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Dirichlet series associated with square of the class numbers

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1 Introduction

For an even integer k and a complex number σ such that $2\Re\sigma+k>3$, the real analytic Siegel-Eisenstein series of degree 2 and weight k is defined by

$$E_{2,k}(Z,\sigma)=\sum_{\{C,D\}}\det(CZ+D)^{-k}|\det(CZ+D)|^{-2\sigma},\qquad Z\in H_{2},$$

where the sum is taken over all non-associated coprime symmetric pairs $\{C, D\}$ of degree 2 and $H_{2}=\{Z={}^{t}Z\in M_{2}(\mathbf{C});\Im Z>O\}$ is the Siegel upper half-space. Let

$$E_{2,k}(Z,\sigma)=\sum_{T}C(T,\sigma,Y)e(\mathrm{tr}(TX)),\qquad Z=X+iY$$

be the Fourier expansion, where the summation extends over all half-integral symmetric matrices of size two and $e(x)=e^{2\pi ix}$ as usual. For any non-degenerate T, it is known that

$$C(T,\sigma,Y)=b(T,k+2\sigma)\xi(Y,T,\sigma+k,\sigma),$$

where $b(T,k+2\sigma)$ is the Siegel series and $\xi(Y,T,\sigma+k,\sigma)$ is the confluent hypergeometric function of degree 2 (see [9], [8]). Moreover, Kaufhold's formula

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[8] tells us that

$$b(T,\sigma)=\frac{1}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d|e(T)}d^{2-\sigma}L_{\frac{-(\det 2T)}{d^2}}(\sigma-1),$$

where $e(T)=(n,r,m)$ for $T=\left(\begin{smallmatrix} n & r/2 \\ r/2 & m\end{smallmatrix}\right)$, $L_D(s)$ is defined for $D\neq 0, d\equiv 0,1 \pmod 4$ by

$$L_D(s)=L(s,\chi_{D_K})\sum_{a|f}\mu(a)\chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a).$$

Here the natural number f is defined by $D=d_Kf^2$ with the discriminant d_K of $K=\mathbf{Q}(\sqrt{D})$, χ_K is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n)=\sum_{d|n}d^s$.

Following Arakawa [1] and Ibukiyama-Katsurada [6], the Koecher-Maass series for positive-definite Fourier coefficients of the real analytic Siegel-Eisenstein series of degree 2 and weight 2 is defined by

$$\sum_{T\in L_2^+/SL_2(\mathbf{Z})}\frac{b(T,2)}{\sharp E(T)(\det T)^s},$$

where L_2^+ is the set of all half-integral positive-definite symmetric matrices of size 2, the summation extends over all $T\in L_2^+$ modulo the action $T\to T[U]={}^tUTU$ of $SL_2(\mathbf{Z})$ and $E(T)=\{U\in SL_2(\mathbf{Z}); T[U]=T\}$ is the the unit group of T.

In order to consider the case associated with indefinite Fourier coefficients, denote by $(L_2^-)'$ the set of all half-integral indefinite symmetric matrices of size 2 such that $\sqrt{-\det(T)}\notin\mathbf{Q}$. To any $T=\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)\in(L_2^-)'$, we associate the geodesic semicircle $S_T=\{\tau=u+iv; v>0,\ a(u^2+v^2)+bu+c=0\}$. The unit group $E(T)$ acts on S_T. Then Siegel ([21], [22]) defined the quantity $\mu(T)$ as the non-Euclidean length of a fundamental domain on S_T for $E(T)$. Note here that, when $\sqrt{-\det(T)}\in\mathbf{Q}$, such a quantity is not finite.

Similar to the case associated with positive-definite Fourier coefficients, we consider the following series associated with indefinite Fourier coefficients

$$\sum_{T\in(L_2^-)'/SL_2(\mathbf{Z})}\frac{\mu(T)b(T,2)}{|\det T|^s},$$

where the summation extends over all $T \in (L_2^-)'$ modulo the action $T \to T[U]$ of $SL_2(\mathbf{Z})$.

First of all, by Böcherer, these Dirichlet series are proportional to

$$\sum_{d>0} \frac{L_{-d}(1)^2}{d^{s-1/2}},$$

$$\sum_{d<0,-d\neq\square} \frac{L_{-d}(1)^2}{|d|^{s-1/2}}.$$

Hence, we shall study these two Dirichlet series.

These Dirichlet series might be called as square analogues of the Shintani zeta functions, which arise in the theory of prehomogeneous zeta functions and are defined by

$$\sum_{T\in L_2^+/SL_2(\mathbf{Z})} \frac{1}{\sharp E(T)(\det T)^s},$$

$$\sum_{T\in(L_2^-)'/SL_2(\mathbf{Z})} \frac{\mu(T)}{|\det T|^s}.$$

These series are proportional to

$$\sum_{d>0} \frac{L_{-d}(1)}{d^{s-1/2}},$$

$$\sum_{d<0,-d\neq\square} \frac{L_{-d}(1)}{|d|^{s-1/2}}.$$

More precisely, Shintani [20] studied the Dirichlet series

$$\xi_-(s) = \frac{1}{\pi}\sum_{d>0} \frac{L_{-d}(1)}{d^{s-1/2}}, \qquad \xi_-^*(s) = \frac{1}{\pi}\sum_{\substack{d>0 \\ d\equiv 0 \pmod 4}} \frac{L_{-d}(1)}{d^{s-1/2}},$$

$$\xi_+(s) = \sum_{d<0,-d\neq\square} \frac{L_{-d}(1)}{|d|^{s-1/2}} + \zeta(2s-1)\left(\frac{\zeta'}{\zeta}(2s) - \frac{\zeta'}{\zeta}(2s-1)\right),$$

$$\begin{aligned} \xi_+^*(s) &= \sum_{\substack{d<0,-d\neq\square \\ d\equiv 0 \pmod 4}} \frac{L_{-d}(1)}{|d|^{s-1/2}} \\ &+ 2^{1-2s}\zeta(2s-1)\left(\frac{\zeta'}{\zeta}(2s) - \frac{\zeta'}{\zeta}(2s-1) + 2^{-1}(1-2^{-2s})^{-1}\log 2\right). \end{aligned}$$

He discovered the following theorems. See also Datsukovsky [5], Ibukiyama-Saito [7], Peter [14], Saito [16], Sato [17], Strum [23], Yukie [24].

Theorem 1. *The Dirichlet series $\xi_-(s)$ and $\xi_-^*(s)$ can be meromorphically continued to the whole s-plane. They satisfy the functional equation*

$$\begin{aligned}\xi_-(3/2-s) &= 2^{2s-1}\pi^{1/2-2s}\Gamma(s-1/2)\Gamma(s)(\cos\pi s)\xi_-^*(s)\\ &\quad - 2^{-1}\pi^{1/2-2s}\Gamma(s-1/2)\Gamma(s)\zeta(2s-1).\end{aligned}$$

Theorem 2. *The Dirichlet series $\xi_+(s)$ and $\xi_+^*(s)$ can be meromorphically continued to the whole s-plane. They satisfy the functional equation*

$$\begin{aligned}\xi_+(3/2-s) &= 2^{2s-1}\pi^{1/2-2s}\Gamma(s-1/2)\Gamma(s)(\cos\pi s)(\pi\xi_-^*(s)+(\sin\pi s)\xi_+^*(s))\\ &+ 2^{-1}\pi^{1/2-2s}\Gamma(s-1/2)\Gamma(s)\zeta(2s-1)(\sin\pi s)\left(\frac{\Gamma'}{\Gamma}(s)-\frac{\Gamma'}{\Gamma}(s-1/2)\right).\end{aligned}$$

Analogously, our main results are meromorphic continuations and functional equations of the square analogues. In the case associated with positive-definite Fourier coefficients, define

$$\Xi_-(s)=\frac{1}{\pi^2}\sum_{d>0}\frac{L_{-d}(1)^2}{d^{s-1}}.$$

Then put

$$\Xi_-^*(s)=\pi^{-2s}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1)\Xi_-(s).$$

In [12], we gave the following result.

Theorem 3. *The Dirichlet series $\Xi_-^*(s)$ can be meromorphically continued to the whole s-plane. It satisfies the functional equation*

$$\Xi_-^*(2-s)=\Xi_-^*(s)+2^{-5}\pi^{-3/2}\frac{\Gamma(s)}{(\cos\pi s)\Gamma(s-1)}\zeta^*(2s-1)\zeta^*(2s-2).$$

Theorem 3 has been proved in our previous paper [12]. At this RIMS conference, the author was informed from Professor Sato that Professor Arakawa got Theorem 3 in his unpublished notebooks [3] pp. 151-152.

In the case associated with indefinite Fourier coefficients, define

$$\begin{aligned}\Xi_+(s) &= \sum_{d<0,-d\neq\square} \frac{L_{-d}(1)^2}{|d|^{s-1}} - \zeta(2s-2)\sum_p \left(\frac{\log p}{1-p^{2s}} - \frac{\log p}{1-p^{2s-1}}\right)^2 \\ &\quad +\zeta(2s-2)\left\{\left(\frac{\zeta'}{\zeta}\right)'(2s) - \left(\frac{\zeta'}{\zeta}\right)'(2s-1) + 2\left(\frac{\zeta'}{\zeta}\right)'(2s-2)\right\}.\end{aligned}$$

Here, we used the notation

$$\left(\frac{\zeta'}{\zeta}\right)'(s) = \frac{\zeta''(s)\zeta(s)-(\zeta'(s))^2}{\zeta(s)^2}, \qquad \left(\frac{\Gamma'}{\Gamma}\right)'(s) = \frac{\Gamma''(s)\Gamma(s)-(\Gamma'(s))^2}{\Gamma(s)^2}.$$

The following is our main result.

Theorem 4. *The Dirichlet series $\Xi_+(s)$ can be meromorphically continued to the whole s-plane. It satisfies the functional equation*

$$\begin{aligned}&\Xi_+(3/2-s) \\ = \; & \pi^{-2s}\varphi(1-s)\frac{\cos\pi s}{\sin\pi s}\Gamma(s-1/2)\Gamma(s+1/2) \\ & \quad \times \{2\pi^2\Xi_-(s+1/2) + (\sin\pi s)\Xi_+(s+1/2)\} \\ + \; & 2^{-1}\pi^{-2s}\varphi(1-s)(\cos\pi s)\Gamma(s-1/2)\Gamma(s+1/2)\zeta(2s-1) \\ & \quad \times \left\{-\frac{\pi^2}{(\sin\pi s)^2(\cos\pi s)^2} + \left(\frac{\Gamma'}{\Gamma}\right)'(s-1/2) + \left(\frac{\Gamma'}{\Gamma}\right)'(s+1/2)\right\},\end{aligned}$$

where $\varphi(s) = \zeta^*(2-2s)/\zeta^*(2s)$ with $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

As application, we can now define a Koecher-Maass series for indefinite Fourier coefficients of the real analytic Siegel-Eisenstein series of degree 2 and weight 2 by

$$\begin{aligned}& M_{2,2}^{(1)}(s,0) \\ = \; & \zeta(2)^2 \sum_{T\in(L_2^-)'/SL_2(\mathbf{Z})} \frac{\mu(T)b(T,2)}{|\det T|^{s-1/2}} \\ & -2^{2s}\zeta(2s-1)\zeta(2s-2)\sum_p \left(\frac{\log p}{1-p^{2s}} - \frac{\log p}{1-p^{2s-1}}\right)^2 \\ & +2^{2s}\zeta(2s-1)\zeta(2s-2)\left\{\left(\frac{\zeta'}{\zeta}\right)'(2s) - \left(\frac{\zeta'}{\zeta}\right)'(2s-1) + 2\left(\frac{\zeta'}{\zeta}\right)'(2s-2)\right\}.\end{aligned}$$

where the summation extends over all $T \in (L_2^-)'$ modulo the action $T \to T[U]$ of $SL_2(\mathbf{Z})$. Then we have

Theorem 5. *The Koecher-Maass series $M_{2,2}^{(1)}(s,0)$ can be meromorphically continued to the whole s-plane. It satisfies a functional equation similar to Theorem 4.*

1.1 Proof of Theorem 4

All of the above Dirichlet series can be regarded as two kinds of Dirichlet series associated with real analytic Cohen's Eisenstein series introduced by Ibukiyama and Saito [7]. One is the Mellin transform and the other is the Rankin-Selberg convolution. In fact, Ibukiyama-Saito proved Theorem 1 and 2 by taking the Mellin transform of real analytic Cohen's Eisenstein series. See also Strum [23] for Theorem 1, where Zagier's Eisenstein series is used. We shall prove Theorem 3 and 4 by taking the Rankin-Selberg convolution of real analytic Cohen's Eisenstein series.

First, we summarize about Cohen's Eisenstein series following [7]. See [11] for a relation with the real analytic Jacobi-Eisenstein series defined by Arakawa [2].

For an odd integer k, $\sigma \in \mathbf{C}$ such that $-k + 2\Re\sigma - 4 > 0$ and $\tau \in H = \{u + iv; v > 0\}$, the Cohen type Eisenstein series is defined by Ibukiyama and Saito [7] as

$$F(k,\sigma,\tau) = E(k,\sigma,\tau) + 2^{k/2-\sigma}(e^{2\pi i\frac{k}{8}} + e^{-2\pi i\frac{k}{8}})E(k,\sigma,-1/(4\tau))(-2i\tau)^{k/2},$$

$$E(k,\sigma,\tau) = (\Im\tau)^{\sigma/2} \sum_{d=1,\mathrm{odd}}^{\infty} \sum_{c=-\infty}^{\infty} (\frac{4c}{d})\epsilon_d^{-k}(4c\tau + d)^{k/2}|4c\tau + d|^{-\sigma},$$

where $j(\gamma,\tau) = (\frac{4c}{d})\epsilon_d^{-1}(4c\tau + d)^{1/2}$ is the usual automorphic factor on $\Gamma_0(4)$ [18]. This is a real analytic modular form of weight $-k/2$ on $\Gamma_0(4)$ and has a Fourier expansion

$$F(k,\sigma,\tau) = v^{\sigma/2} + v^{\sigma/2} \sum_{d=-\infty}^{\infty} c(d,\sigma,k)e^{2\pi i d u}\tau_d(v, \frac{\sigma-k}{2}, \frac{\sigma}{2}), \qquad \tau = u + iv,$$

where $\tau_d(v, \alpha, \beta)$ is the function defined by

$$\tau_d(v, \alpha, \beta) = \int_{-\infty}^{\infty} e^{-2\pi i d u} \tau^{-\alpha} \overline{\tau}^{-\beta} du \tag{1}$$

and its meromorphic continuation to all $(\alpha, \beta) \in \mathbf{C}^2$ (see [19], [10]), the d-th Fourier coefficient $c(d, \sigma, k)$ is given by

$$c(d, \sigma, k) = 2^{k+3/2-2\sigma} e^{(-1)^{(k+1)/2} \frac{\pi i}{4}} \frac{L_{(-1)^{(k+1)/2} d}(\sigma - \frac{k+1}{2})}{\zeta(2\sigma - k - 1)}.$$

Here

$$L_D(s) = \begin{cases} \zeta(2s-1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a) \chi_{D_K}(a) a^{-s} \sigma_{1-2s}(f/a), & D \neq 0,\ D \equiv 0, 1 \bmod 4 \\ 0, & D \equiv 2, 3 \bmod 4, \end{cases}$$

where the natural number f is defined by $D = d_K f^2$ with the discriminant d_K of $K = \mathbf{Q}(\sqrt{D})$, χ_K is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n) = \sum_{d|n} d^s$.

Put

$$S^+_\infty(F, s) = 2^{5-4\sigma} \pi^{\sigma-1/2} \frac{\Gamma(\sigma/2 - 1/2)^{-2}}{\zeta(2\sigma-2)^2} \sum_{d<0} \frac{L_{-d}(\sigma-1)^2}{|d|^{s-\sigma+3/2}},$$

$$S^-_\infty(F, s) = 2^{5-4\sigma} \pi^{\sigma-1/2} \frac{\Gamma(\sigma/2)^{-2}}{\zeta(2\sigma-2)^2} \sum_{d<0} \frac{L_{-d}(\sigma-1)^2}{|d|^{s-\sigma+3/2}}.$$

Note that if σ belongs to any compact subset (without poles) in σ-plane, then the series converge absolutely and uniformly for $\Re(s)$ being sufficiently large. Moreover, put

$$\mathcal{A}(s, \sigma) = \frac{2\pi \cos \pi s}{\sin \pi s} \frac{\Gamma(s - \sigma + 3/2)\Gamma(s + \sigma - 3/2)}{\Gamma(\sigma/2)^2 \Gamma(3/2 - \sigma/2)^2} S^+_\infty(F, s),$$

$$\mathcal{B}(s, \sigma) = \pi^{-1} \left(\cos \pi s - \frac{\sin \pi \sigma}{\sin \pi s} \right) \Gamma(s - \sigma + 3/2)\Gamma(s + \sigma - 3/2) S^-_\infty(F, s).$$

Then it follows from the works by Arakawa [2], Pitale [15], Müller [13], Zagier [25] combined with [11] that $S^{\pm}_{\infty}(F, s)$ can be continued meromorphically to all s and σ and satisfy the functional equation

$$S^{-}_{\infty}(F, 1-s) = \pi^{1-2s}\varphi(1-s)\left\{\mathcal{A}(s,\sigma) + \mathcal{B}(s,\sigma)\right\}.$$

The comparison of the reading coefficients of Laurent expansion at $\sigma = 2$ gives the functional equation of $\zeta^*(2s)\zeta^*(2s-1)$. The comparison of the residues at $\sigma = 2$ gives the functional equation of

$$\frac{\zeta'}{\zeta}(2s) - \frac{\zeta'}{\zeta}(2s-1).$$

The comparison of the constant terms of Laurent expansion at $\sigma = 2$ gives Theorem 4.

Note that this approach is taken from Ibukiyama-Saito [7]. They discovered this method in order to prove Theorem 2.

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