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# On weakly $\Phi$-like of order $\alpha$ with respect to certain analytic functions

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**Abstract**

For analytic functions $f(z)$ in the open unit disk $E$, weakly $\Phi$-like of order $\alpha$ with respect to a function $g(z)$ is introduced. The purpose of the present paper is to drive univalency for weakly $\Phi$-like of order $\alpha$ with respect to $g(z)$.

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## 1. Introduction

Let $A$ be the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A$ is called starlike if $f(z)$ satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in E). \tag{1.2}$$

For $f(z)$ given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, let $\Phi(f(z), g(z))$ be analytic on $(f(E), g(E)) \in \mathbb{C}^2$ with $\Phi(f(0), g(0)) = 0$, $\Phi(f(z), g(z)) \neq 0$ and $f(z) \neq 0$ in $0 < |z| < 1$, and for arbitrary $\omega \in f(E)$, $\Phi(\omega, g(re^{i\theta}))$ $(0 < r < 1$ and $0 \leq \theta \leq 2\pi)$ satisfy

$$\frac{d}{d\theta} \arg \Phi(\omega, g(re^{i\theta})) < \frac{1}{2}(3 - \alpha) \quad (z \in E)$$

where $1 < \alpha < 2$.

A function is said to be weakly $\Phi$-like of order $\alpha$ with respect to a functin $g(z)$ which satisfies the above condition of upper order $\frac{1}{2}(3 - \alpha)$ if it satisfies

$$\left|\arg \frac{zf'(z)}{\Phi(f(z), g(z))}\right| < \frac{\pi}{2}\alpha \quad (z \in E) \tag{1.3}$$

for $1 < \alpha < 2$ (cf. [1] and [2]).

## 2. Main result

**Theorem 1.** *If $f(z)$ is a weakly $\Phi$-like of order $\alpha$ with respect to a function $g(z)$ of upper order $\frac{1}{2}(3 - \alpha)$ for $1 < \alpha < 2$, then $f(z)$ is univalent in $E$.*

**Proof.** We will prove it by reductive absurdity. Let us suppose that there exists a positive real number $r$ $(0 < r < 1)$ for which $f(z)$ is univalent in $|z| < r$, but $f(z)$ is not univalent on $|z| = r$.

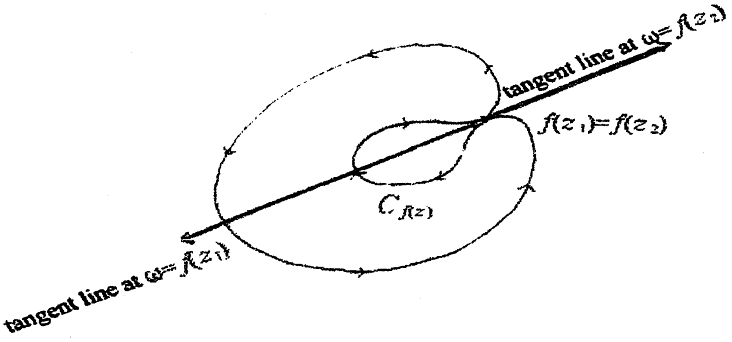


Figure 1

In view of Figure 1, we know that there are two points $z_1$ and $z_2$ $(z_1 \neq z_2)$ such that

$$|z_1| = |z_2| = r, \ \ z_1 = re^{i\theta_1}, \ \ z_2 = re^{i\theta_2}, \ \ 0 < \beta = \theta_2 - \theta_1,$$

for which $f(z_1) = f(z_2)$. Let us put $C = \{z : z = re^{i\theta}, \theta_1 < \theta \leq \theta_2\}$ and $C_{f(z)} = \{f(z) : z \in C\}$.

On the other hand, from the assumption of the theorem, we have

$$\begin{aligned}
&\int_{|z|=r} d\arg d\frac{zf'(z)}{\phi(f(z),g(z))} \\
&= \int_{|z|=r} d\arg z + \int_{|z|=r} d\arg df(z) - \int_{|z|=r} d\arg dz - \int_{|z|=r} d\arg \phi(f(z),g(z)) \\
&= 2\pi + \int_{|z|=r} d\arg df(z) - 2\pi - \int_{|z|=r} d\arg \phi(f(z),g(z))
\end{aligned}$$

and

$$\pi\alpha > \int_{|z|=r} d\arg df(z) - \int_{|z|=r} d\arg \phi(f(z),g(z)) > -\pi\alpha.$$

Now then, it is trivial that

$$\int_{|z|=r} d\arg \phi(f(z),g(z)) = 2\pi.$$

This shows that

$$4\pi > \pi\alpha + 2\pi > \int_{|z|=r} d\arg df(z) > 2\pi - \pi\alpha > 0$$

and therefore it must be

$$\int_{|z|=r} d\arg df(z) = 2\pi. \tag{2.1}$$

Now, we have

$$\int_{C_{f(z)}} d\arg df(z) = \int_{C} d\arg f'(z)dz = -\pi. \tag{2.2}$$

Putting $L = \{z : |z| = r\}$, then from the assumption of the theorem, we have

$$\pi\alpha > \int_{L-C} d\arg \frac{zf'(z)}{\Phi(f(z),g(z))} > -\pi\alpha$$

and so, we have

$$\pi\alpha > \int_{L-C} d\arg df(z) - \int_{L-C} d\arg \Phi(f(z),g(z)) > -\pi\alpha.$$

It follows that

$$
\begin{aligned}
&\int_{L-C} d\arg df(z) \\
&< \pi\alpha + \arg\Phi(f(z_1), g(re^{i(\theta_1+2\pi)})) - \arg\Phi(f(z_2), g(re^{i\theta_2})) \\
&= \pi\alpha + \arg\Phi(f(z_2), g(re^{i(\theta_1+2\pi)})) - \arg\Phi(f(z_2), g(re^{i\theta_2})) \\
&= \pi\alpha + \int_{\theta_2}^{\theta_1+2\pi} \frac{d}{d\theta}\arg\Phi(f(z_2), g(re^{i\theta}))d\theta \\
&< \pi\alpha + \int_{\theta_2}^{\theta_1+2\pi} \frac{1}{2}(3-\alpha)d\theta \\
&= \pi\alpha + \frac{1}{2}(3-\alpha)(2\pi-\beta) \\
&< \pi\alpha + (3-\alpha)\pi = 3\pi
\end{aligned}
$$

This show that

$$
\int_{|z|=r} d\arg df(z) - \int_{C_{f(z)}} d\arg df(z) < 3\pi. \tag{2.3}
$$

From (2.1), (2.2) and (2.3), we have contradiction. This completes the proof of theroem.

**Remak.** When $f(z)$ satisfies the hypothesis of Theorem 1, the real part of the function $zf'/\Phi(f(z), g(z))$ can be negative.

**Theorem 2.** *Let $\Phi(f(z), g(z))$ be analytic on $(f(E), g(E))$ with $\Phi(f(0), g(0)) = 0$, $\Phi(f(z), g(z)) \neq 0$ and $f(z) \neq 0$ in $0 < |z| < 1$ and for arbitrary $w \in f(E)$, $\Phi(w, g(re^{i\theta}))$ satisfies the following condition*

$$
\frac{d}{d\theta}\arg\Phi(w, g(re^{i\theta})) > -\frac{1}{2} \quad (z \in E)
$$

*where $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. Then, if $f(z)$ satisfies the following conditon*

$$
\mathrm{Re}\frac{z^2(f'(z))^2}{\Phi(f(z), g(z))} > 0 \quad (z \in E)
$$

*then $f(z)$ is univalent in $E$.*

**Proof.** Applying the same method as the proof of Theorem 1, let us suppose that there exists a positive real number $r$ $(0 < r < 1)$ for which $f(z)$ is univalent in $|z| < r$, but $f(z)$ is not univalent on $|z| = r$. Also, in view of Figure 1, we know that there are two points $z_1$ and $z_2$ $(z_1 \neq z_2)$ such that $|z_1| = |z_2| = r$, $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$, $0 < \theta_2 - \theta_1$, for which $f(z_1) = f(z_2)$. From the assumption of the theorem, we have

$$\begin{aligned}
\pi &> \int_C d\arg\frac{z^2(f'(z))^2}{\Phi(f(z),g(z))}\\
&= \int_C d\arg z^2 + \int_C d\arg(f'(z))^2 - \int_C d\arg\Phi(f(z),g(z))\\
&= 2\int_C d\arg z + 2\int_C d\arg f'(z) - \int_C d\arg\Phi(f(z),g(z))\\
&= 2\int_C d\arg z + 2\int_C d\arg df(z) - 2\int_C d\arg dz - \int_C d\arg\Phi(f(z),g(z))\\
&= 2\int_C d\arg df(z) - (\arg\Phi(f(z_2),g(re^{i\theta_2})) - \arg\Phi(f(z_1),g(re^{i\theta_1})))\\
&= 2\int_C d\arg df(z) - (\arg\Phi(f(z_1),g(re^{i\theta_2})) - \arg\Phi(f(z_1),g(re^{i\theta_1})))\\
&> -\pi.
\end{aligned}$$

It follow that

$$\begin{aligned}
&2\int_C d\arg df(z)\\
&> (\arg\Phi(f(z_1),g(re^{i\theta_2})) - \arg\Phi(f(z_1),g(re^{i\theta_1}))) - \pi\\
&= \int_{\theta_1}^{\theta_2}\frac{d}{d\theta}\arg\Phi(f(z_1),g(re^{i\theta}))d\theta - \pi\\
&> -\frac{1}{2}\int_{\theta_1}^{\theta_2} d\theta - \pi\\
&> -\pi - \pi = -2\pi
\end{aligned}$$

and therefore, we have

$$\int_C d\arg df(z) > -\pi,$$

but from the assumption, we have

$$\int_C d\arg df(z) = -\pi.$$

This is a contradiction and it completes the proof.

Applying the proof of Theroem 2, we have following corollary:

**Corollary 1.** *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *be analytic in* $E$ *and suppose that*

$$\mathrm{Re}\frac{z^2(f'(z))^2}{f(z)^{2-\beta}g(z)^{\beta}} > 0 \quad (z \in E)$$

*where* $\beta > 0$ *and*

$$\text{Re}\frac{zg'(z)}{g(z)} > -\frac{1}{2\beta} \quad \text{in } E, \tag{2.4}$$

*then* $f(z)$ *is univalent in* $E$.

**Remark.** If $g(z)$ satisfies the condition (2.4), then $g(z)$ is not necessarily a starlike function.

## References

[1] L. Brickman, *Φ-like analytic fucntions*, Bull. Amer. Math. Soc. **79** (1973), 555–558.

[2] M. Nunokawa, S. Owa, O. S. Kwon and N. E. Cho, *On Φ-like with respect to certain starlike functins*, to appear.