数理解析研究所講究録
第 1777 巻 2012 年 43-49

Handlebody-links and Heegaard splittings of link complements

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In this article, we give an observation on certain relations between handlebody-links and Heegaard splittings of link exteriors (or bridge presentations of links).

1. Handlebody-links

A *spatial graph* is a graph embedded in the 3-sphere S^3. Two spatial graphs are said to be *equivalent* if there exists an orientation-preserving homeomorphism $f : S^3 \to S^3$ which sends one of the spatial graphs to the other. It is known, by [14] and [24] for example, that two spatial graphs are equivalent if and only if their diagrams can be carried into each other by a finite sequence of elementary moves in Figure 1.

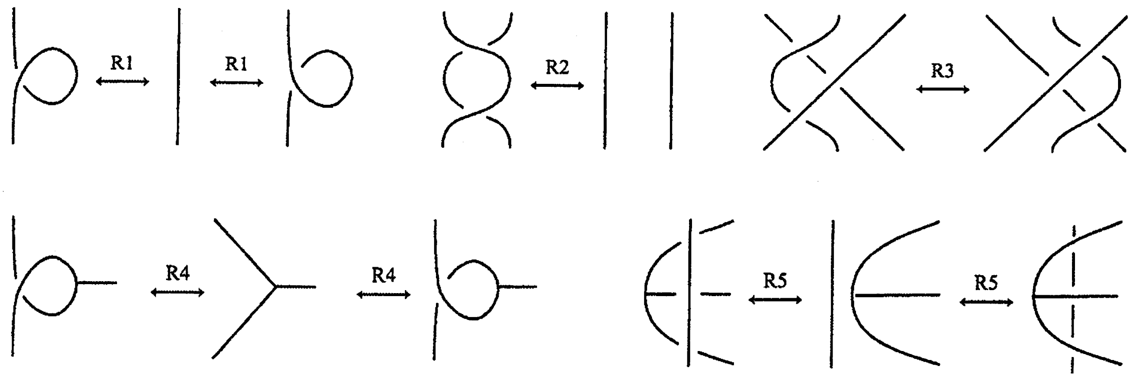

Figure 1:

A *handlebody-link* is a union of handlebodies embedded in the S^3 considered up to ambient isotopy, namely, two handlebody-links are said to be *equivalent* if there exists an orientation-preserving homeomorphism $f : S^3 \to S^3$ which sends one of the handlebody-links to the other. By considering the spines of handlebodies, handlebody-links can be regarded as spatial graphs up to certain local move in a 3-ball, called *IH-move*, replacing an I-shaped part with an H-shaped part. Throughout this article, we do not distinguish between handlebody-links and their spines. Ishii [5] proved that two handlebody-links are equivalent if and only if their diagrams can be carried into each other by a finite sequence of elementary moves in Figure 1 and Figure 2.

Remark 1 Let K and K' be two graphs embedded in S^3. Apparently, K and K' are equivalent as handlebody-links if they are equivalent as spatial graphs.

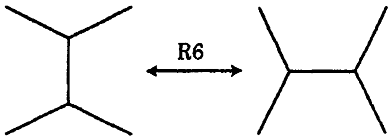

Figure 2:

However, the converse does not hold in general. For example, two spacial graphs represented by the diagrams in Figure 3 are not equivalent (as spatial graphs) while they are equivalent as handlebody-links.

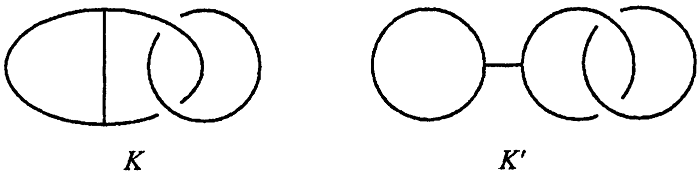

Figure 3:

Compared to the fact that various kinds of invariants are known for spacial graphs, not many invariants are known for handlebody-links yet. Invariants of handlebody-links defined by using "quandles" have been given in [6, 7, 13], and Mizusawa and Murakami [16] introduced another invariant of handlebody-knots by using certain quantum invariant for spatial graphs. Ishii and Masuoka [8] also introduced invariants of handlebody-knots each of which is defined from unimodular Hopf algebra.

2. Unknotting tunnel systems of knots and Heegaard splittings of knot exteriors

For a knot K in the 3-sphere S^3, let $N(K)$ be the regular neighborhood of K in S^3. We denote the exterior of K in S^3 by $E(K)$, that is, $E(K)$ is the closure of $S^3 \setminus N(K)$ in S^3. Let τ be the union of disjoint arcs $t_1, t_2, \dots, t_n$ embedded in S^3 so that $K \cap \tau = \partial\tau$. We call τ an *unknotting tunnel system* of K if the closure of $S^3 \setminus N'(K \cup \tau)$ is a genus-$(n+1)$ handlebody, which we denote by H_τ, where $N'(K \cup \tau)$ denotes a regular neighborhood of $K \cup \tau$ in S^3 which is taken to contain $N(K)$ in its interior. In particular, we call τ an *unknotting tunnel* of K when n is 1. Note that $N'(K \cup \tau) \cap E(K)$, or the closure of $E(K) \setminus H_\tau$, is a "compression-body". Thus the decomposition of $E(K)$ into H_τ and the closure of $E(K) \setminus H_\tau$ is a so-called (genus-(n+1)) *Heegaard splitting* of $E(K)$.

We say that two unknotting tunnel systems τ and τ' of a knot K are *equivalent* if there exists an orientation-preserving homeomorphism $f : E(K) \to E(K)$ such that $f(\tau \cap E(K)) = \tau' \cap E(K)$. It can be easily seen that every Heegaard splitting of $E(K)$ can be obtained from an unknotting tunnel system of K and that two unknotting tunnel systems of a knot K are equivalent if and only if the Heegaard splittings of $E(K)$ obtained from the unknotting tunnel systems are *equivalent*, that is, there is an orientation-preserving homeomorphism of $E(K)$ carrying one of the Heeagaard splittings to the other. Thus we do not distinguish between the two concepts, unknotting tunnel systems of knots and Heegaard splittings of knot exteriors, in this article. Unknotting tunnels of knots or Heegaard splittings of knot exteriors were studied by various mathematician (see [1, 2, 4, 15, 17, 18, 21] for example),

When a knot K in S^3 and its unknotting tunnel system τ are given, we can obtain from them a handlebody-link $N(K) \cup H_\tau$. We denote this handlebody-link by $H(K, \tau)$. Then we have the following proposition.

Proposition 2 *Let K be a knot in S^3, and let τ and τ' be unknotting tunnel systems of K. If the two unknotting tunnel systems τ and τ' are equivalent then $H(K, \tau)$ and $H(K, \tau')$ are equivalent as handlebody-links.*

Proof. Assume that τ_1 and τ_2 are equivalent. By the definition of equivalence of unknotting tunnel systems, we can find an orientation-preserving homeomorphism $f : E(K) \to E(K)$ such that $f(\tau \cap E(K)) = \tau' \cap E(K)$. This implies that f sends $N'(K \cup \tau) \cap E(K)$ to $N'(K \cup \tau') \cap E(K)$, and hence $H(\tau)$ to $H(\tau')$. By [3], the homeomorphism f extends to a self-homeomorphism $\bar{f}$ of S^3 preserving K. Namely, we have an orientation-preserving homeomorphism $\bar{f} : S^3 \to S^3$ which carries $H(K, \tau)$ to $H(K, \tau')$. ∎

Remark 3 In the above proof, we used the fact that every orientation-preserving homeomorphism of a knot exterior extends to a homeomorphism of S^3 preserving the knot, which is not always true for links. However, we may generalize arguments in this section to those for links by considering homeomorphisms of link exteriors preserving the "peripheral structures". We also need to assume that one of the two compression bodies of a given Heegaard splitting of a link exterior is a handlebody.

3. Bridge presentations of links

An *n-string trivial tangle* is a pair (B, t) of the 3-ball B and n arcs t properly embedded in B so that they are parallel to the boundary of B. Namely, t bounds disjoint disks in B together with n arcs on the boundary of B. An *n-bridge sphere* of a link L in S^3 is a 2-sphere which meets L in $2n$ points and cuts (S^3, L) into two n-string trivial tangles, (B_1, t_1) and (B_2, t_2). We call this decomposition of (S^3, L) into n-string trivial tangles an *n-bridge presentation* (or an *n-bridge decomposition*) of L and denote it by $(B_1, t_1) \cup_S (B_2, t_2)$. It is known that every link admits an n-bridge presentation for some positive integer n. We call a link L an *n-bridge link* if L admits an n-bridge presentation and does not admit an $(n-1)$-bridge presentation.

We say that two n-bridge presentations $(B_1, t_1) \cup_S (B_2, t_2)$ and $(B_1', t_1') \cup_{S'} (B_2', t_2')$ (or S and S' in brief) of a link L are *equivalent* (or *homeomorphic*) if there exists an orientation-preserving self-homeomorphism f of S^3 which preserves L and sends S to S'. We say that S and S' are *isotopic* if f is isotopic to the identity map by an isotopy preserving L. Equivalence classes or isotopy classes of bridge spheres of links have been studied in [9, 10, 11, 12, 19, 20, 22, 23] and so on. Most of the results are on the uniqueness of bridge spheres of given links, and in [12] the author gave an isotopy classification of 3-bridge spheres of 3-bridge arborescent links. (We can also obtain a homeomorphism classification of the bridge spheres from their isotopy classification.) In the paper, 3-bridge spheres were distinguished by using the "commutator invariants" of the genus-2 Heegaard surfaces obtained as the pre-images of the 3-bridge spheres in the double branched coverings of S^3 branched along given links. However, when n becomes bigger, the correspondence between n-bridge spheres of links and genus-$(n-1)$ Heegaard surfaces of 3-manifolds gets more complicated and we cannot directly apply the methods used in the case where n is 3. One of the ideas which might be useful to treat higher-index bridge presentations is using the relation between bridge presentations of links and handlebody-links as seen in the rest of this section.

Let $(B_1, t_1) \cup_S (B_2, t_2)$ be an n-bridge presentation of a link L in S^3. Note that the closure of $B_1 \setminus N'(L)$ is a handlebody of genus-n, which we denote by H_{B_1}, and the closure of $(B_2 \cup N'(L)) \setminus N(L)$ is a compression-body, where $N(L)$ and $N'(L)$ are regular neighborhoods of L in S^3 such that $N(L)$ is contained in the interior of $N'(L)$ (see Figure 4). We denote by $H(L, S; B_1)$ the handlebody-link $N(L) \cup H_{B_1}$. Another handlebody-link $H(L, S; B_2)$ can be defined similarly, by switching the roles of B_1 and B_2. In this way, we obtain two handlebody-links from a given n-bridge presentation of a link. Then the following proposition can be proved by an argument similar to that for Proposition 2.

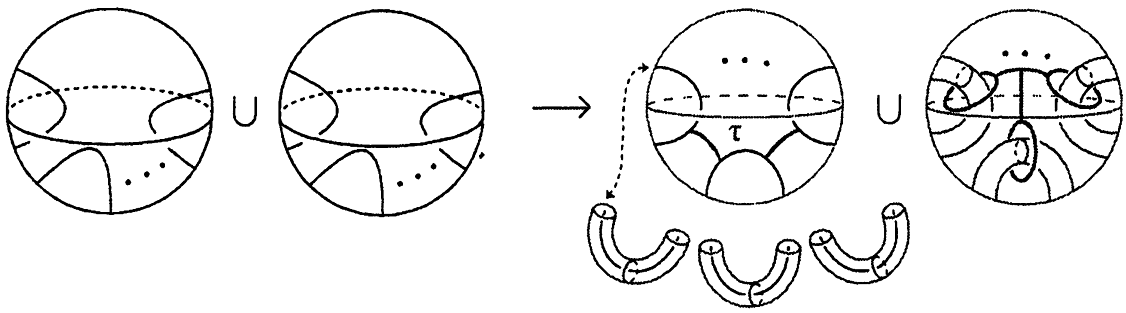

Figure 4:

Proposition 4 *For a knot L in S^3, suppose that two n-bridge presentations $(B_1, t_1) \cup_S (B_2, t_2)$ and $(B_1', t_1') \cup_{S'} (B_2', t_2')$ of L are equivalent. Then $\{H(L, S; B_1), H(L, S; B_2)\}$ and $\{H(L, S'; B_1'), H(L, S'; B_2')\}$ are equivalent, namely,*

(i) handlebody-links $H(L, S; B_1)$ and $H(L, S; B_2)$ are equivalent to $H(L, S'; B_1')$ and $H(L, S'; B_2')$, respectively, or

(ii) handlebody-links $H(L, S; B_1)$ and $H(L, S; B_2)$ are equivalent to $H(L, S'; B_2')$ and $H(L, S'; B_1')$, respectively.

As seen in Figure 4, each bridge presentation of a link gives two unknotting tunnel systems of the link (or two Heegaard splittings of the link exterior). Moreover, we can see that an equivalence class of bridge presentations of a link gives the set of two equivalence classes, possibly same, of unknotting tunnel systems of the link.

4. Towards applications

Towards applications of handlebody-links to unknotting tunnel systems or bridge presentations of links, one can start from giving alternative proofs for some results already known. For example, a classification of the unknotting tunnels of 2-bridge knots (up to equivalence as in this article) is given in [15]. (We remark that the isotopy classification of the unknotting tunnels of 2-bridge knots is given in [18].) In particular, the two unknotting tunnels τ_1 and τ_2 of a 2-bridge knot in Figure 5 are not homeomorphic by [15]. The handlebody-links $H(K, \tau_1)$ and $H(K, \tau_2)$ obtained from the two unknotting tunnels are also illustrated in the figure. Thus, if $H(K, \tau_1)$ and $H(K, \tau_2)$ are not equivalent as handlebody-links, then it implies that τ_1 and τ_2 are not equivalent by Proposition 2.

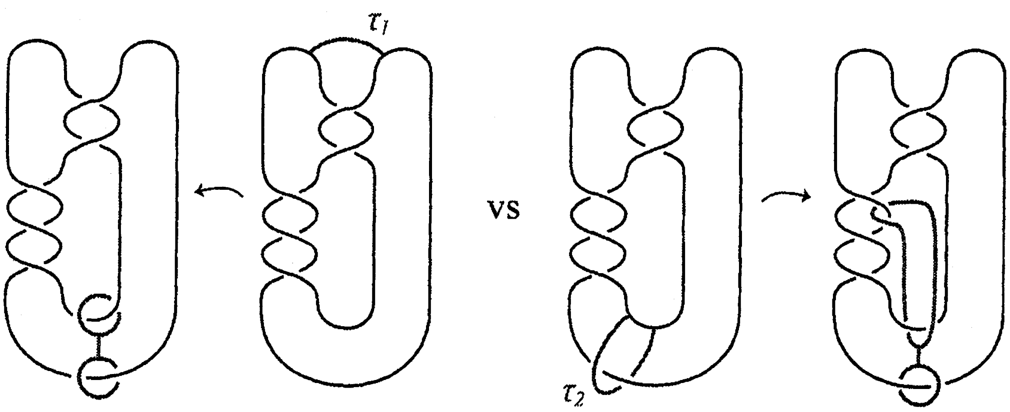

Figure 5:

Problem 5 (1) Prove that $H(K, \tau_1)$ and $H(K, \tau_2)$ in Figure 5 are not equivalent as handlebody-links. How about for other unknotting tunnels of 2-bridge knots?

(2) Distinguish unknotting tunnel systems of n-bridge knots ($n > 2$) by using the handlebody-links corresponding to them.

We remark that invariants which can be derived from the fundamental groups of the complements of handlebody-links cannot distinguish the handlebody-links obtained from unknotting tunnel systems since all such handlebody-links have complements with isomorphic fundamental groups, as seen in the following proposition.

Proposition 6 *Let $H(K, \tau)$ be the handlebody-link obtained from a given knot K and an unknotting tunnel system τ of K. Then the fundamental group of the complement of $H(K, \tau)$ is isomorphic to the free product of the free abelian group of rank* 2 *and the infinite cyclic group.*

Proof. Note that the exterior of $H(K,\tau)$ in S^3 is the compression body $N'(K\cup\tau)\setminus N(K)$ which can be decomposed by a disk into a solid torus and the product of a torus and an interval. Then the desired result can be obtained by applying van Kampen's theorem. ∎

Problem 7 *Distinguish n-bridge presentations of links by comparing the sets of handlebody-links corresponding to them.*

Recall that the unknot and a 2-bridge link admit a unique n-bridge sphere for each natural number n which is bigger than or equal to 1 and 2, respectively. (see [19, 20, 23]). Thus the simplest links admitting mutually non-equivalent (minimal) bridge spheres must be at least 3-bridge links. In fact, such examples of 3-bridge links were given in [9, 12]. We can start with these examples to see how useful the handlebody-links are to distinguish bridge presentations, or study more complicated links.

We would like to thank Professor Makoto Sakuma for mentioning about Proposition 6 which enabled us to find out mistakes in the examples given in the talk by the author.

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