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Torus-covering links and their triple linking numbers

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Abstract

This report is a survey of [14]. A torus-covering link is an oriented surface link in the form of a covering over the standard torus. Triple linking number is an invariant defined for an oriented surface link with at least three components, analogous to the linking number of classical links. A torus-covering T^2-link is determined from two commutative classical braids, which we call basis braids. We present the triple linking numbers of a torus-covering T^2-link by using the linking numbers of the closures of its basis braids, in the case when the basis braids are pure braids.

1 Introduction

A *surface link* is a smooth embedding of a closed surface into the Euclidean 4-space $\mathbb{R}^4$. A T^2*-link* is a surface link whose each component is of genus one. In this paper we consider a certain "torus-covering T^2-link", which is an m-component T^2-link determined from two commutative pure m-braids a and b. The triple linking number of an oriented surface link is defined in [1] as an analogical notion of the linking number of a classical link. The aim of this paper is to present the triple linking number of such a T^2-link, by using the linking numbers of the closures of a and b. Further, we study the triple point number. The triple linking numbers give a lower bound of the triple point number. In some cases, we can determine the triple point number, which is a multiple of four.

This paper is organized as follows. In Section 2, we give the definition of a torus-covering link. In Section 3, we review the linking numbers of a classical link and the triple linking numbers of an oriented surface link. In Section 4, we give the main theorem (Theorem 3). In Section 5, we give applications to the triple point numbers (Theorems 4 and 5).

2 Torus-covering links

We consider torus-covering T^2-links, i.e. torus-covering links whose each component is of genus one. In this section we review the definition of a torus-covering T^2-link. See [13] for the original definition and properties of torus-covering links.

Let T be the standard torus in $\mathbb{R}^4$, i.e. the boundary of the standard solid torus in $\mathbb{R}^3 \times \{0\}$. Let $N(T)$ be a tubular neighborhood of T in $\mathbb{R}^4$.

Definition 1. A *torus-covering T^2-link* is a surface link F in $N(T) \subset \mathbb{R}^4$ such that $p|_F : F \to T$ is an unbranched covering map of degree m, where $p : N(T) \to T$ is the projection.

Remark. It is known [10, 11] that any oriented surface link can be presented in the form of a simple branched covering over the standard 2-sphere S^2 i.e. in the form of a surface link embedded in a tubular neighborhood of S^2 in such a way that the projection of it to S^2 is a simple branched covering over S^2. A torus-covering link is an oriented surface link in the form of a simple branched covering over the standard torus T (see [13]), introduced by considering the standard torus instead of the standard 2-sphere in this fact.

Let us fix a point x_0 of T, and take a meridian μ and a longitude λ of T with the base point x_0. A meridian is an oriented simple closed curve on T which bounds the 2-disk of the solid torus whose boundary is T. A longitude is an oriented simple closed curve on T which is null-homologous in the complement of the solid torus in the three space $\mathbb{R}^3 \times \{0\}$. For a torus-covering T^2-link F, we obtain classical m-braids by cutting $F \cap p^{-1}(\mu)$ and $F \cap p^{-1}(\lambda)$ at the 2-disk $p^{-1}(x_0)$. We call them *basis braids.*

Lemma 2 ([13]). *(1) The basis braids are commutative.*

(2) For any commutative m-braids a and b, there is a unique torus-covering T^2-link with basis braids a and b.

Thus a torus-covering T^2-link is determined from basis braids. We denote by $\mathcal{S}_m(a, b)$ the torus-covering T^2-link with basis m-braids a and b.

3 Linking numbers and triple linking numbers

The triple linking number of an oriented surface link is defined in [1] as an analogical notion of the linking number of a classical link. In this section, we review the linking numbers of a classical link and the triple linking numbers of a surface link.

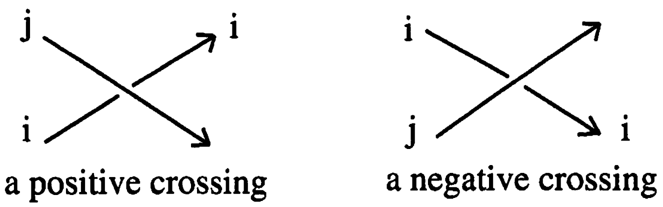

Figure 1: A positive crossing and a negative crossing.

3.1 Linking numbers of a classical link

We review the linking number of an oriented classical link L as follows. For i and j with $i \neq j$, the *linking number* of the ith and jth components of L is the total number of positive crossings minus the total number of negative crossings of a diagram of L such that the under-arc (resp. over-arc) is from the ith (resp. jth) component; see Fig. 1. We denote it by $\mathrm{Lk}_{i,j}(L)$. It is known [16] that $\mathrm{Lk}_{j,i}(L) = \mathrm{Lk}_{i,j}(L)$.

3.2 Triple linking number of a surface link

The triple linking number of an oriented surface link F is defined as follows (see [1, Definition 9.1], see also [3]). For i, j, and k with $i \neq j$ and $j \neq k$, the *triple linking number* of the ith, jth, and kth components of F is the total number of positive triple points minus the total number of negative triple points of a surface diagram of F such that the top, middle, and bottom sheet is from the ith, jth, and kth component of F respectively [1]; see Fig. 2. We denote it by $\mathrm{Tlk}_{i,j,k}(F)$.

We enumerate several properties of triple linking numbers.

Property 1 ([1]). $\mathrm{Tlk}_{k,j,i}(F) = -\mathrm{Tlk}_{i,j,k}(F)$ *if* i, j, k *are mutually distinct, and otherwise* $\mathrm{Tlk}_{i,j,k}(F) = 0$.

Property 2 ([1]). $\mathrm{Tlk}_{1,2,3}(F) + \mathrm{Tlk}_{2,3,1}(F) + \mathrm{Tlk}_{3,1,2}(F) = 0$.

Property 3 ([1]). *From the above two properties, it is seen that for any three-component surface link F, there exists a pair of integers α and β such that*

$$\begin{cases} \mathrm{Tlk}_{3,1,2}(F) = -\mathrm{Tlk}_{2,1,3}(F) = \alpha, \\ \mathrm{Tlk}_{1,2,3}(F) = -\mathrm{Tlk}_{3,2,1}(F) = -(\alpha+\beta), \\ \mathrm{Tlk}_{2,3,1}(F) = -\mathrm{Tlk}_{1,3,2}(F) = \beta. \end{cases}$$

Let us denote the ith component of F by K_i.

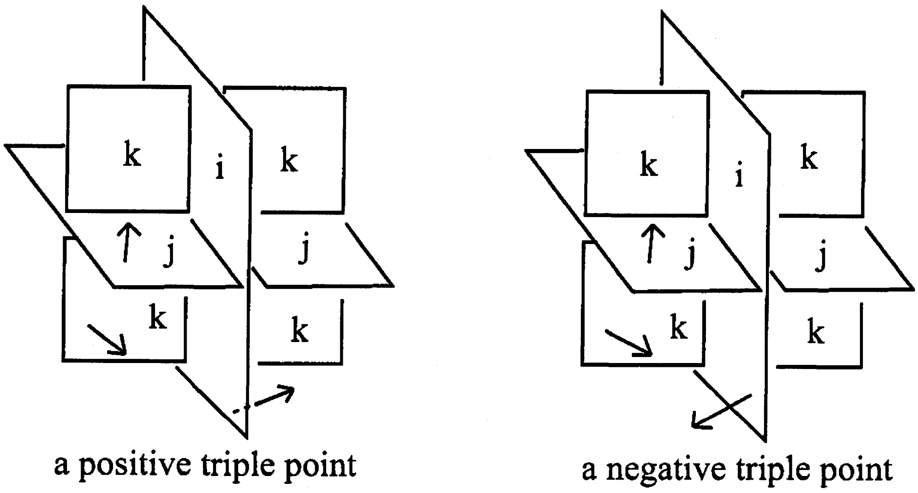

Figure 2: A positive triple point and a negative triple point, where we denote the orientations of sheets by normals.

Property 4 ([3]). *If K_2 is homeomorphic to a 2-sphere, then* $\mathrm{Tlk}_{1,2,3}(F) = 0$, *and if both of K_1 and K_3 are homeomorphic to a 2-sphere, then* $\mathrm{Tlk}_{1,2,3}(F) = 0$.

In other words: if $\alpha \neq 0$ and $\beta = 0$, then $g(K_i) \geq 1 (i = 1, 2)$, and if $\alpha \neq 0$, $\beta \neq 0$ and $\alpha + \beta \neq 0$, then $g(K_i) \geq 1 (i = 1, 2, 3)$, where $g(K_i)$ denotes the genus of K_i. There is a surface link which realizes this (see [3], see also [4]).

Property 5 ([4]). *Triple linking number is a link bordism invariant.*

4 Main Result

Here we consider a torus-covering T^2-link for the case when the basis braids are pure m-braids for $m \geq 3$. Then the triple linking numbers of the T^2-link is presented by the linking numbers of the closures of the basis braids. For an m-braid c, let us denote by $\hat{c}$ the closure of c.

Theorem 3 ([14]). *Let a and b be commutative pure m-braids for $m \geq 3$. Then the triple linking number $Tlk_{i,j,k}(\mathcal{S}_m(a, b))$ $(i \neq j$ and $j \neq k)$ is given by*

$$Tlk_{i,j,k}(\mathcal{S}_m(a, b)) = -\mathrm{Lk}_{i,j}(\hat{a})\mathrm{Lk}_{j,k}(\hat{b}) + \mathrm{Lk}_{i,j}(\hat{b})\mathrm{Lk}_{j,k}(\hat{a}),$$

where $\mathrm{Lk}_{i,j}(\hat{a})$ (resp. $\mathrm{Lk}_{i,j}(\hat{b})$) is the linking number of the ith and jth components of $\hat{a}$ (resp. $\hat{b}$). Here we define the lth component of $\mathcal{S}_m(a, b)$ (resp. $\hat{c}$

for $c = a$ or b) by the component containing the lth string of the basis braids (resp. c) for $l = 1, 2, \ldots, m$.

5 Application

The *triple point number* of a surface link F, denoted by $t(F)$, is the minimal number of triple points among all possible generic projections of F. By definition, we can see that $t(F) \geq \sum_{i \neq j, j \neq k} |\mathrm{Tlk}_{i,j,k}(F)|$; thus the triple linking numbers of F give a lower bound of the triple point number of F. In particular, we have the following theorem.

Theorem 4 ([14]). *Let $m \geq 3$. Let b be a pure m-braid, and let Δ be a full twist of a bundle of m parallel strings. Put $\mu = \sum_{i<j} |\mathrm{Lk}_{i,j}(\hat{b})|$, and let $\nu = \sum_{i<j<k}(\nu_{i,j,k} + \nu_{j,k,i} + \nu_{k,i,j})$, where $\nu_{i,j,k} = \min_{i,j,k}\{|\mathrm{Lk}_{i,j}(\hat{b})|, |\mathrm{Lk}_{j,k}(\hat{b})|\}$ if $\mathrm{Lk}_{i,j}(\hat{b})\mathrm{Lk}_{j,k}(\hat{b}) > 0$ and otherwise zero. Then*

$$t(\mathcal{S}_m(b, \Delta^n)) \geq 4n(\mu(m-2) - \nu).$$

In some cases, we can determine the triple point number. Let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of the m-braid group.

Theorem 5 ([14]). *Let $m \geq 3$. Let b be an m-braid presented by a braid word which consists of $\sigma_i^{2(-1)^i}$ ($i = 1, 2, \ldots, m-1$); note that b is a pure braid. Then*

$$t(\mathcal{S}_m(b, \Delta^n)) = 4n(m-2)(\sum_{i<j} |\mathrm{Lk}_{i,j}(\hat{b})|).$$

Further the triple point number is realized by a surface diagram in the form of a covering over the torus.

It is known [4] (see also [5]) that any oriented surface link is bordant to the split union of oriented "necklaces", and [5] any surface link is unorientedly bordant to the split union of necklaces and connected sums of standard projective planes; see also [17]. A necklace has the triple point number $4n$ (see [4]). For other examples of surface links (not necessarily orientable) which realize large triple point numbers, see [6, 12, 15, 18]. In the papers [6, 12, 15] (resp. [18]), they use quandle cocycle invariants (resp. normal Euler numbers) to give lower bounds of triple point numbers. Quandle cocycle invariants [1, 2, 3] can be regarded as an extended notion of triple linking numbers ([1, 4]), useful to give lower bounds of triple point numbers; see [6, 8, 9, 12, 15, 20, 21].

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