q)(1-t^{-1})}{1+p^n}\frac{x^n}{n}\Big).$$

Next we introduce a representation of this algebra. For $h \in \mathbb{C}$, let $|h\rangle$ be a vector and define the action of $\mathrm{Vir}_{q,t}$ by $1\,|h\rangle = |h\rangle$, $T_0\,|h\rangle = h\,|h\rangle$, $T_n\,|h\rangle = 0$ ($\forall n \in \mathbb{Z}_{>0}$). Then the Verma module $M_h$ for $\mathrm{Vir}_{q,t}$ is defined to be the $\mathrm{Vir}_{q,t}$-module generated by $|h\rangle$.

The dual (right) module $M_h^*$ is similarly defined. It is generated by the highest weight vector $\langle h|$ satisfying $\langle h|\,1 = \langle h|$, $\langle h|\,T_0 = h\,\langle h|$, and $\langle h|\,T_n = 0$ for any $n \in \mathbb{Z}_{<0}$.

Let us introduce the outer grading operator $d$ satisfying $[d, T_n] = -nT_n$. Defining the action of $d$ on $M_h$ by $d\,|h\rangle = 0$, we have the direct decomposition $M_h = \oplus_{n\in\mathbb{Z}_{\geq 0}} M_{h,n}$ with respect to this grading. $M_{h,n}$ has a basis consisting of $T_{-\lambda}\,|h\rangle := T_{-\lambda_1}T_{-\lambda_2}\cdots T_{-\lambda_l}\,|h\rangle$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $|\lambda| = n$. The dual representation $M_h^*$ also has a similar grading structure $M_h^* = \oplus_{n\in\mathbb{Z}_{\geq 0}} M_{h,n}^*$, and $M_h^*$ has a basis consisting of the vectors $\langle h|\,T_\lambda := \langle h|\,T_{\lambda_l}\cdots T_{\lambda_2}T_{\lambda_1}$ indexed by partitions of $n$.

Let us denote by $\widehat{M_h}$ and $\widehat{M_h^*}$ the completions of $M_h$ and $M_h^*$ with respect to the grading above. The deformed Gaiotto state $|G; q, t\rangle \in \widehat{M_h}$ is defined to be a vector satisfying

$$T_1\,|G; q, t\rangle = \Lambda^2\,|G; q, t\rangle\,, \quad T_n\,|G; q, t\rangle = 0 \ (n \geq 2), \tag{4.3}$$

where $\Lambda^2$ is a (non-zero) complex number. The dual vector $\langle G; q, t| \in \widehat{M_h^*}$ is defined similarly:

$$\langle G; q, t|\,T_{-1} = \Lambda^2\,\langle G; q, t|\,, \quad \langle G; q, t|\,T_n = 0 \ (n \leq -2).$$

4.2.2. *Recollection of the $K$-theoretic Nekrasov partition function.* The instanton part of the $\mathrm{SU}(N)$ $K$-theoretic Nekrasov partition function was defined as the integration in the equivariant $K$-theory on the moduli space of framed rank $N$ torsion free sheaves on $\mathbb{P}^2$ in [N] (see also [NY2]). By the localization theorem for equivariant $K$-theory, it becomes a summation over the fixed point contributions. The fixed points are parametrised by $N$-tuples of Young diagrams, and one obtains a combinatorial form of the partition function.

Recall the definition of the Nekrasov factor $N_{\lambda,\mu}(u)$ in Definition 2.10. The Nekrasov partition function for the $\mathrm{SU}(2)$ case is given as follows:

$$Z^{\mathrm{inst}K}_{\mathrm{pure\,SU(2)}}(Q, q, t; \Lambda) = \sum_{\lambda,\mu\in\mathcal{P}} (\Lambda^4 t/q)^{|\lambda|+|\mu|} Z_{\lambda,\mu}(Q, q, t),$$

$$Z_{\lambda,\mu}(Q, q, t) := \Big[N_{\lambda,\lambda}(1) N_{\mu,\mu}(1) N_{\lambda,\mu}(Q) N_{\mu,\lambda}(Q^{-1})\Big]^{-1}.$$

4.2.3. *$K$-theoretic AGT conjecture.* Now the main conjecture in [AY1] is: under the correspondence $h = Q^{1/2} + Q^{-1/2}$ we have

$$\langle G; q, t|G; q, t\rangle \stackrel{?}{=} Z^{\mathrm{inst}K}_{\mathrm{pure\,SU(2)}}(Q, q, t; \Lambda). \tag{4.4}$$

Here the pairing of $M_h$ and $M_h^*$ is determined by $\langle h|h\rangle = 1$.

[AY1] also proposes a factorized expansion of $|G;q,t\rangle$ in terms of the Macdonald symmetric functions: under certain identification of $M_h$ and the Fock space $\mathcal{F} \simeq \Lambda_{\mathbb{F}}$,

$$|G;q,t\rangle \overset{?}{=} \sum_{\lambda} \Lambda^{2|\lambda|} P_\lambda(x;q,t) \prod_{\square\in\lambda} \frac{Q^{1/2}}{1-Qq^{i(\square)}t^{-j(\square)}} \frac{q^{a_\lambda(\square)}}{1-q^{a_\lambda(\square)+1}t^{\ell_\lambda(\square)}}. \tag{4.5}$$

Here the index $\lambda$ runs over the set of arbitrary partitions. Note that the four dimensional version of this expansion is proved in [Y].

A remark is in order here. In the expansion (4.5), the pairing in (4.4) is not consistent with the pairing on $\mathcal{F}$ induced from the Macdonald inner product (2.1) on $\Lambda_{\mathbb{F}}$. The former expressed in terms of power-sum symmetric function is $\langle p_m, p_n\rangle = -n(t^n+q^n)(q/t)^n \frac{1-q^n}{1-t^n}\delta_{m,n}$ (see [AY1, (3.20)]), but the latter is $\langle p_m, p_n\rangle_{q,t} = n\frac{1-q^n}{1-t^n}\delta_{m,n}$.

## 5. Whittaker vectors for Ding-Iohara algebra

The purpose of this section is the introduction of the Whittaker vector for the level $m$ representation of $\mathcal{U}$. First we give a construction of the Whittaker vector $|G\rangle$ for the level one representation, and show that the expansion of $|G\rangle$ in terms of the Macdonald symmetric function has factorized coefficients. Next we define its higher level analogue $|G;\Lambda,\boldsymbol{\alpha}\rangle$. We give a conjecture on the factorized coefficients of $|G;\Lambda,\boldsymbol{\alpha}\rangle$ expanded in the basis $(|P_{\boldsymbol{\lambda}}\rangle)$.

5.1. **Level one case.** We introduce the Whittaker (or Gaiotto) vector for the level one representation of the Ding-Iohara algebra. Our argument here is based on Macdonald's homomorphism $\varepsilon_{u,t}\Lambda_{\mathbb{F}} \to \mathbb{F}$ defined by $\varepsilon_{u,t}(p_r) = (1-u^r)/(1-t^r)$ [M, Chap. VI, (6.16)], and the factorization formula for $\varepsilon_{u,t}(P_\lambda)$ [M, Chap. VI, (6.17)].

5.1.1. *Whittaker vector* $|G\rangle$. For two parameters $\alpha, \beta$, let $|G\rangle$ be the vector

$$|G\rangle := \exp\Big(\sum_{n=0}^{\infty} \frac{1}{n}\frac{\beta^n-\alpha^n}{1-q^n} a_{-n}\Big) |0\rangle, \tag{5.1}$$

in the completed Fock space $\widehat{\mathcal{F}}_u$. From the permutation relation

$$\eta(z)\exp\Big(\sum_{n=0}^{\infty} \frac{1}{n}\frac{\beta^n-\alpha^n}{1-q^n} a_{-n}\Big) = \frac{1-\beta/z}{1-\alpha/z}\exp\Big(\sum_{n=0}^{\infty} \frac{1}{n}\frac{\beta^n-\alpha^n}{1-q^n} a_{-n}\Big)\eta(z),$$

we have

$$(1-\alpha/z)\eta(z)\,|G\rangle = (1-\beta/z)\exp\Big(\sum_{n=0}^{\infty} \frac{1}{n}\frac{\beta^n-\alpha^n}{1-q^n} a_{-n}\Big)\eta(z)\,|0\rangle.$$

In Fourier modes, we have

$$(\eta_{n+1}-\alpha\eta_n)\,|G\rangle = \begin{cases} -\beta\,|G\rangle & n=0, \\ 0 & n=1,2,3,\dots. \end{cases} \tag{5.2}$$

Therefore $|G\rangle$ is a joint eigenfunction with respect to the set of operators $\eta_{n+1}-\alpha\eta_n$ ($n = 0,1,2,\cdots$). Note that these equations resemble the defining conditions of Whittaker vectors (see (4.1) and (4.3)).

Note that by setting $\alpha = v$ and $\beta = tu/q$, we have $|G\rangle = \Phi(w)\,|0\rangle$, where $\Phi(w)$ is the vertex operator in Proposition 2.13. Then (5.2) can be regarded as a good starting point for guessing the correct permutation relations for $\Phi(w)$ stated in Definition 2.12.

5.1.2. *Factorized coefficients of Whittaker vector.* We show that the expansion of $|G\rangle$ in terms of $P_\lambda$ has factorized coefficients. One may compare this with (4.5). Note that, however, here we treat the level one case and (4.5) is related with the deformed Virasoro algebra i.e. the level two representation from the point of view of $\mathcal{U}$.

Recall the following specialization of $P_\lambda(x;q,t)$ [M, VI, (6.17)]. Let $u \in \mathbb{F}$. Then under the homomorphism

$$\varepsilon_{u,t} : \Lambda_{\mathbb{F}} \to \mathbb{F}, \quad p_n \mapsto \frac{1-u^n}{1-t^n},$$

we have

$$\varepsilon_{u,t} P_\lambda = \prod_{\square\in\lambda} \frac{t^{i(\square)-1} - q^{j(\square)-1}u}{1 - q^{a_\lambda(\square)} t^{\ell_\lambda(\square)+1}}. \tag{5.3}$$

For simplicity of display, let

$$\widetilde{\varepsilon}_{\alpha,\beta,t} : \Lambda_{\mathbb{F}} \to \mathbb{F}, \quad p_n \mapsto \frac{\beta^n - \alpha^n}{1-t^n}.$$

From (5.3) and (2.2), we have

$$\widetilde{\varepsilon}_{\alpha,\beta,t} Q_\lambda = \beta^{|\lambda|} \cdot \varepsilon_{\alpha/\beta,t} Q_\lambda = (\beta^{|\lambda|} c_\lambda / c'_\lambda) \cdot \varepsilon_{\alpha/\beta,t} P_\lambda = \prod_{\square\in\lambda} \frac{t^{i(\square)-1}\beta - q^{j(\square)-1}\alpha}{1 - q^{a_\lambda(\square)+1} t^{\ell_\lambda(\square)}}. \tag{5.4}$$

Next we recall the Cauchy-type kernel function [M, VI §2, (4.13)]:

$$\exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y)\Big) = \sum_\lambda P_\lambda(x;q,t) Q_\lambda(y;q,t), \tag{5.5}$$

where the index $\lambda$ runs over the set of arbitrary partitions.

Now let us apply the specialization $\widetilde{\varepsilon}_{\alpha,\beta,t}$ to the $y$-variables in (5.5). We have

$$\exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\beta^n - \alpha^n}{1-q^n} p_n(x)\Big) = \sum_\lambda P_\lambda(x;q,t) \cdot \prod_{\square\in\lambda} \frac{t^{i(\square)-1}\beta - q^{j(\square)-1}\alpha}{1 - q^{a_\lambda(\square)+1} t^{\ell_\lambda(\square)}}.$$

Since the left hand side is $|G\rangle$ under the identification $\mathcal{F}_u \xrightarrow{\sim} \Lambda_{\widetilde{\mathbb{F}}}$ (2.4), we have

**Proposition 5.1.** We have

$$|G\rangle = \sum_\lambda |P_\lambda(x;q,t)\rangle \cdot \prod_{\square\in\lambda} \frac{t^{i(\square)-1}\beta - q^{j(\square)-1}\alpha}{1 - q^{a_\lambda(\square)+1} t^{\ell_\lambda(\square)}}.$$

5.2. **Conjecture for the Higher level cases.** For the higher level representations, we introduce the Whittaker vector as follows.

**Definition 5.2.** For an $m$-tuple of parameters $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ and a parameter $\Lambda$, let us define the state $|G;\Lambda,\boldsymbol{\alpha}\rangle \in \widehat{\mathcal{F}}_{\mathbf{u}}$ of the completed Fock space by the condition

$$\Big(X^{(k)}_n - \Lambda_{\boldsymbol{\alpha}} X^{(k)}_{n-1} + e_k(\boldsymbol{\alpha}/v)\Lambda_{\boldsymbol{\alpha}}\delta_{n,1}\Big)\,|G;\Lambda,\boldsymbol{\alpha}\rangle = 0 \quad (n \in \mathbb{Z}_{\geq 1},\ k \in \mathbb{Z}_{\geq 0}), \tag{5.6}$$

with the normalization condition

$$|G;\Lambda,\boldsymbol{\alpha}\rangle = |0\rangle + \cdots.$$

Here we used the symbols

$$v := \sqrt{q/t}, \quad \Lambda_{\boldsymbol{\alpha}} := \Lambda \prod_{i=1}^{m} v u_i/\alpha_i,$$

and $e_k(\boldsymbol{\alpha}/v)$ is the $k$-th elementary symmetric polynomial in $(\alpha_1/v, \alpha_2/v, \dots, \alpha_m/v)$.

The condition (5.6) can be rewritten as follows. Let

$$|G;\Lambda,\boldsymbol{\alpha}\rangle=\sum_{n=0}^{\infty}\Lambda_{\boldsymbol{\alpha}}^{n}\,|G;\Lambda,\boldsymbol{\alpha};n\rangle\,,$$

be the expansion of $|G;\Lambda,\boldsymbol{\alpha}\rangle$ with $|G;\Lambda,\boldsymbol{\alpha};n\rangle\in\mathcal{F}_{\mathbf{u},n}$. Here $\mathcal{F}_{\mathbf{u},n}$ is the homogeneous component of $\mathcal{F}_{\mathbf{u}}$ whose degree is induced by that of $\mathcal{F}=\oplus_n\mathcal{F}_n$. In other words, $\mathcal{F}_{\mathbf{u},n}:=\oplus_{n_1+\cdots+n_m=n}\mathcal{F}_{n_1}\otimes\cdots\otimes\mathcal{F}_{n_m}$. Then (5.6) is equivalent to

$$X_n^{(k)}\,|G;\Lambda,\boldsymbol{\alpha};n\rangle-X_{n-1}^{(k)}\,|G;\Lambda,\boldsymbol{\alpha};n-1\rangle+e_k(\boldsymbol{\alpha}/v)\delta_{n,1}\,|G;\Lambda,\boldsymbol{\alpha};n-1\rangle=0.$$

The dual state is defined as follows. For an $m$-tuple of parameters $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_m)$, let $\langle G;\Lambda,\boldsymbol{\beta}|\in\widehat{\mathcal{F}}_{\mathbf{u}}^{*}$ be an element such that

$$\langle G;\Lambda,\boldsymbol{\beta}|\left((q/t)^{1-k}X_{-n}^{(k)}-\Lambda_{\boldsymbol{\beta}}X_{1-n}^{(k)}+e_i(\boldsymbol{\beta}/v)\Lambda_{\boldsymbol{\beta}}\delta_{n,1}\right)=0$$

with

$$\Lambda_{\boldsymbol{\beta}}:=\Lambda\prod_{i=1}^{m}vu_i/\beta_i$$

and the normalization condition $\langle G;\Lambda,\boldsymbol{\beta}|=\mathbf{1}+\cdots$.

Now we state our conjecture.

**Conjecture 5.3.** (1) For generic parameters, $|G;\Lambda,\boldsymbol{\alpha}\rangle$ exists uniquely. We have the expansion

$$|G;\Lambda,\boldsymbol{\alpha}\rangle=\sum_{\boldsymbol{\lambda}}(q/t)^{\sum_{k=1}^{m}\frac{1-k}{2}|\lambda^{(k)}|}C_{\boldsymbol{\lambda}}(\Lambda,\mathbf{u},\boldsymbol{\alpha};q,t)\,|P_{\boldsymbol{\lambda}}\rangle\,,$$

where

$$\begin{aligned}C_{\boldsymbol{\lambda}}(\Lambda,\mathbf{u},\boldsymbol{\alpha};q,t)&\stackrel{?}{=}\prod_{k=1}^{m}\Big[\prod_{l=k+1}^{m}(u_k/u_l)^{|\lambda^{(l)}|}\times\prod_{\square\in\lambda^{(k)}}\Big[\frac{-\Lambda q^{-j(\square)}(-q^{1-j(\square)}t^{i(\square)-1})^{k-1}}{1-q^{-a_{\lambda^{(k)}}(\square)-1}t^{-\ell_{\lambda^{(k)}}(\square)}}\\&\qquad\times\frac{\prod_{l=1}^{m}\Big(1-v\frac{u_k}{\alpha_l}q^{j(\square)-1}t^{1-i(\square)}\Big)}{\prod_{l=1}^{k-1}\Big(1-\frac{u_l}{u_k}q^{a_{\lambda^{(l)}}(\square)}t^{\ell_{\lambda^{(k)}}(\square)+1}\Big)\prod_{l=k+1}^{m}\Big(1-\frac{u_k}{u_l}q^{-a_{\lambda^{(l)}}(\square)-1}t^{-\ell_{\lambda^{(k)}}(\square)}\Big)}\Big]\Big]\\&=\Big[\prod_{k=1}^{m}\prod_{\square\in\lambda^{(k)}}\frac{-\Lambda q^{-j(\square)}(-q^{1-j(\square)}t^{i(\square)-1})^{k-1}}{1-q^{-a_{\lambda^{(k)}}(\square)-1}t^{-\ell_{\lambda^{(k)}}(\square)}}\Big]\cdot\Big[\frac{\prod_{k,l=1}^{m}N_{\lambda^{(k)},\emptyset}(vu_k/\alpha_l)}{\prod_{1\le k<l\le m}\big(\frac{u_k}{u_l}\big)^{-|\lambda^{(l)}|}N_{\lambda^{(k)},\lambda^{(l)}}(u_k/u_l)}\Big].\end{aligned}$$

(2) For generic parameters, the element $\langle G;\Lambda,\boldsymbol{\beta}|$ exists uniquely. We have

$$\langle G;\Lambda,\boldsymbol{\beta}|=\sum_{\boldsymbol{\lambda}}(q/t)^{\sum_{k=1}^{m}\frac{k-1}{2}|\lambda^{(k)}|}\overline{C}_{\boldsymbol{\lambda}}(\Lambda,\mathbf{u},\boldsymbol{\beta};q,t)\,\langle P_{\boldsymbol{\lambda}}|\,,$$

where

$$\begin{aligned}\overline{C}_{\boldsymbol{\lambda}}(\Lambda,\mathbf{u},\boldsymbol{\beta};q,t)&\stackrel{?}{=}C_{\overline{\boldsymbol{\lambda}}}(\Lambda,\overline{\mathbf{u}},\boldsymbol{\beta};q,t)\stackrel{?}{=}C_{\overline{\boldsymbol{\lambda}}}(\Lambda_{\boldsymbol{\beta}},1/\overline{\mathbf{u}},v^2/\boldsymbol{\beta};q^{-1},t^{-1})\\&\stackrel{?}{=}\Big[\prod_{k=1}^{m}\prod_{\square\in\lambda^{(k)}}\frac{-\Lambda q^{-j(\square)}(-q^{1-j(\square)}t^{i(\square)-1})^{m-k}}{1-q^{-a_{\lambda^{(k)}}(\square)-1}t^{-\ell_{\lambda^{(k)}}(\square)}}\Big]\cdot\Big[\frac{\prod_{k,l=1}^{m}N_{\lambda^{(k)},\emptyset}(vu_k/\beta_l)}{\prod_{1\le k<l\le m}\big(\frac{u_l}{u_k}\big)^{-|\lambda^{(k)}|}N_{\lambda^{(l)},\lambda^{(k)}}(u_l/u_k)}\Big]\\&=\Big[\prod_{k=1}^{m}\prod_{\square\in\lambda^{(k)}}\frac{-\Lambda_{\boldsymbol{\beta}}q^{j(\square)-1}(-q^{1-j(\square)}t^{i(\square)-1})^{1-k}}{1-q^{a_{\lambda^{(k)}}(\square)+1}t^{\ell_{\lambda^{(k)}}(\square)}}\Big]\cdot\Big[\frac{\prod_{k,l=1}^{m}N_{\emptyset,\lambda^{(k)}}(v\beta_l/u_k)}{\prod_{1\le k<l\le m}\big(\frac{u_l}{u_k}\big)^{-|\lambda^{(l)}|}N_{\lambda^{(l)},\lambda^{(k)}}(u_l/u_k)}\Big].\end{aligned}$$

Here we used $\overline{\boldsymbol{\lambda}} := (\lambda^{(m)}, \dots, \lambda^{(2)}, \lambda^{(1)})$, $\overline{\mathbf{u}} := (u_m, \dots, u_2, u_1)$, $1/\overline{\mathbf{u}} := (1/u_m, \dots, 1/u_2, 1/u_1)$, and $v^2/\boldsymbol{\beta} := (v^2/\beta_1, \dots, v^2/\beta_m)$.

This conjecture implies

$$\langle G; \Lambda, \boldsymbol{\beta} | G; \Lambda, \boldsymbol{\alpha} \rangle = \sum_{\boldsymbol{\lambda}} C_{\boldsymbol{\lambda}}(\Lambda, \mathbf{u}, \boldsymbol{\alpha}; q, t) \overline{C}_{\boldsymbol{\lambda}}(\Lambda, \mathbf{u}, \boldsymbol{\beta}; q, t) \prod_{k=1}^{m} \frac{c'_{\lambda^{(k)}}}{c_{\lambda^{(k)}}}$$
$$\stackrel{?}{=} \sum_{\boldsymbol{\lambda}} (\Lambda \Lambda_{\boldsymbol{\beta}})^{|\boldsymbol{\lambda}|} \prod_{k,l=1}^{m} \frac{N_{\lambda^{(k)}, \emptyset}(v u_k/\alpha_l) N_{\emptyset, \lambda^{(k)}}(v \beta_l / u_k)}{N_{\lambda^{(k)}, \lambda^{(l)}}(u_k/u_l)}.$$

This is equal to the instanton part of the five dimensional $\mathrm{U}(m)$ Nekrasov partition function with $N_f = 2m$ fundamental matters (see [AY2, §5]).

## 6. Examples of the Matrix elements of the Level One Vertex Operator

From the point of view of the Whittaker vector considered in the last section (in particular the expression (5.1)), one may be interested in the operator

$$\psi(z) = \psi(z; \alpha, \beta, \kappa, \delta) := \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\beta^n - \alpha^n}{1 - q^n} a_{-n} w^n\Big) \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\delta^n - \kappa^n}{1 - q^n} a_n w^{-n}\Big), \tag{6.1}$$

where $\alpha$, $\beta$, $\kappa$ and $\delta$ are parameters. However, one may easily find that we need to have some relations in the parameters $\alpha$, $\beta$, $\kappa$ and $\delta$ if we demand that all the matrix elements with respect to the Macdonald functions be factorized.

In this section, we give some examples of the calculation of the matrix elements, in which some transformation formulas for the basic hypergeometric series can be applied. Recall (2.6), where we have $\Phi(w) : \mathcal{F}_u \to \mathcal{F}_v$ and

$$\Phi(w) = \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{v^n - (t/q)^n u^n}{1 - q^n} a_{-n} w^n\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n v^{-n} - q^n u^{-n}}{1 - q^n} a_n w^{-n}\Big). \tag{6.2}$$

Namely, we have $\Phi(w) = \psi(w; v, tu/q, q/v, q/u)$. We use the notations

$$(x; q)_n := \prod_{i=1}^{n} (1 - x q^{i-1}), \quad (x_1, x_2, \dots, x_m; q)_n := \prod_{i=1}^{m} (x_i; q)_n,$$

$$_3\phi_2(a_1, a_2, a_3; b_1, b_2; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n (a_3, q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n} z^n.$$

### 6.1. The case $\lambda = (j)$, $\mu = (k)$.

We will compute $\langle Q_{(j)} | \, \psi(w) \, | Q_{(k)} \rangle$ for $j, k \in \mathbb{Z}_{\geq 0}$.

We have the generating function of $Q_{(r)}$

$$\sum_{r=0}^{\infty} Q_{(r)}(x; q, t) y^r = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) y^n\Big).$$

Using the isomorphism $\iota : \Lambda_{\widetilde{\mathbb{F}}} \xrightarrow{\sim} \mathcal{F}$ (2.4) we have

$$\sum_{j,k=0}^{\infty} \langle Q_{(j)} | \, \psi(w) \, | Q_{(k)} \rangle x^{-j} y^k$$
$$= \Big\langle \exp\Big(\sum_{k=1}^{\infty} \frac{1}{k} \frac{1 - t^k}{1 - q^k} a_k x^{-k}\Big) \psi(w) \exp\Big(\sum_{k=1}^{\infty} \frac{1}{k} \frac{1 - t^k}{1 - q^k} a_{-k} y^k\Big) \Big\rangle$$
$$= \frac{(\alpha w/x; q)_\infty}{(\beta w/x; q)_\infty} \frac{(ty/x; q)_\infty}{(y/x; q)_\infty} \frac{(\kappa y/w; q)_\infty}{(\delta y/w; q)_\infty}$$

$$= \sum_{l,m,n=0}^{\infty} w^{l-n} x^{-l-m} y^{m+n} \frac{(\alpha/\beta;q)_l (t;q)_m (\kappa/\delta;q)_n}{(q;q)_l (q;q)_m (q;q)_n} \beta^l \delta^n.$$

Here we have used the $q$-binomial formula $(az;q)_\infty/(z;q)_\infty = \sum_{n=0}^{\infty} z^n (a;q)_n/(q;q)_n$. Hence we have

$$\begin{aligned}\langle Q_{(j)} | \psi(w) | Q_{(k)} \rangle &= w^{j-k} \beta^j \delta^k \frac{(\alpha/\beta;q)_j (\kappa/\delta;q)_k}{(q;q)_j (q;q)_k} \\ &\times \sum_{m=0}^{\infty} \frac{(q^{-j};q)_m (t;q)_m (q^{-k};q)_m}{(q^{-j+1}\beta/\alpha;q)_m (q;q)_m (q^{-k+1}\delta/\kappa;q)_m} q^{2m} \alpha^{-m} \kappa^{-m}. \qquad (6.3)\end{aligned}$$

Recall the $q$-analogue of Saalschütz's summation formula for terminating balanced ${}_3\phi_2$ series [GR, §1.7]:

$$ {}_3\phi_2(a, b, q^{-k}; c, abc^{-1}q^{1-k}; q, q) = \frac{(c/a, c/b; q)_k}{(c, c/ab; q)_k}. \qquad (6.4)$$

Let

$$a = q^{-j}, \qquad b = t, \qquad c = q^{-j+1}\beta/\alpha.$$

Then we have the two conditions

$$\alpha\kappa = q, \qquad \beta\delta = t \qquad (6.5)$$

to identify the two ${}_3\phi_2$ series in (6.3) and (6.4). Setting $\alpha = v$ and $\beta = tu/q$, we have $\kappa = q/v$ and $\delta = q/u$ from (6.5). Thus we conclude that we have factorized matrix elements with respect to the one row Macdonald function $Q_{(n)}$'s for the operator $\Phi(w) = \psi(w; v, tu/q, q/v, q/u)$.

Noting that $c'_{(n)} = (q;q)_n$ and simplifying the formulas, we have

$$\langle J_{(j)} | \Phi(w) | J_{(k)} \rangle = (q^{-k+1}v/tu;q)_j (q^{1+j-k}v/u;q)_k w^{j-k} (tu/q)^j (-v/q)^{-k} q^{k(k-1)/2},$$

which agrees with Proposition 2.14.

6.2. **The case $\lambda = (1^j)$, $\mu = (k)$.** Next we treat the case when the partition $\lambda$ is one column and $\mu$ is one row.

In this case we have $J_{(1^j)} = c_{(1^j)} P_{(1^j)} = c_{(1^j)} e_j$, where $e_j$ is the $j$-th elementary symmetric function. The generating function is given by

$$\sum_{r=0}^{\infty} e_r(x) y^r = \exp\Big( - \sum_{n=1}^{\infty} \frac{1}{n} p_n(x) (-y)^n \Big).$$

We have

$$\begin{aligned}&\sum_{j,k=0}^{\infty} \langle P_{(1^j)} | \Phi(w) | Q_{(k)} \rangle x^{-j} y^k \\ &= \Big\langle \exp\Big( - \sum_{k=1}^{\infty} \frac{1}{k} a_k (-x)^{-k} \Big) \Phi(w) \exp\Big( \sum_{k=1}^{\infty} \frac{1}{k} \frac{1-t^k}{1-q^k} a_{-k} y^k \Big) \Big\rangle \\ &= \frac{(-twu/qx;t)_\infty}{(-vw/x;t)_\infty} \Big(1 + \frac{y}{x}\Big) \frac{(qy/vw;q)_\infty}{(qy/uw;q)_\infty} \\ &= \Big[ \sum_{m=0}^{\infty} \frac{(tu/qv;t)_m}{(t;t)_m} \Big( -\frac{vw}{x} \Big)^m \Big] \cdot \Big[ 1 + \frac{y}{x} \Big] \cdot \Big[ \sum_{n=0}^{\infty} \frac{(u/v;q)_n}{(q;q)_n} \Big( \frac{qy}{uw} \Big)^n \Big].\end{aligned}$$

Hence we have

$$\langle P_{(1^j)} | \Phi(w) | Q_{(k)} \rangle = (-vw)^j (q/uw)^k \frac{(tu/qv;t)_j}{(t;t)_j} \frac{(u/v;q)_k}{(q;q)_k} \frac{(1-u/qv)(1-q^{k-1}t^j u/v)}{(1-q^{k-1}u/v)(1-t^j u/qv)}.$$

Recalling $c_{(1^j)} = (t;t)_j$ and $c'_{(k)} = (q;q)_k$, we have

$$\langle J_{(1^j)} | \, \Phi(w) \, | J_{(k)} \rangle = (1 - q^{1-k}t^{-j}v/u)(qt^{1-j}v/u;t)_{j-1}(q^{2-k}v/u;q)_k \\ \times w^{j-k}(tu/q)^j(-v/q)^{-k}t^{j(j-1)/2}q^{k(k-1)/2}.$$

6.3. **The case** $\lambda = (1^j)$, $\mu = (1^k)$. This case is similar to the first case $\lambda = (j)$, $\mu = (k)$. The generating function we consider is

$$\sum_{j,k=0}^{\infty} \langle P_{(1^j)} | \, \Phi(w) \, | P_{(1^k)} \rangle \, x^{-j} y^k \\ = \Big\langle \exp\Big( - \sum_{k=1}^{\infty} \frac{1}{k} a_k (-x)^{-k} \Big) \Phi(w) \exp\Big( - \sum_{k=1}^{\infty} \frac{1}{k} a_{-k} (-y)^k \Big) \Big\rangle \\ = \frac{(-tuw/qx;t)_\infty}{(-vw/x;t)_\infty} \frac{(qy/x;t)_\infty}{(y/x;t)_\infty} \frac{(-qy/uw;t)_\infty}{(-qy/vw;t)_\infty}.$$

Then we have

$$\langle P_{(1^j)} | \, \Phi(w) \, | P_{(1^k)} \rangle = (-vw)^j(-q/vw)^k \frac{(tu/qv;t)_j}{(t;t)_j} \frac{(v/u;t)_k}{(t;t)_k} \\ \times \sum_{m=0}^{\infty} \frac{(q;t)_m (t^{-j};t)_m (t^{-k},t)_m}{(t;t)_m (t^{-j}qv/u;t)_m (t^{-k+1}u/v;t)_m} t^m.$$

Using the $q$-Saalschütz's formula (6.4), we have

$$\langle J_{(1^j)} | \, \Phi(w) \, | J_{(1^k)} \rangle = (t^{-j}v/u;t)_k (qt^{-j+k}v/u;t)_j \\ \times w^{j-k}(tu/q)^j(-v/q)^{-k}t^{j(j-1)/2}.$$

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HA: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan
*E-mail address*: awata@math.nagoya-u.ac.jp

BF: Landau Institute for Theoretical Physics, Russia, Chernogolovka, 142432, prosp. Akademika Semenova, 1a,, Higher School of Economics, Russia, Moscow, 101000, Myasnitskaya ul., 20, and, Independent University of Moscow, Russia, Moscow, 119002, Bol'shoi Vlas'evski per., 11
*E-mail address*: bfeigin@gmail.com

AH. Department of Mathematics, Sophia University, Kioicyo, Tokyo, 102-8554, Japan
*E-mail address*: ayumu-h@sophia.ac.jp

MK,JS: Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan
*E-mail address*: kanai@ms.u-tokyo.ac.jp
*E-mail address*: shiraish@ms.u-tokyo.ac.jp

SY: Kobe University, Department of Mathematics, Rokko, Kobe 657-8501, Japan
*E-mail address*: yanagida@math.kobe-u.ac.jp