数理解析研究所講究録
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Problem session

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1 Cone deformation

This is just a remark. Here is one way of understanding cone deformation. Consider the elliptic transformation T of the upper half plane with center λi and angle 2θ. (Fig. 1) We investigate the hyperbolic cone of angle 4θ whose monodromy is given by T^2.

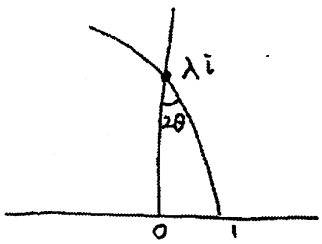


Figure 1: Rotation T of angle 2θ around λi

The Möbius transformation T is represented by

$$\begin{aligned} T &= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \lambda\sin\theta \\ -\lambda^{-1}\sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

We impose the following assumption.

$$T(0) = \lambda\tan\theta = 1$$

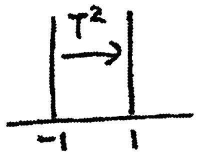
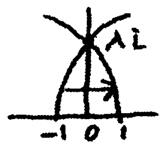
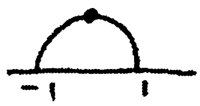
Thus, λ and θ are functions of each other. We then obtain

$$T = \begin{pmatrix} \cos\theta & \cos\theta \\ -\frac{\sin\theta}{\cos^2\theta} & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta \\ \cos\theta - \frac{1}{\cos\theta} & \cos\theta \end{pmatrix},$$

and

$$T^2 = \begin{pmatrix} 2\cos^2\theta - 1 & 2\cos^2\theta \\ 2\cos^2\theta - 2 & 2\cos^2\theta - 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \cos 2\theta + 1 \\ \cos 2\theta - 1 & \cos 2\theta \end{pmatrix}.$$

The following table illustrates the various limiting situations. Here, we denote $c = \cos\theta$ and $c' = \cos 2\theta$.

θ	$\cos\theta$	T	T^2	
0	1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
θ	c	$\begin{pmatrix} c & c \\ c - \frac{1}{c} & c \end{pmatrix}$	$\begin{pmatrix} c' & c'+1 \\ c'-1 & c' \end{pmatrix}$	
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	
$\frac{\pi}{2}$	0	–	$\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$